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ABSTRACT

We prove an extension of the homology version of the Hofer–Zehnder conjecture proved by Shelukhin to the weighted projective spaces which are symplectic orbifolds. In particular, we prove that if the number of fixed points counted with their isotropy order as multiplicity of a non-degenerate Hamiltonian diffeomorphism of such a space is larger than the minimum number possible, then there are infinitely many periodic points.

1. Introduction

We are interested in the study of the Hofer–Zehnder conjecture in a specific class of symplectic orbifolds: the weighted projective spaces. These spaces appear as symplectic reductions: let us fix a tuple of weights $\mathbf{q} = (q_0, \dots, q_d) \in (\mathbb{N}^*)^{d+1}$ and define the Hamiltonian map $K_{\mathbf{q}} : \mathbb{C}^{d+1} \rightarrow \mathbb{R}$,

$$K_{\mathbf{q}}(z) := \pi \sum_{j=0}^d q_j |z_j|^2,$$

this Hamiltonian induces the S^1 -action on \mathbb{C}^{d+1} defined by $\bar{t} \cdot (z_j) := (e^{2i\pi q_j t} z_j)$ and preserving the weighted sphere $\mathbb{S}(\mathbf{q}) := K_{\mathbf{q}}^{-1}(\pi)$, the weighted projective space with weights \mathbf{q} is the symplectic orbifold $\mathbb{CP}(\mathbf{q}) := \mathbb{S}(\mathbf{q})/S^1$. The symplectic form ω of $\mathbb{CP}(\mathbf{q})$ is the unique 2-form that satisfies $p^*\omega = i^*\Omega$ where $p : \mathbb{S}(\mathbf{q}) \rightarrow \mathbb{CP}(\mathbf{q})$ is the quotient map, $i : \mathbb{S}(\mathbf{q}) \hookrightarrow \mathbb{C}^{d+1}$ is the inclusion and $\Omega := \sum_j dx_j \wedge dy_j$ is the canonical symplectic form of \mathbb{C}^{d+1} . The study of Hamiltonian dynamics on $\mathbb{CP}(\mathbf{q})$ is equivalent to the study of the Hamiltonian dynamics on $\mathbb{C}^{d+1} \setminus 0$ restricted to the flows induced by positively 2-homogeneous Hamiltonians (H_t) commuting with $K_{\mathbf{q}}$. We study Hamiltonian diffeomorphisms $\varphi \in \text{Ham}(\mathbb{CP}(\mathbf{q}))$, which are time-one flows of this dynamics. The case where all the weights $q_j = 1$ corresponds to the dynamics on the complex projective space \mathbb{CP}^d , it was proved by Fortune and Weinstein that the number of fixed points of such a diffeomorphism is always $\geq d + 1$, as was conjectured by Arnol'd [For85]. We first show the following generalization of Fortune–Weinstein theorem. Let $|\mathbf{q}| := \sum_j q_j$ and for $z \in \mathbb{CP}(\mathbf{q})$ let us denote $\text{order}(z) \in \mathbb{N}$ the order of the isotropy group of any lift of z to $\mathbb{S}(\mathbf{q})$.

THEOREM 1.1. *Every Hamiltonian diffeomorphism φ of $\mathbb{CP}(\mathbf{q})$ satisfies*

$$\sum_{x \in \text{Fix}(\varphi)} \text{order}(x) \geq |\mathbf{q}|.$$

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Surprisingly, the number of fixed points is replaced by a weighted count of the fixed points. Of course, when $\mathbf{q} = (1, \dots, 1)$, we get the Fortune–Weinstein theorem back. In [Lu08], the author claims that the $d + 1$ lower bound is also satisfied by the unweighted count of the fixed points. However, the proof contains gaps where the orders of isotropy groups should intervene.¹ Let us remark that Theorem 1.1 is obvious when the q_j are prime with each other two by two because for every $j \in \{0, \dots, d\}$, there is only one point $z_j \in \mathbb{C}\mathbb{P}(\mathbf{q})$ such that $\text{order}(z_j) = q_j$ and every diffeomorphism preserves the order. Similarly, Theorem 1.1 is a consequence of the Fortune–Weinstein theorem when \mathbf{q} satisfies for all i, j , either $q_i = q_j$ or q_i is prime with q_j (by considering the restriction of φ to the weighted projective subspace $\{z \mid \text{order}(z) = q_j\}$ for each j).

The main subject of our article is the study of periodic points of $\varphi \in \text{Ham}(\mathbb{C}\mathbb{P}(\mathbf{q}))$, which are fixed points of φ^k for some $k \in \mathbb{N}^*$. On the tori \mathbb{T}^{2d} , Conley conjectured that every Hamiltonian diffeomorphism has infinitely many periodic points. This statement was proven by Hingston [Hin09] after decades of advances [CZ86, SZ92, FH03, LeC06] and then generalized to a large class of symplectic manifolds by Ginzburg [Gin10], Ginzburg and Gürel [GG12, GG15, GG19] and Orita [Ori19]. However, the Conley conjecture does not hold in $\mathbb{C}\mathbb{P}(\mathbf{q})$: the Hamiltonian diffeomorphism

$$[z_0 : z_1 : \dots : z_d] \mapsto [e^{2i\pi a_0} z_0 : e^{2i\pi a_1} z_1 : \dots : e^{2i\pi a_d} z_d],$$

with rationally independent $a_0, \dots, a_d \in \mathbb{R}/\mathbb{Z}$, has only $d + 1$ periodic points, which is the projection of the canonical base. In this case, one has equality at Theorem 1.1. Hofer and Zehnder conjectured that the only case for which a Hamiltonian diffeomorphism of a symplectic manifold can have finitely many periodic points is when its periodic points are fixed and in the minimal number possible [HZ94, p. 263]. The conjecture was inspired by a theorem of Franks showing that every area-preserving homeomorphism of $\mathbb{C}\mathbb{P}^1$ isotopic to the identity has two or infinitely many periodic points (which implies the Hofer–Zehnder conjecture in $\mathbb{C}\mathbb{P}^1$) [Fra92, Fra96]. Collier *et al.* gave a proof of the Franks theorem in the case of $\text{Ham}(\mathbb{C}\mathbb{P}^1)$ using symplectic tools [CKR⁺12]. The higher achievement in proving this conjecture is Shelukhin’s theorem showing a homology version of this conjecture in a class of symplectic manifolds including $\mathbb{C}\mathbb{P}^d$ (see [She22] and also [All22a, CGG22]). In this article, we prove an extension of this theorem to the weighted projective spaces. Following Shelukhin, we introduce a homology count of the fixed points of $\varphi \in \text{Ham}(\mathbb{C}\mathbb{P}(\mathbf{q}))$:

$$N(\varphi; \mathbb{F}) := \sum_{x \in \text{Fix}(\varphi)} O(x; \mathbb{F}) \in \mathbb{N},$$

where the $O(x; \mathbb{F}) \in \mathbb{N}$ are numbers linked to the local homology groups of the fixed points x and depending on a coefficient field \mathbb{F} (see (7) for a precise definition of the homology count, the precise definition of $N(\varphi; \mathbb{F})$ also depends on the choice of isotopy from id to φ). In the case where the fixed point x is non-degenerate (i.e. 1 is not an eigenvalue of $d\varphi(x)$), $O(x; \mathbb{F}) = \text{order}(x)$. Therefore, if every fixed point is non-degenerate, $N(\varphi; \mathbb{F})$ is the weighted count of fixed points used in Theorem 1.1.

THEOREM 1.2. *Every Hamiltonian diffeomorphism φ of $\mathbb{C}\mathbb{P}(\mathbf{q})$ such that $N(\varphi; \mathbb{F}) > |\mathbf{q}|$ for some field \mathbb{F} whose characteristic is either zero or prime with each q_j has infinitely many periodic points. Moreover, when φ has finitely many fixed points, if \mathbb{F} has characteristic zero in the former assumption, there exists $A \in \mathbb{N}$ such that, for all prime $p \geq A$, φ has a p -periodic point*

¹ Just before defining the family Ω_z in the beginning of the proof, the fact that $\Pi \circ \tilde{z}_1 = \Pi \circ \tilde{z}_2$ should imply $\text{order}(z_1)(\lambda_1 - \lambda_2) \in 2\pi\mathbb{Z}$ rather than $\lambda_1 - \lambda_2 \in 2\pi\mathbb{Z}$.

that is not a fixed point; if \mathbb{F} has characteristic $p \neq 0$, φ has infinitely many periodic points whose period belongs to $\{p^k \mid k \in \mathbb{N}\}$.

Using the fact that $O(z; \mathbb{F}) = \text{order}(z)$, we obtain the following answer to the generalized Hofer–Zehnder conjecture in the non-degenerate case.

COROLLARY 1.3. *Every Hamiltonian diffeomorphism φ of $\mathbb{C}P(\mathbf{q})$ such that*

$$\sum_{\substack{x \in \text{Fix}(\varphi) \\ x \text{ non-degenerate}}} \text{order}(x) > |\mathbf{q}|$$

has infinitely many periodic points.

The proof of Theorem 1.2 is an adaptation of our proof of the theorem of Shelukhin in the case of $\mathbb{C}P^d$ (see [All22a]). It is based on the proofs given by Givental and Th eret of the Fortune–Weinstein theorem using generating functions [Giv90, The98], we mention that these proofs can be easily adapted to show Theorem 1.1. The technical base of our proof is prior to Floer theory and does not appeal to the J -holomorphic curves theory: it relies on finite-dimensional critical point theory and classical algebraic topology. The key ideas of the proof of Theorem 1.2 are due to Shelukhin: we are studying a barcode that we can associate to the persistence module $(G_*^{(-\infty, t)}(\varphi; \mathbb{F}))_t$ induced by the generating functions homology of φ (as for $N(\varphi; \mathbb{F})$, it also depends on the choice of isotopy from id to φ). The definition of such homology groups was introduced in [All22a] in the case of $\mathbb{C}P^d$ and is a generating functions counterpart to the Floer homology groups $HF_*^{(-\infty, t)}(\varphi)$ inspired by previous constructions of Viterbo [Vit92] and Traynor [Tra94] in the case of compactly supported Hamiltonian diffeomorphisms of \mathbb{C}^d . The theory of barcodes in symplectic topology was introduced by Polterovich and Shelukhin in [PS16]. Adapting Shelukhin’s proof, we show that if the homology count $N(\varphi; \mathbb{F})$ is greater than $|\mathbf{q}|$, the barcode of $(\varphi; \mathbb{F})$ must contain finite bars. As each finite bar must have a length lower than 1 whereas the sum of the length of (representatives of) finite bars $\beta_{\text{tot}}(\varphi^k; \mathbb{F})$ is diverging to $+\infty$ for good choices of powers k and fields \mathbb{F} according to a Smith-type inequality, the number of finite bars diverges (more precisely: the number of \mathbb{Z} -orbits of finite bars diverges) and so does the number of periodic points.

Major works in symplectic topology recently used symplectic orbifolds [CHM+21, PS21] and we are hoping that this study will contribute to a better understanding of what one should expect of an orbifold Hamiltonian Floer homology theory. Indeed, our ‘weighted’ result is not the first intriguing phenomenon observed in this topic: a recent work extending the Floer homology to global quotient orbifolds (i.e. orbifolds obtained as quotient of a manifold by a finite group, in contrast to $\mathbb{C}P(\mathbf{q})$) relates this homology theory to the Chen–Ruan homology [Mor19].

Organization of the paper

In § 2, we discuss preliminary results needed for the construction of the generating functions homology: homology projective join on weighted projective spaces and generating functions of $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms. In § 3, we extend the construction of the generating functions homology for complex projective spaces to weighted projective spaces. We then study the spectral invariants associated with these homology groups and derive Theorem 1.1 and the universal bound on the length of finite bars (Theorem 3.9). In § 4, we show the Smith-type inequality satisfied by the length of the finite bars of the barcode associated with a diffeomorphism φ . In § 5, we prove Theorem 1.2. In Appendix A, we discuss an extension of the Thom isomorphism and the Gysin long exact sequence to orbibundles that is needed in our article.

2. Preliminaries

2.1 The category of weighted projective spaces

Let us fix the conventions and notation for weighted projective spaces.

All spaces considered here are finite dimensional. Let E be a complex vector space and let $\rho : S^1 \rightarrow GL(E)$ be a smooth group morphism defining a linear S^1 -action of E . Formally, a weighted projective space will consist of the data (E, ρ) , where E and ρ are as above, and be denoted $\mathbb{P}(E, \rho)$. The group $\mathbb{R}_+^* \times S^1$ acts on $E \setminus 0$ by

$$(\lambda, t) \cdot z := \lambda \rho(t)z, \quad \forall (\lambda, t) \in \mathbb{R}_+^* \times S^1, \quad \forall z \in E \setminus 0.$$

The induced orbifold $(E \setminus 0)/(\mathbb{R}_+^* \times S^1)$ is naturally associated with $\mathbb{P}(E, \rho)$ and we will often identify $\mathbb{P}(E, \rho)$ with this space, by a slight abuse of notation (see [LT97] for an introduction to the notion of orbifold in symplectic geometry). A morphism from $\mathbb{P}(E, \rho)$ to $\mathbb{P}(E', \rho')$ is a class of injective S^1 -equivariant linear morphisms $(E, \rho) \rightarrow (E', \rho')$ under the equivalence relationship \sim defined by

$$f \sim g \Leftrightarrow \exists (\lambda, t) \in \mathbb{R}_+^* \times S^1, f = \lambda \rho'(t) \circ g.$$

Every morphism induces a natural orbifold map, we identify morphism and induced map by a slight abuse of notation. A weighted projective subspace $P \subset \mathbb{P}(E, \rho)$ is a projective space $P = \mathbb{P}(F, \rho')$ induced by an S^1 -invariant subspace $F \subset E$ with ρ' the natural restricted action.

Given an S^1 -action $\rho : S^1 \rightarrow GL(E)$, there exists a base (v_0, \dots, v_d) and integers $q_0, \dots, q_d \in \mathbb{Z}$ such that

$$\rho(t)v_j := e^{2i\pi tq_j}v_j, \quad \forall t \in S^1, \quad \forall j \in \{0, \dots, d\},$$

seeing S^1 as \mathbb{R}/\mathbb{Z} . The multiset $\{q_0, \dots, q_d\}$ is uniquely defined by (E, ρ) and called the weights of $\mathbb{P}(E, \rho)$: it defines a functor from the category of weighted projective spaces to the category of multisets. We only study weighted projective spaces with positive weights. A usual (or ‘unweighted’) projective space is a weighted projective space whose weights are all equal to one. Given $\mathbf{q} \in (\mathbb{N}^*)^{d+1}$, let $\rho_{\mathbf{q}} : S^1 \rightarrow GL_{d+1}(\mathbb{C})$ be such that

$$\rho_{\mathbf{q}}(t)\varepsilon_j := e^{2i\pi tq_j}\varepsilon_j, \quad \forall t \in S^1, \quad \forall j \in \{0, \dots, d\}, \tag{1}$$

where (ε_j) is the canonical base. We use the notation $\mathbb{CP}(\mathbf{q}) := \mathbb{P}(\mathbb{C}^{d+1}, \rho_{\mathbf{q}})$. Every weighted projective space with weights \mathbf{q} is isomorphic to $\mathbb{CP}(\mathbf{q})$: the category of weighted projective spaces up to isomorphism is equivalent to the category of multisets of \mathbb{N}^* .

2.2 Projective join

Given two weighted projective spaces $P_j := \mathbb{P}(E_j, \rho_j)$, $j \in \{1, 2\}$, their projective join $P_1 * P_2$ is the weighted projective space $\mathbb{P}(E_1 \times E_2, \rho_1 \times \rho_2)$. The spaces P_1 and P_2 are naturally included in $P_1 * P_2$ via $E_1 \times 0 \subset E_1 \times E_2$ and $0 \times E_2 \subset E_1 \times E_2$. Given subsets $A_j \subset P_j$, one can also define the projective join $A_1 * A_2 \subset P_1 * P_2$ by $A_1 \cup A_2 \cup \pi(\tilde{A}_1 \times \tilde{A}_2)$, where $\pi : (E_1 \times E_2) \setminus 0 \rightarrow P_1 * P_2$ is the quotient map and the $\tilde{A}_j \subset E_j$ are the inverse images of the A_j under $E_j \setminus 0 \rightarrow P_j$. Given points $a_j \in A_j$, the projective line $(a_1 a_2) \subset P_1 * P_2$ will refer to the weighted projective line $\{a_1\} * \{a_2\}$.

Given a topological space or pair X , $H_*(X)$ and $H^*(X)$ denote the singular homology and cohomology groups. When we need to make explicit the ring of coefficients R , it will be written $H_*(X; R)$ and $H^*(X; R)$. In [All22a, Appendix A], we defined a natural morphism $\text{pj}_* : H_*(A \times B) \rightarrow H_{*+2}(A * B)$ in the unweighted case called the homology projective join. Let us extend

this natural map. Given $A \subset P$ and $B \subset P'$ subsets of weighted projective spaces, let us define

$$E_{A,B} := \{(a, b, c) \in A \times B \times (A * B) \mid c \in (ab)\},$$

and projection maps $p_1 : E_{A,B} \rightarrow A \times B$ and $p_2 : E_{A,B} \rightarrow A * B$.

LEMMA 2.1. *At the topological level, the map p_1 defines a $\mathbb{C}P^1$ -orbibundle (in the sense of Appendix A), with the natural orientation induced by the complex structure.*

Proof. We refer to Appendix A for the statement of the triviality condition we must show. In order to work with coordinates, one can assume $A \subset \mathbb{C}P(\mathbf{q})$ and $B \subset \mathbb{C}P(\mathbf{q}')$, $\mathbf{q} \in \mathbb{N}^{d+1}$, $\mathbf{q}' \in \mathbb{N}^{d'+1}$. To simplify notation, let us rather express $\mathbb{C}P(\mathbf{q})$ as the quotient of $\mathbb{C}^{d+1} \setminus 0$ under the equivalence relation

$$z \sim z' \Leftrightarrow z_j = \lambda^{q_j} z'_j, \quad \exists \lambda \in \mathbb{C}^*, \forall j, \tag{2}$$

and similarly for $\mathbb{C}P(\mathbf{q}')$ and $\mathbb{C}P(\mathbf{q}, \mathbf{q}')$. The covering $(V_{k,l})$ of $A \times B$ is

$$V_{k,l} := \{([a], [b]) \in A \times B \mid a_k \neq 0 \text{ and } b_l \neq 0\}.$$

The associated sets $U_{k,l} \subset \mathbb{C}^{d+d'}$ are the maximal subsets such that the map

$$\tilde{\varphi}_{k,l} : (a_0, \dots, \hat{a}_k, \dots, a_d, b_0, \dots, \hat{b}_l, \dots, b_{d'}) \mapsto ([a_0, \dots, a_d], [b_0, \dots, b_{d'}]),$$

with $a_k := 1$ and $b_l := 1$ in the right-hand side (the symbol \hat{a}_k means that the symbol a_k is erased from the sequence), are well-defined $U_{k,l} \rightarrow V_{k,l}$. Let us use $\mathbb{U}_k \subset \mathbb{C}$ to denote the group of the k th roots of unity. Then $\Gamma_{k,l} := \mathbb{U}_{q_k} \times \mathbb{U}_{q'_l}$ acts linearly on $U_{k,l}$ by

$$(\zeta, \zeta') \cdot (a, b) := (\zeta^{q_0} a_0, \dots, \widehat{\zeta^{q_k} a_k}, \dots, \zeta^{q_d} a_d, (\zeta')^{q'_0} b_0, \dots, \widehat{(\zeta')^{q'_l} b_l}, \dots, (\zeta')^{q'_{d'}} b_{d'}).$$

The maps $\tilde{\varphi}_{k,l}$ induce homeomorphisms $\varphi_{k,l} : U_{k,l}/\Gamma_{k,l} \rightarrow V_{k,l}$.

Let us now define the $\Gamma_{k,l}$ -invariant map $\tilde{\chi}_{k,l} : U_{k,l} \times \mathbb{C}P^1 \rightarrow p_1^{-1}(V_{k,l})$ by

$$(a, b, [u : v]) \mapsto (\tilde{\varphi}_{k,l}(a, b), [u^{q_0} a_0, \dots, u^{q_d} a_d, v^{q'_0} b_0, \dots, v^{q'_{d'}} b_{d'}]),$$

with $a_k := 1$ and $b_l := 1$ in the right-hand side. These maps are invariant under the following actions of the $\Gamma_{k,l}$:

$$(\zeta, \zeta') \cdot ((a, b), [u : v]) := ((\zeta, \zeta') \cdot (a, b), [\zeta^{-1} u : (\zeta')^{-1} v])$$

and induce homeomorphisms $\chi_{k,l} : (U_{k,l} \times \mathbb{C}P^1)/\Gamma_{k,l} \rightarrow p_1^{-1}(V_{k,l})$. □

As $\mathbb{C}P^1 \simeq \mathbb{S}^2$, there is a natural Gysin morphism $p_1^* : H_*(A \times B) \rightarrow H_{*+2}(E_{A,B})$ according to Corollary A.2. We can now extend the definition of pj_* to the weighted case by setting $\text{pj}_* := (p_2)_* \circ p_1^*$. Let us now get all the properties stated in [All22a, Appendix A] back in the weighted case.

Here, let us express $\mathbb{C}P(\mathbf{q})$ as the quotient of $\mathbb{C}^{d+1} \setminus 0$ under the equivalence relation (2) in order to simplify the notation in the following definition. Following Kawasaki [Kaw73], let us consider the map $g_{\mathbf{q}} : \mathbb{C}P^d \rightarrow \mathbb{C}P(\mathbf{q})$,

$$g_{\mathbf{q}}([z_0 : \dots : z_d]) := [z_0^{q_0}, \dots, z_d^{q_d}].$$

Let $\mathbb{U}_k \subset \mathbb{C}$ denote the groups of the k th roots of unity and $\mathbb{U}_{\mathbf{q}} := \mathbb{U}_{q_0} \times \dots \times \mathbb{U}_{q_d}$ acting coordinate-wise on $\mathbb{C}P^d$. The map $g_{\mathbf{q}}$ induces a homeomorphism $\mathbb{C}P^d/\mathbb{U}_{\mathbf{q}} \simeq \mathbb{C}P(\mathbf{q})$ (beware that it is not an isomorphism of orbifolds). Let us recall the following classical result of singular homology.

LEMMA 2.2 [Bor60, IV.3.4(c)]. Given a finite G -action on a topological space X , the morphism induced by the quotient map in homology (and in cohomology)

$$H_*(X; R)^G \rightarrow H_*(X/G; R)$$

is an isomorphism when the characteristic of R is either zero or prime with the order of G .

Let us remark that $\mathbb{U}_{\mathbf{q}}$ acts trivially on $H_*(\mathbb{C}P^d)$. Therefore, according to this lemma, when the characteristic of R is either zero or prime with the order of $\mathbb{U}_{\mathbf{q}}$, that is when it is prime with any of the weights q_j , $g_{\mathbf{q}}$ induces the isomorphism

$$(g_{\mathbf{q}})_* : H_*(\mathbb{C}P^d; R) \xrightarrow{\cong} H_*(\mathbb{C}P(\mathbf{q}); R), \tag{3}$$

More generally, for every $A \subset \mathbb{C}P(\mathbf{q})$, $g_{\mathbf{q}}$ induces the isomorphism

$$(g_{\mathbf{q}})_* : H_*(g_{\mathbf{q}}^{-1}(A); R)^{\mathbb{U}_{\mathbf{q}}} \xrightarrow{\cong} H_*(A; R).$$

In order to get the properties stated in [All22a, Appendix A] back, let us show the commutativity of the following diagram:

$$\begin{CD} H_*(\tilde{A} \times \tilde{B}; R)^{\mathbb{U}_{\mathbf{q}} \times \mathbb{U}_{\mathbf{q}'}} @>{pj_*}>> H_{*+2}(\tilde{A} * \tilde{B}; R)^{\mathbb{U}_{\mathbf{q}} \times \mathbb{U}_{\mathbf{q}'}} \\ @V{(g_{\mathbf{q}} \times g_{\mathbf{q}'})_*}VV @VV{(g_{(\mathbf{q}, \mathbf{q}')})_*}V \\ H_*(A \times B; R) @>{pj_*}>> H_{*+2}(A * B; R) \end{CD} \tag{4}$$

where $\tilde{A} := g_{\mathbf{q}}^{-1}(A)$ and $\tilde{B} := g_{\mathbf{q}'}^{-1}(B)$ and the actions of the group $\mathbb{U}_{\mathbf{q}} \times \mathbb{U}_{\mathbf{q}'}$ on $\tilde{A} \times \tilde{B}$ and $\tilde{A} * \tilde{B}$ are coordinate-wise. Let us show that the top pj_* in (4) is well-defined, that is, $pj_* : H_*(\tilde{A} \times \tilde{B}) \rightarrow H_{*+2}(\tilde{A} * \tilde{B})$ is $\mathbb{U}_{\mathbf{q}} \times \mathbb{U}_{\mathbf{q}'}$ -equivariant. The group $\mathbb{U}_{\mathbf{q}} \times \mathbb{U}_{\mathbf{q}'}$ acts on $E_{\tilde{A}, \tilde{B}}$ by restriction of its diagonal action on $(\tilde{A} \times \tilde{B}) \times (\tilde{A} * \tilde{B})$. Both associated projection maps p_1 and p_2 are equivariant under these actions, so the equivariance of pj_* follows.

By naturality of the properties stated in [All22a, Appendix A] and naturality of the homology projective joins, diagram (4) implies that these properties are still verified for our extension of the homology projective join to the weighted case: for instance, the homology projective join is associative,

$$pj_*(pj_*(\alpha \times \beta) \times \gamma) = pj_*(\alpha \times pj_*(\beta \times \gamma)), \quad \forall \alpha, \beta, \gamma,$$

and it satisfies $pj_*([P] \times [P']) = [P * P']$ for every (disjoint) weighted projective spaces P and P' .

2.3 Generating functions of $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms

In this section, we recall definitions and properties already discussed in [All22b, Section 5] and [All22a, Section 3.2] in the case of unweighted projective space and that generalize directly to our ‘weighted’ case. Let us fix once for all the weights $\mathbf{q} = (q_0, \dots, q_d) \in (\mathbb{N}^*)^{d+1}$, the S^1 -action of \mathbb{C}^{d+1} will always refer to the action induced by $\rho_{\mathbf{q}}$ defined in (1).

Given a Hamiltonian map $(h_s) : S^1 \times \mathbb{C}P(\mathbf{q}) \rightarrow \mathbb{R}$, let (H_s) be the Hamiltonian map of \mathbb{C}^{d+1} that is 2-homogeneous, S^1 -invariant and whose restriction to $\mathbb{S}(\mathbf{q})$ lifts (h_s) (see [LT97] for an introduction to the notion of symplectic orbifold, here we only need to see an orbifold map of $\mathbb{C}P(\mathbf{q})$ as a quotient of an S^1 -invariant map of $\mathbb{S}(\mathbf{q})$). Let (Φ_s) be the associated Hamiltonian flow. An $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphism will refer to the time-one map of

such a flow. These are smooth diffeomorphisms of $\mathbb{C}^{d+1} \setminus 0$ that are S^1 -equivariant and positively homogeneous and extends to homeomorphisms of \mathbb{C}^{d+1} . When the restriction of such a diffeomorphism σ to $\mathbb{S}(\mathbf{q})$ is C^1 -close to the identity, there exists a unique map $f : \mathbb{C}^{d+1} \rightarrow \mathbb{R}$ such that $f(0) = 0$ and

$$\forall z \in \mathbb{C}^{d+1}, \exists! w \in \mathbb{C}^{d+1}, \quad w = \frac{z + \sigma(z)}{2} \quad \text{and} \quad \nabla f(w) = i(z - \sigma(z)),$$

where ∇f denotes the gradient of f (the existence of $g := \nabla f$ is a consequence of the implicit function theorem and g is a gradient because it is the graph of a Lagrangian submanifold). The map f is called the elementary generating function of σ , and such σ will be called *small* $\mathbb{R}_+^* \times S^1$ -equivariant diffeomorphisms. It is smooth away from 0 where it is only C^1 , it is S^1 -invariant and positively 2-homogeneous (this is a consequence of the definition of f and the equivariance of σ). In general, an $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphism Φ can be written as $\Phi = \sigma_n \circ \dots \circ \sigma_1$ where every $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphism σ_j is sufficiently small so that it admits an elementary generating function f_j . For all $n \in \mathbb{N}^*$, we say that the n -tuple $\sigma = (\sigma_1, \dots, \sigma_n)$ is associated with the Hamiltonian flow (Φ_s) if there exist real numbers $0 = t_0 \leq t_1 \leq \dots \leq t_n = 1$ such that $\sigma_k = \Phi_{t_k} \circ \Phi_{t_{k-1}}^{-1}$. For all $k \in \mathbb{N}$, we denote by ε^k the k -tuple

$$\varepsilon^k := (\text{id}, \dots, \text{id}).$$

More generally, given a tuple σ and an integer $n \in \mathbb{N}$, σ^n denotes the n -fold concatenation. A continuous family of such tuples (σ_s) will denote a family of tuples of the same size $n \geq 1$, $\sigma_s =: (\sigma_{1,s}, \dots, \sigma_{n,s})$ such that the maps $s \mapsto \sigma_{k,s}$ are C^1 -continuous. We denote by F_σ the following function $(\mathbb{C}^{d+1})^n \rightarrow \mathbb{R}$:

$$F_\sigma(v_1, \dots, v_n) := \sum_{k=1}^n f_k \left(\frac{v_k + v_{k+1}}{2} \right) + \frac{1}{2} \langle v_k, i v_{k+1} \rangle,$$

with convention $v_{n+1} = v_1$, each $f_k : \mathbb{C}^{d+1} \rightarrow \mathbb{R}$ being the elementary generating function associated with σ_k . When n is odd, F_σ is a generating function of $\sigma_n \circ \dots \circ \sigma_1$. Therefore, every $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphism admits a generating function. Generating functions are S^1 -invariant and positively 2-homogenous. The $\mathbb{R}_+^* \times S^1$ -orbits (for the diagonal action) of critical points of a generating function of Φ are in bijection with the $\mathbb{R}_+^* \times S^1$ -orbits of fixed points of Φ through the map $(v_1, \dots, v_n) \mapsto v_1$.

Given generating functions $F : \mathbb{C}^{d+1} \times \mathbb{C}^k \rightarrow \mathbb{R}$ and $G : \mathbb{C}^{d+1} \times \mathbb{C}^l \rightarrow \mathbb{R}$ of Φ and Ψ , respectively, the fiberwise sum of F and G denotes the map

$$(F + G)(x; \xi, \eta) := F(x; \xi) + G(x; \eta). \tag{5}$$

Although this is not a generating function of $\Phi \circ \Psi$, the critical points of $F + G$ are also in bijection with the fixed points of $\Phi \circ \Psi$ via $(x; \xi, \eta) \mapsto x - i\partial_x G(x; \eta)/2$.

When the small equivariant diffeomorphisms σ_j are linear, the associated f_j are quadratic forms and so is the resulting F_σ . When $\sigma_n \circ \dots \circ \sigma_1$ also admits an elementary (quadratic) generating function, we have the following unicity lemma, the proof of which follows that of [The96, Prop. 35].

LEMMA 2.3. *Let $Q : \mathbb{C}^{d+1} \times \mathbb{C}^k \rightarrow \mathbb{R}$ be a quadratic generating function generating the same linear Hamiltonian diffeomorphism as the elementary generating function $q : \mathbb{C}^{d+1} \rightarrow \mathbb{R}$. Then, there exists a linear fibered isomorphism A of $\mathbb{C}^{d+1} \times \mathbb{C}^k$ which is isotopic to the identity through linear fiberwise isomorphisms such that $Q \circ A = q \oplus R$ for some quadratic form $R : \mathbb{C}^k \rightarrow \mathbb{R}$.*

More precisely, if $Q(z) = \langle \tilde{Q}z, z \rangle$ with

$$\tilde{Q} = \begin{bmatrix} a & b \\ t_b & c \end{bmatrix},$$

then c is invertible and $A(x; \xi) := (x; \xi - c^{-1}tbx)$ so that $Q \circ A(x; \xi) = q(x) + {}^t\xi c\xi$.

3. Generating functions homology

In this section, we define the generating functions homology of a Hamiltonian diffeomorphism φ in $\mathbb{C}P(\mathbf{q})$ and give its main properties. The constructions and proofs are very close to those of the ‘unweighted’ case $\mathbb{C}P^d$, so we mainly refer to [All22a] and emphasize the key changes.

3.1 Action and generating functions

Let (Φ_s) be the $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian flow lifting a Hamiltonian flow (φ_s) of $\mathbb{C}P(\mathbf{q})$ generated by the Hamiltonian map (h_s) . Let $x \in \mathbb{C}P(\mathbf{q})$ be a fixed point of $\varphi := \varphi_1$ and $u : \mathbb{D}^2 \rightarrow \mathbb{C}P(\mathbf{q})$ be an orbifold map such that $u|_{\partial\mathbb{D}^2}$ corresponds to $s \mapsto \varphi_s(x)$ (a capping of x). The action of the capped fixed point (x, u) is the real number

$$a(x, u) := -\frac{1}{\pi} \left(\int_{\mathbb{D}^2} u^* \omega + \int_0^1 h_s \circ \varphi_s(x) ds \right).$$

Proposition 5.8 in [The98] gives a characterization of the action values of the cappings of x in terms of the lifted dynamics that directly extends to our weighted case. Let $z \in \mathbb{S}(\mathbf{q})$ be a lift of x , it is not necessarily a fixed point of $\Phi := \Phi_1$ but there exist numbers $t \in \mathbb{R}$ such that $\rho_{\mathbf{q}}(-t)\Phi(z) = z$. Following Théret, we see that such t corresponds exactly to the action values of the cappings of x . With this characterization, it is clear that the set of action values of x is invariant under \mathbb{Z} -translations. Here is the major difference between the weighted and unweighted case: the set of action values of x equals $t_0 + (1/\text{order}(x))\mathbb{Z}$ so the number of action values inside $[0, 1)$ depends on $\text{order}(x)$. This is ultimately the reason why our version of the Fortune–Weinstein theorem, which follows Givental and Théret’s steps, give multiplicity to fixed points.

In order to study the fixed points of φ , we define continuous families of generating functions F_t associated with $\rho_{\mathbf{q}}(-t)\Phi$ for compact intervals I of t . A fixed point of action $t \in I$ corresponds to an $\mathbb{R}_+^* \times S^1$ -orbit of critical points of such an F_t . Given a positively 2-homogeneous map F that is invariant under $\rho_{\mathbf{q}'}$ (in our case \mathbf{q}' will often be a concatenation \mathbf{q}^n), we define its projectivization $\hat{F} : \mathbb{C}P(\mathbf{q}') \rightarrow \mathbb{R}$ by factoring the restriction of F to $\mathbb{S}(\mathbf{q}')$ under $\mathbb{S}(\mathbf{q}') \rightarrow \mathbb{C}P(\mathbf{q}')$. Fixed points with action $t \in I$ now correspond to critical points of \hat{F}_t with value zero.

Let us now recall the precise construction of the families of generating functions $F_t = F_{\sigma_{m,t}}$. For $m \in \mathbb{N}$, we define continuous tuples $t \mapsto \sigma_{m,t}$ associated with $\rho_{\mathbf{q}}(-t)\Phi$ for $t \in [-m, m]$ in the following way. Let (δ_t) be the family of small $\mathbb{R}_+^* \times S^1$ -equivariant diffeomorphisms $\delta_t(z) := \rho_{\mathbf{q}}(-t)z$, $|t| < 1/(2 \max q_j)$. The associated elementary generating function is

$$w \mapsto - \sum_j \tan(q_j \pi t) |w_j|^2$$

(this function is an elementary generating function of δ_t as long as it is well defined for the fixed t , we use it for a fixed t of absolute value smaller than $1/(2 \max q_j)$ in the proof of Proposition 4.2).

Let us fix once for all an even number $n_0 \geq 4 \max q_j$ and let $(\delta_t^{(1)})$ be the family of n_0 -tuples $(\delta_{t/n_0}, \dots, \delta_{t/n_0})$ generating $z \mapsto \rho_{\mathbf{q}}(-t)z$ for $t \in (-2, 2)$. For all $m \in \mathbb{N}^*$, let $(\delta_t^{(m)})$ be a family

of mn_0 -tuples generating $z \mapsto \rho_{\mathbf{q}}(-t)z$ for $t \in (-m - 1, m + 1)$ and satisfying

$$\delta_t^{(m+1)} = (\delta_t^{(m)}, \varepsilon^{n_0}), \quad \forall t \in [-m, m]. \tag{6}$$

More precisely, let $\chi_m : \mathbb{R} \rightarrow \mathbb{R}$ be an odd smooth non-decreasing map such that $\chi_m \equiv \text{id}$ on $[-m - 1/4, m + 1/4]$ and $\chi_m \equiv m + 1/2$ on $[m + 3/4, +\infty)$. We set

$$\delta_t^{(m+1)} = (\delta_{\chi_m(t)}^{(m)}, \delta_{t-\chi_m(t)}^{(1)}), \quad \forall t \in (-m - 2, m + 2).$$

Finally, we can set

$$\sigma_{m,t} := (\sigma, \delta_t^{(m)}), \quad \forall t \in [-m, m].$$

As \tan is increasing on $(-\pi/2, \pi/2)$, we deduce that $\partial_t F_{\sigma_{m,t}} \leq 0$ by a straightforward computation (here it is crucial that each weight q_j is positive).

3.2 Homology of sublevel sets and local homology of a fixed point

Here and throughout this paper, $H_*(X)$ and $H^*(X)$ denote the singular homology and the singular cohomology, respectively, of a topological space or pair X over an indeterminate ring R whose characteristic is zero or prime with any of the weights q_j (that have been fixed once for all). If one needs to specify the ring R , one writes $H_*(X; R)$ and $H^*(X; R)$ instead. The following notation naturally extends the one used in the unweighted case. Let σ be an n -tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms (as defined in §2.3). We denote by $Z(\sigma) \subset \mathbb{C}P(\mathbf{q}^n)$ the sublevel set

$$Z(\sigma) := \{\hat{F}_\sigma \leq 0\},$$

where \hat{F}_σ denotes the projectivization of the generating function associated with σ . We denote by $HZ_*(\sigma)$ the shifted homology group

$$HZ_*(\sigma) := H_{*+(n-1)(d+1)}(Z(\sigma)),$$

and if $Z(\sigma') \subset Z(\sigma)$, with σ' an n -tuple, we set

$$HZ_*(\sigma, \sigma') := H_{*+(n-1)(d+1)}(Z(\sigma), Z(\sigma')).$$

For $m \in \mathbb{N}^*$ and $a \leq b$ in $[-m, m]$, one has $F_{\sigma_{m,b}} \leq F_{\sigma_{m,a}}$ so $Z(\sigma_{m,a}) \subset Z(\sigma_{m,b})$ and we can set

$$G_*^{(a,b)}(\sigma, m) := HZ_*(\sigma_{m,b}, \sigma_{m,a}),$$

when a and b are not action values of σ . We define in the same way the cohomology analogues by

$$G_{(a,b)}^*(\sigma, m) := HZ^*(\sigma_{m,b}, \sigma_{m,a}) = H^{*+(n-1)(d+1)}(Z(\sigma_{m,b}), Z(\sigma_{m,a})).$$

This homology group can be naturally identified to the homology of sublevel sets of a map (see [All22b, Section 5.4])

$$G_*^{(a,b)}(\sigma, m) \simeq H_{*+(n-1)(d+1)}(\{\mathcal{T} \leq b\}, \{\mathcal{T} \leq a\}),$$

for some C^1 -map $\mathcal{T} : M \rightarrow \mathbb{R}$ defined on a manifold M and that is smooth in the neighborhood of its critical points. The function \mathcal{T} is some kind of finite-dimensional action: critical points of \mathcal{T} are in one-to-one correspondence with capped fixed points of φ with action value inside $[-m, m]$. In the unweighted case at least, this correspondence sends critical value to action value and Morse index up to a $(n - 1)(d + 1)$ shift in degree to the Conley–Zehnder index. More generally, the local homology of \mathcal{T} (up to the same shift in degree) is isomorphic to the local Floer homology

of the corresponding capped orbit (in the unweighted case at least). Let us denote by $C_*(f; x)$ the local homology of the critical point x of a map f :

$$C_*(f; x) := H_*(\{f \leq f(x)\}, \{f \leq f(x)\} \setminus x).$$

We can define up to isomorphism

$$C_*(\sigma; z, t) \simeq C_*(\hat{F}_{\sigma_m, t}; \zeta) \simeq C_{*+(n-1)(d+1)}(\mathcal{T}; (\zeta, t)),$$

where $\zeta \in \mathbb{CP}(\mathbf{q}^n)$ is the critical point of $\hat{F}_{\sigma_m, t}$ associated with the fixed point $z \in \mathbb{CP}(\mathbf{q})$ of action $t \in [-m, m]$. Let us briefly justify the independence on m , we refer to [All22b, Section 5.5] for the technical details. In §3.3, we define an isomorphism θ_m^{m+1} between $G_*^{(a,b)}(\sigma, m)$ and $G_*^{(a,b)}(\sigma, m + 1)$. For $\varepsilon > 0$ small enough,

$$G_*^{(t-\varepsilon, t+\varepsilon)}(\sigma, m) \simeq \bigoplus_z C_*(\sigma; z, t, m),$$

for $z \in \mathbb{CP}(\mathbf{q})$ fixed points with action t (here we temporarily keep the dependency on m explicit). In this case, one shows that θ_m^{m+1} respects this decomposition, which gives the desired isomorphisms. Similarly, the local homology groups $C_*(\sigma; z, t)$ and $C_*(\sigma; z, t + 1)$ are isomorphic up to a $2|\mathbf{q}|$ shift in degree by the local version of the periodicity isomorphism defined at (10). However, when $\text{order}(z) \neq 1$, it is not clear whether the local homology groups associated with action values that do not differ by an integer are isomorphic (up to a shift in degree). For these reasons, when the grading is irrelevant, we only specify the action value up to an integer.

We can now define precisely $N(\sigma; \mathbb{F})$ for a choice of tuple σ and of field \mathbb{F} (whose characteristic is either 0 or prime with any of the weights) by

$$N(\sigma; \mathbb{F}) := \sum_{z \in \text{Fix}(\varphi)} \sum_{j=1}^{\text{order}(z)} \dim C_*(\sigma; z, t_j(z); \mathbb{F}) \in \mathbb{N}, \tag{7}$$

where $(t_j(z))$ is the increasing sequence of action values that z takes inside $[0, 1)$. The integer $O(z; \mathbb{F})$ defined in the introduction for a fixed point $z \in \text{Fix}(\varphi)$ is then

$$O(z; \mathbb{F}) := \sum_{j=1}^{\text{order}(z)} \dim C_*(\sigma; z, t_j(z); \mathbb{F}).$$

We recall that an integer $k \in \mathbb{N}^*$ is said to be admissible for φ at a fixed point z if $\lambda^k \neq 1$ for all eigenvalues $\lambda \neq 1$ of $d\varphi(z)$. Until the end of the section, φ is associated with a tuple σ and the periodic points of φ are isolated in order to simplify the statements. The proofs do not differ from the unweighted case [All22a, Proposition 3.1 and Corollary 3.2].

PROPOSITION 3.1. *Let $k \in \mathbb{N}^*$ be an admissible iteration of φ at the fixed point z . Then as graded modules over a ring whose characteristic is prime with any of the weights,*

$$C_*(\sigma^k; z) \simeq C_{*-i_k}(\sigma; z),$$

for some shift in degree $i_k \in \mathbb{Z}$.

COROLLARY 3.2. *For every fixed point z of φ , there exists $B > 0$ such that, for all prime p*

$$\dim C_*(\sigma^p; z, t_j(z); \mathbb{F}_p) < B, \quad \forall j \in \{1, \dots, \text{order}(z)\}.$$

3.3 Composition morphisms and the direct system of $G_*^{(a,b)}(\sigma)$

In [All22a, Section 4.1], we observed that the linear embedding,

$$\tilde{B}_{n,m}(\mathbf{w}, \mathbf{w}') := \left(\mathbf{w}, \sum_{k=1}^m (-1)^{k+1} w'_k, \mathbf{w}' \right), \quad \forall \mathbf{w} \in (\mathbb{C}^{d+1})^n, \forall \mathbf{w}' \in (\mathbb{C}^{d+1})^m,$$

expressed in coordinates $w_k := (v_k + v_{k+1})/2$, satisfies for all n -tuples σ and m -tuples σ'

$$F_{(\sigma, \varepsilon, \sigma')}(\tilde{B}_{n,m}(\mathbf{w}, \mathbf{w}')) = F_{\sigma}(\mathbf{w}) + F_{\sigma'}(\mathbf{w}').$$

Therefore, the induced map $B_{n,m} : \mathbb{C}P(\mathbf{q}) * \mathbb{C}P(\mathbf{q}') \hookrightarrow \mathbb{C}P(\mathbf{q}, 1, \mathbf{q}')$ defines a natural morphism $H_*(Z(\sigma) * Z(\sigma')) \rightarrow H_*(Z(\sigma, \varepsilon, \sigma'))$ and the composition with the homology projective join $\alpha \otimes \beta \mapsto \text{pj}_*(\alpha \times \beta)$ define the composition morphism

$$HZ_*(\sigma) \otimes HZ_*(\sigma') \rightarrow HZ_{*-2d}((\sigma, \varepsilon, \sigma'))$$

denoted by $\alpha \otimes \beta \mapsto \alpha \otimes \beta$. For the same formal reasons as in the unweighted case, this morphism admits relative versions and it is associative [All22a, Section 4.1].

As for the unweighted case, for a fixed $m \in \mathbb{N}$, the long exact sequence of the triple induces inclusion and boundary morphisms fitting into a long exact sequence:

$$\dots \xrightarrow{\partial_{*+1}} G_*^{(a,b)}(\sigma, m) \rightarrow G_*^{(a,c)}(\sigma, m) \rightarrow G_*^{(b,c)}(\sigma, m) \xrightarrow{\partial_*} G_{*-1}^{(a,b)}(\sigma, m) \rightarrow \dots$$

with $-m \leq a \leq b \leq c \leq m$ and a, b, c not action values of σ . Using the composition morphism \otimes , one can define canonical isomorphisms

$$\theta_m^{m+1} : G_*^{(a,b)}(\sigma, m) \rightarrow G_*^{(a,b)}(\sigma, m+1), \tag{8}$$

for $-m \leq a \leq b \leq m$, that commute with the previously mentioned inclusion and boundary morphisms. One can then define $G_*^{(a,b)}(\sigma)$ as the direct limit of the direct system induced by $(\theta_m^{m+1})_m$:

$$G_*^{(a,b)}(\sigma) := \varinjlim G_*^{(a,b)}(\sigma, m).$$

We then have inclusion and boundary morphisms fitting into a long exact sequence

$$\dots \xrightarrow{\partial_{*+1}} G_*^{(a,b)}(\sigma) \rightarrow G_*^{(a,c)}(\sigma) \rightarrow G_*^{(b,c)}(\sigma) \xrightarrow{\partial_*} G_{*-1}^{(a,b)}(\sigma) \rightarrow \dots$$

for all $a \leq b \leq c$ that are not action values; one can thus set

$$G_*^{(-\infty, b)}(\sigma) := \varprojlim G_*^{(a,b)}(\sigma), \quad a \rightarrow -\infty,$$

and one can then define $G_*^{(-\infty, +\infty)}(\sigma)$ by taking a direct limit in a similar way. The definition of (8) is the natural extension of the unweighted case, let us make it explicit. For an odd n , the space $Z(\varepsilon^n)$ retracts on the projectivization of the maximal non-positive linear subspace of F_{ε^n} which has the same homology as a $\mathbb{C}P^{N-1}$ with $N = (d+1)(n+1)/2$. Therefore,

$$HZ_*(\varepsilon^n) = \bigoplus_{k=-(d+1)(n-1)/2}^d Ra_k^{(n)} \simeq H_{*+(n-1)(d+1)}(\mathbb{C}P^{(d+1)(n+1)/2-1}),$$

where $a_k^{(n)}$ is the generator of degree $2k$ identified with the class $[\mathbb{C}P^l]$ of appropriate degree $2l = 2k + (n-1)(d+1)$ under the isomorphism induced by the inclusion of a maximal complex projective subspace of $Z(\varepsilon^n)$ and (3). We now define (8) by

$$\theta_m^{m+1}(\alpha) := \alpha \otimes a_d^{(n_0-1)} \in G_*^{(a,b)}(\sigma, m+1), \quad \forall \alpha \in G_*^{(a,b)}(\sigma, m).$$

With the same proof as in the unweighted case, one shows that θ_m^{m+1} is an isomorphism.

In [All22a, Section 4.4], we defined a second composition morphism \diamond with the goal (only partially reached) to imitate the composition morphism of the Hamiltonian Floer homology. Let us fix two tuples σ and σ' of odd respective sizes n and n' , $a, b, c \in \mathbb{R}$ that are not action values of σ and σ' respectively. For sufficiently large $m, m' \in \mathbb{N}$, this composition morphism $\alpha \otimes \beta \mapsto \alpha \diamond \beta$,

$$HZ_*(\sigma'_{m',c}) \otimes G_*^{(a,b)}(\sigma, m) \rightarrow G_{*-2d}^{(a+c,b+c)}((\sigma', \varepsilon, \sigma), m + m'),$$

naturally generalizes with the same construction. By the same formal arguments as in the unweighted case, it is associative and it commutes with the morphisms θ_m^{m+1} ultimately defining

$$HZ_*(\sigma'_{m',c}) \otimes G_*^{(a,b)}(\sigma) \rightarrow G_{*-2d}^{(a+c,b+c)}((\sigma', \varepsilon, \sigma)). \tag{9}$$

3.4 Properties of the generating functions homology

Let us first focus on the special case $\sigma = \varepsilon$, i.e. $\varphi_s \equiv \text{id}$. Let us denote by $T_{m,t}$ the family of generating functions associated with $(\varepsilon_{m,t})_t$. As the elementary generating function of δ_s is a quadratic form, so is the map $T_{m,t}$. As $T_{m,0}$ is a generating function of the identity, its kernel as a quadratic form has dimension $2(d + 1)$ and $\text{ind}T_{m,0} = mn_0(d + 1)$ (see [All22a, Proposition 4.1]). The variation of index is governed by the Maslov index of

$$t \mapsto \rho_{\mathbf{q}}(-t) = \bigoplus_{j=0}^d e^{-2i\pi q_j t},$$

so that

$$\text{ind}T_{m,t} - \text{ind}T_{m,0} = 2 \sum_{j=0}^d [q_j t]$$

(see [All22b, Section 3 and Lemma 5.5]). Similarly to the unweighted case, we deduce that the persistence module $(H_*(Z(\varepsilon_{m,t})))$ is isomorphic to the persistence module $(H_*(\mathbb{C}P^{N(t)}))$, $-m < t < m$, induced by the family of non-decreasing projective subspaces of complex dimension $N(t) := m(d + 1)n_0/2 + \sum_j [q_j t]$. We recall that the coefficient ring R has characteristic zero or prime with the q_j (see (3), otherwise the family $(\mathbb{C}P^{N(t)})$ of non-decreasing unweighted projective subspaces must be replaced by a family of non-decreasing *weighted* projective subspaces). Thus, as a graded R -module,

$$HZ_*(\varepsilon_{m,t}) = \bigoplus_{k=-(d+1)mn_0/2}^{d+\sum_j [q_j t]} Ra_k^{(mn_0+1)}(t),$$

where $a_k^{(mn_0+1)}(t)$ is the generator of degree $2k$ identified with the class $[\mathbb{C}P^l]$ of appropriate degree $2l = 2k + (d + 1)mn_0$ under the previous persistence modules isomorphism. The inclusion morphism $HZ_*(\varepsilon_{m,t}) \rightarrow HZ_*(\varepsilon_{m,s})$ maps each $a_k^{(mn_0+1)}(t)$ to $a_k^{(mn_0+1)}(s)$ (for $-m \leq t \leq s \leq m$). Hence,

$$G_*^{(a,b)}(\varepsilon, m) = \bigoplus_{k=d+\sum_j [q_j a]}^{d+\sum_j [q_j b]} R\alpha_k^{(m)}(a, b),$$

for $-m < a \leq b < m$, where $\alpha_k^{(m)}(a, b)$ is the image of $a_k^{(mn_0+1)}(b)$ under the inclusion morphism $HZ_*(\varepsilon_{m,b}) \rightarrow G_*^{(a,b)}(\varepsilon, m)$. Similarly to the unweighted case, one has $\theta_m^{m+1}\alpha_k^{(m)}(a, b) = \alpha_k^{(m+1)}(a, b)$. We set $\alpha_k(a, b) := \theta_m^\infty \alpha_k^{(m)}(a, b)$. For $a < b < c$, if $\alpha_k(b, c)$ is well-defined, then

$\alpha_k(a, c)$ is also well-defined and sent to the former; there exists a well-defined $\alpha_k(-\infty, c) \in G_{2k}^{(-\infty, c)}(\varepsilon)$ sent to $\alpha_k(a, c)$ for all $a \leq c$. Let α_k be the image of $\alpha_k(-\infty, c)$ under $G_{2k}^{(-\infty, c)}(\varepsilon) \rightarrow G_{2k}^{(-\infty, +\infty)}(\varepsilon)$, then

$$G_*^{(-\infty, +\infty)}(\varepsilon) = \bigoplus_{k \in \mathbb{Z}} R\alpha_k.$$

The ‘periodicity’ isomorphism naturally extends to the weighted case as

$$G_*^{(a, b)}(\sigma) \xrightarrow{\cong} G_{*+2|\mathbf{q}|}^{(a+1, b+1)}(\sigma), \tag{10}$$

(we recall that $|\mathbf{q}| := \sum_j q_j$). Similarly to the unweighted case, it is defined using composition morphisms \diamond (9), with slight changes in the degree of the generators a_k involved, let us precise them. Let us set $a_d := a_d^{(mn_0+1)}(0) \in HZ_{2d}(\varepsilon_{m,0})$ and $a_{d+|\mathbf{q}|} := a_{d+|\mathbf{q}|}^{(mn_0+1)}(1) \in HZ_{2(d+|\mathbf{q}|)}(\varepsilon_{m,1})$. The morphism $G_*^{(a+1, b+1)}(\sigma) \rightarrow G_*^{(a+1, b+1)}(\varepsilon^2, \sigma)$, $\alpha \mapsto a_d \diamond \alpha$, is an isomorphism, let us write $\alpha \mapsto a_d^{-1} \diamond \alpha$ its inverse morphism. We define the morphism (10) by $\alpha \mapsto a_d^{-1} \diamond a_{d+|\mathbf{q}|} \diamond \alpha$. The proof of [All22a, Proposition 4.10] applies with these formal adaptations (in the proof, the generator a_{-1} of degree -2 must also be replaced by the generator $a_{d-|\mathbf{q}|}$ of degree $2(d - |\mathbf{q}|)$).

PROPOSITION 3.3. *The morphism (10) is an isomorphism commuting with inclusion and boundary morphisms.*

Following the proof in the unweighted case, one can now define spectral invariants $c_k(\sigma)$ in our generalized setting.

THEOREM 3.4. *Let σ be a tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms associated with the Hamiltonian diffeomorphism φ of $\mathbb{C}P(\mathbf{q})$. As a graded module over a coefficient ring R whose characteristic is either zero or prime with the weights,*

$$G_*^{(-\infty, +\infty)}(\sigma) = \bigoplus_{k \in \mathbb{Z}} R\alpha_k$$

for some non-zero α_k ’s with $\deg \alpha_k = 2k$. For all $k \in \mathbb{Z}$, let

$$c_k(\sigma) := \inf \{ t \in \mathbb{R} \mid \alpha_k \in \text{im}(G_*^{(-\infty, t)}(\sigma) \rightarrow G_*^{(-\infty, +\infty)}(\sigma)) \}.$$

Then for all $k \in \mathbb{Z}$, $c_k(\sigma) \in \mathbb{R}$ is an action value of σ and $c_{k+|\mathbf{q}|}(\sigma) = c_k(\sigma) + 1$. Moreover,

$$c_k(\sigma) \leq c_{k+1}(\sigma)$$

for all $k \in \mathbb{Z}$, and if there exists $k \in \mathbb{Z}$ such that $c_k(\sigma) = c_{k+1}(\sigma)$, then φ has infinitely many fixed points of action $c_k(\sigma)$. If $d + 1$ consecutive $c_k(\sigma)$ are equal, then $\varphi = \text{id}$.

As a corollary, we obtain the generalization of the Fortune-Weinstein theorem stated in the introduction.

Proof of Theorem 1.1. Let σ be a tuple associated with the Hamiltonian diffeomorphism $\varphi \in \text{Ham}(\mathbb{C}P(\mathbf{q}))$ that has finitely many fixed points. The spectral values $c_k(\sigma)$ must all be distinct. As $(c_k(\sigma))$ is increasing and satisfies $c_{k+|\mathbf{q}|}(\sigma) = c_k(\sigma) + 1$, there are exactly $|\mathbf{q}|$ distinct spectral values inside $[0, 1)$. However, the number of action values inside $[0, 1)$ associated with a fixed point z equals $\text{order}(z)$, so the conclusion follows. \square

The Poincaré duality in singular homology implies the following duality, according to the proof of [All22a, Proposition 4.13].

PROPOSITION 3.5. *Let σ be a tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms of \mathbb{C}^{d+1} . There exists a duality isomorphism between generating functions homology and cohomology*

$$PD : G_{(a,b)}^*(\sigma) \xrightarrow{\sim} G_{2d-*}^{(-b,-a)}(\sigma^{-1}),$$

with $-\infty \leq a \leq b \leq +\infty$ and a, b not action values. This isomorphism is natural: it commutes with inclusion and boundary maps.

COROLLARY 3.6. *Let σ be a tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms, then*

$$c_k(\sigma^{-1}) = -c_{d-k}(\sigma).$$

Following the proof in the unweighted case, the properties of the composition morphisms imply the sub-additivity of the spectral invariants.

PROPOSITION 3.7. *Given any tuples σ and σ' of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms, one has*

$$c_{k+l-d}((\sigma, \varepsilon, \sigma')) \leq c_k(\sigma) + c_l(\sigma').$$

We can now associate to every σ a persistence module $(G_*^{(-\infty, t)}(\sigma))_t$ that satisfies the ‘periodicity’ property $G_*^{(-\infty, t+1)}(\sigma) \simeq G_{*+2|\mathbf{q}|}^{(-\infty, t)}(\sigma)$, the isomorphism being an isomorphism of persistence module according to the naturality of (10). Let us refer to [All22a, Section 3.1] and references therein for a quick review of persistence modules and barcodes and let us just recall that persistence modules are \mathbb{R} -families of vector spaces $(V^t)_{t \in \mathbb{R}}$ with natural maps $V^t \rightarrow V^s$ for $t \leq s$ and that, under some finiteness and continuity conditions (that are satisfied in our case), one can associate a multiset $\mathcal{B}(V^t)$ of intervals $[a, b)$ called the barcode of (V^t) satisfying, for all $t_0 \in \mathbb{R}$,

$$\text{Card}\{I \in \mathcal{B}(V^t) \mid t_0 \in I\} = \dim V^{t_0}.$$

While discussing barcodes properties of $(G_*^{(-\infty, t)}(\sigma))_t$, we assume that the persistence module is over a field (whose characteristic is either zero or prime with the weights) and that the number of fixed points in $\mathbb{C}\mathbb{P}(\mathbf{q})$ associated with σ is finite. As this periodicity property shifts the degree by a constant positive integer $2|\mathbf{q}|$, it induces a permutation of the bars of the barcode sending a bar $[a, b)$ on a bar $[a + 1, b + 1)$ that generates a free \mathbb{Z} -action on the bars. A family of representatives of the bars is given by the union of the barcodes of $(G_k^{(-\infty, t)}(\sigma))_t$ for $0 \leq k \leq 2|\mathbf{q}| - 1$. The infinite bars of the barcode are exactly the multiset of intervals $I_k := [c_k(\sigma), +\infty)$, $k \in \mathbb{Z}$, the positive generator of the \mathbb{Z} -action sending I_k to $I_{k+|\mathbf{q}|}$, so that there are exactly $|\mathbf{q}|$ \mathbb{Z} -orbits of infinite bars. Still following the proof in the unweighted case, we obtain the following interpretation of the homology count of fixed points $N(\sigma; \mathbb{F})$ in terms of count of bars.

PROPOSITION 3.8. *Given a tuple σ of $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms with a finite number of fixed $\mathbb{R}_+^* \times S^1$ -orbits, for every field \mathbb{F} whose characteristic is either zero or prime with the weights,*

$$N(\sigma; \mathbb{F}) = |\mathbf{q}| + 2K(\sigma; \mathbb{F}),$$

where $K(\sigma; \mathbb{F})$ is the number of \mathbb{Z} -orbits of finite bars of the persistence module of σ over the field \mathbb{F} . In other words, $N(\sigma; \mathbb{F})$ is the number of (finite) extremities of a set of representative bars.

The crucial point here is that $N(\sigma; \mathbb{F}) > |\mathbf{q}|$ means that there exist finite bars. Let us denote $\beta(\sigma; \mathbb{F}) \geq 0$ the maximal length of a finite bar of the barcode and $\beta_{\text{tot}}(\sigma; \mathbb{F}) \geq 0$ the sum of the

lengths of representatives of the \mathbb{Z} -orbits of finite bars. A final key property whose proof does not differ from the unweighted case is the universal bound on the length of finite bars.

THEOREM 3.9. *For every tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms σ generating a Hamiltonian diffeomorphism of $\mathbb{C}P(\mathbf{q})$ with finitely many fixed points and every field \mathbb{F} whose characteristic is either zero or prime with each weight q_j , the longest finite bar of its barcode satisfies*

$$\beta(\sigma; \mathbb{F}) < c_{d+k}(\sigma) - c_k(\sigma), \quad \forall k \in \mathbb{Z}.$$

In particular, the longest finite bar is less than one.

4. Smith inequality

In this section, we show the natural extension of the Smith-type inequality stated in [All22a, Corollary 6.3]. Although the proof is rather similar to the unweighted case, some technical adjustments must be made. The assumption that the prime number used does not divide any of the weights is crucial here.

4.1 $\mathbb{Z}/p\mathbb{Z}$ -action of a p -iterated generating function

Let us fix a prime number $p \geq 3$ that does not divide any of the weights q_j . Let us fix $t \in \mathbb{R}$ and study the generating function of $\rho_{\mathbf{q}}(-t)\Phi$ expressed

$$F_{\sigma_{m,t}}(\mathbf{v}) := \sum_{k=1}^n f_k \left(\frac{v_k + v_{k+1}}{2} \right) + \frac{1}{2} \langle v_k, i v_{k+1} \rangle,$$

where $\mathbf{v} := (v_1, \dots, v_n) \in (\mathbb{C}^{d+1})^n$ and the $f_k : \mathbb{C}^{d+1} \rightarrow \mathbb{R}$ are S^1 -invariant and positively 2-homogeneous. Thus, $F_{\sigma_{m,t}}^p : (\mathbb{C}^{n(d+1)})^p \rightarrow \mathbb{R}$ is invariant under the action of $\mathbb{Z}/p\mathbb{Z}$ by cyclic permutation of coordinates generated by

$$(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p) \mapsto (\mathbf{v}_p, \mathbf{v}_1, \dots, \mathbf{v}_{p-1}),$$

(here $\sigma_{m,t}^p$ means $(\sigma_{m,t})^p$). The induced $\hat{F}_{\sigma_{m,t}}^p : \mathbb{C}P(\mathbf{q}^{np}) \rightarrow \mathbb{R}$ is then invariant under the $\mathbb{Z}/p\mathbb{Z}$ -action by permutation of (weighted) homogeneous coordinates induced by

$$[\mathbf{v}_1 : \mathbf{v}_2 : \dots : \mathbf{v}_p] \mapsto [\mathbf{v}_p : \mathbf{v}_1 : \dots : \mathbf{v}_{p-1}].$$

LEMMA 4.1. *The fixed points $(\mathbb{C}P(\mathbf{q}^{np}))^{\mathbb{Z}/p\mathbb{Z}}$ of the above action are the disjoint union $\bigsqcup_q P_q$ of the p following projective subspaces of weights \mathbf{q}^n :*

$$P_r := \{[\mathbf{v} : \rho_{\mathbf{q}^n}(r/p)\mathbf{v} : \rho_{\mathbf{q}^n}(2r/p)\mathbf{v} : \dots : \rho_{\mathbf{q}^n}((p-1)r/p)\mathbf{v}] \mid [\mathbf{v}] \in \mathbb{C}P(\mathbf{q}^n)\},$$

for $r \in \mathbb{Z}/p\mathbb{Z}$.

Proof. Let $(\mathbf{v}_1, \dots, \mathbf{v}_p) \in \mathbb{S}(\mathbf{q}^{np})$ be a point whose projection is in $(\mathbb{C}P(\mathbf{q}^{np}))^{\mathbb{Z}/p\mathbb{Z}}$. There exists $t \in \mathbb{R}$ such that $\mathbf{v}_{j+1} = \rho_{\mathbf{q}^n}(t)\mathbf{v}_j$ for all j so that every \mathbf{v}_j is non-zero and $\rho_{\mathbf{q}^n}(pt)$ fixes it. Let $m \in \mathbb{N}$ be the order of the isotropy group of \mathbf{v}_1 under the S^1 -action, so that $pt = k/m$ for some $k \in \mathbb{N}$. As m divides one of the weights, it is prime with p so there exists $u, v \in \mathbb{Z}$ such that $um + vp = 1$. Thus, $t = k(u/p + v/m)$. As $\rho_{\mathbf{q}^n}(kv/m)$ fixes \mathbf{v}_1 , one can assume $t = ku/p$ and we deduce that $[\mathbf{v}_1 : \dots : \mathbf{v}_p] \in P_{ku}$.

The other inclusion is clear. □

PROPOSITION 4.2. For $r \in \mathbb{Z}/p\mathbb{Z}$ and $t \in (-m, m)$, let $g_{r,t}$ be the restriction to P_r of $\widehat{F}_{\sigma_{m,t}^p}$. Up to a shift in degree,

$$H_*(\{g_{r,b} \leq 0\}, \{g_{r,a} \leq 0\}) \simeq G_*^{(a+r/p, b+r/p)}(\sigma),$$

when $-m < pa < pb < m$, with $a + r/p$ and $b + r/p$ not action values of σ as well as pa and pb not action values of σ^p .

Proof. Given a family (h_t) of maps $X \rightarrow \mathbb{R}$, we use the notation

$$G(h_t) := H_*(\{h_b \leq 0\}, \{h_a \leq 0\}),$$

so that we must show $G(g_{r,t}) \simeq G(\widehat{F}_{\sigma_{m,t+r/p}})$, up to a shift in degree.

Using the fact that the f_k are S^1 -invariant,

$$\begin{aligned} & \frac{1}{p} F_{\sigma_{m,t}^p}(\mathbf{v}, \rho_{\mathbf{q}^n}(r/p)\mathbf{v}, \dots, \rho_{\mathbf{q}^n}(r(p-1)/p)\mathbf{v}) \\ &= \sum_{k=1}^{n-1} \left[f_k \left(\frac{v_k + v_{k+1}}{2} \right) + \frac{1}{2} \langle v_k, i v_{k+1} \rangle \right] + f_n \left(\frac{v_n + \rho_{\mathbf{q}}(r/p)v_1}{2} \right) + \frac{1}{2} \langle v_n, i \rho_{\mathbf{q}}(r/p)v_1 \rangle. \end{aligned}$$

We apply the linear change of variables $\mathbf{v} \mapsto \mathbf{u}$ given by

$$u_k := v_k + (-1)^k \frac{I - \rho_{\mathbf{q}}(r/p)}{2} v_1$$

so that

$$\begin{cases} u_1 + u_2 = v_1 + v_2, \\ u_2 + u_3 = v_2 + v_3, \\ \vdots \\ u_{n-1} + u_n = v_{n-1} + v_n, \\ u_n + u_1 = v_n + \rho_{\mathbf{q}}(r/p)v_1. \end{cases}$$

A direct computation gives

$$\sum_{k=1}^{n-1} \langle v_k, i v_{k+1} \rangle + \langle v_n, i \rho_{\mathbf{q}}(r/p)v_1 \rangle = \sum_{k=1}^n \langle u_k, i u_{k+1} \rangle - 2 \sum_{j=0}^d \tan \left(\frac{r\pi q_j}{p} \right) |u_{1,j}|^2,$$

for all integer $r \in \mathbb{Z}/p\mathbb{Z}$, so that

$$\begin{aligned} F_{\sigma_{m,t}^p}(\mathbf{v}, \rho_{\mathbf{q}^n}(r/p)\mathbf{v}, \dots, \rho_{\mathbf{q}^n}(r(p-1)/p)\mathbf{v}) &= p \left[F_{\sigma_{m,t}}(\mathbf{u}) - \sum_{j=0}^d \tan \left(\frac{r\pi q_j}{p} \right) |u_{1,j}|^2 \right] \\ &= p [F_{\sigma_{m,t}}(\mathbf{u}) + F_{\delta_{r/p}}(u_1)]. \end{aligned}$$

The last bracket is the fiberwise sum of a generating function of $\rho_{\mathbf{q}}(-t)\Phi$ and the elementary generating function of $\delta_{r/p} = \rho_{\mathbf{q}}(-r/p)$, that we denote $F_{\sigma_{m,t}} + F_{\delta_{r/p}}$, evaluated at \mathbf{u} . From this change of coordinates, we deduce the isomorphism

$$G(g_{r,t}) \simeq G(\widehat{F_{\sigma_{m,t}} + F_{\delta_{r/p}}}).$$

We recall that in this case, an $\mathbb{R}_+^* \times S^1$ -orbit of critical points of the fiberwise sum is in one-to-one correspondence with an $\mathbb{R}_+^* \times S^1$ -orbit of fixed points of the composed diffeomorphism $\rho_{\mathbf{q}}(-t - r/p)\Phi$ (see the paragraph surrounding (5)). Let us identify r with its representative in $\{0, 1, \dots, p-1\}$. In contrast to the unweighted case, the path $s \mapsto F_{\delta_{sr/p}}$, $s \in [0, 1]$, is not

well-defined in general and one must add auxiliary variables to this generating function. The generating function $F_{\varepsilon_{1,r/p}}$ is a quadratic form generating the same Hamiltonian diffeomorphism as the elementary generating function $F_{\delta_{r/p}}$. According to Lemma 2.3, there exists a linear fiberwise isomorphism $(x; \xi) \mapsto (x; L(x, \xi))$ such that

$$F_{\varepsilon_{1,r/p}}(x; L(x, \xi)) = F_{\delta_{r/p}}(x) + R(\xi),$$

where R is a non-degenerate quadratic form. Applying the fiberwise isomorphism $(x; \eta, \xi) \rightarrow (x; \eta, L(x; \xi))$, one obtains

$$G(\widehat{F_{\sigma_{m,t}} + F_{\varepsilon_{1,r/p}}}) \simeq G(\widehat{F_{\sigma_{m,t}} + (F_{\delta_{r/p}} \oplus R)}).$$

As 0 is a regular value of $\widehat{F_{\sigma_{m,t}} + (F_{\delta_{r/p}} \oplus R)}$ for $t \in \{a, b\}$ by assumption, [Giv90, Proposition B.1] implies

$$G(\widehat{F_{\sigma_{m,t}} + (F_{\delta_{r/p}} \oplus R)}) \simeq G(\widehat{F_{\sigma_{m,t}} + F_{\delta_{r/p}}}),$$

up to a shift in degree (the index of R). Therefore, we can now replace $F_{\delta_{r/p}}$ by $F_{\varepsilon_{1,r/p}}$.

Let $(f_{s,t})$ be the family of well-defined maps

$$f_{s,t} := F_{\sigma_{m,t+(1-s)r/p}} + F_{\varepsilon_{1,sr/p}}, \quad s \in [0, 1].$$

The function $f_{s,t}$ is the fiberwise sum of a generating function of $\rho_{\mathbf{q}}(-t - (1-s)r/p)\Phi$ and a generating function of $\rho_{\mathbf{q}}(-sr/p)$ so 0 is a regular value of $f_{s,t}$ if and only if $\rho_{\mathbf{q}}(-t - r/p)\Phi$ does not have any $\mathbb{R}_+^* \times S^1$ -orbit of fixed points, that is, if and only if $t + r/p$ is not an action value of σ . According to [All22a, Proposition 4.7], $G(f_{0,t}) \simeq G(f_{1,t})$ so that

$$G(g_{r,t}) \simeq G(\widehat{F_{\sigma_{m,t+r/p}} + F_{\varepsilon_{1,0}}}).$$

As $F_{\varepsilon_{1,0}}$ is generating the identity as the zero map, by the same argument as before, one can replace $\widehat{F_{\sigma_{m,t+r/p}} + F_{\varepsilon_{1,0}}}$ in the last expression with $F_{\sigma_{m,t+r/p}}$, the conclusion follows. \square

4.2 Application of Smith inequality and computation of β_{tot}

Let X be a locally compact space or pair such that $H_*(X; \mathbb{F}_p)$ is finitely generated. According to Smith inequality, if the group $\mathbb{Z}/p\mathbb{Z}$ acts on X ,

$$\dim H_*(X; \mathbb{F}_p) \geq \dim H_*(X^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{F}_p) \tag{11}$$

(see, for instance, [Bor60, Chapter IV, §4.1]). Here $\dim H_*$ means the total dimension $\sum_k \dim H_k$.

PROPOSITION 4.3. *Given any tuple σ of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms, for every prime number p that is prime with any of the weights and every $a \leq b$ such that $a + r/p$ and $b + r/p$ are not action values of σ and pa and pb are not action values of σ^p ,*

$$\dim G_*^{(pa,pb)}(\sigma^p; \mathbb{F}_p) \geq \sum_{(1-p)/2 \leq r \leq (p-1)/2} \dim G_*^{(a+r/p,b+r/p)}(\sigma; \mathbb{F}_p).$$

We only prove the case $p \geq 3$ in order to simplify the exposition. In order to treat the case $p = 2$, one should modify the argument given in [All22a, Section 6.4] with arguments similar to those used here.

Proof. Let us take $m > \max(|a|, |b|)$. Now, we apply the Smith inequality (11) to the couple

$$X := (\{\hat{F}_{\sigma_{m,b}^p} \leq 0\}, \{\hat{F}_{\sigma_{m,a}^p} \leq 0\}).$$

Similarly to the unweighted case, for some $i_0 \in \mathbb{N}$,

$$H_{*+i_0}(X) = HZ_*(\sigma_{m,b}^p, \sigma_{m,a}^p) \simeq G_*^{(pa,pb)}(\sigma^p, pm) \simeq G_*^{(pa,pb)}(\sigma^p). \tag{12}$$

According to Lemma 4.1,

$$X^{\mathbb{Z}/p\mathbb{Z}} \simeq \bigsqcup_{(1-p)/2 \leq r \leq (p-1)/2} (\{\hat{F}_{\sigma_{m,b}^p}|_{P_r} \leq 0\}, \{\hat{F}_{\sigma_{m,a}^p}|_{P_r} \leq 0\}).$$

According to Proposition 4.2, up to a shift in degree,

$$H_*(X^{\mathbb{Z}/p\mathbb{Z}}; \mathbb{F}_p) \simeq \bigoplus_{(1-p)/2 \leq r \leq (p-1)/2} G_*^{(a+r/p, b+r/p)}(\sigma; \mathbb{F}_p).$$

Therefore, Smith inequality (11) together with (12) bring the conclusion. □

Deducing the Smith-type inequality for β_{tot} is now identical to the unweighted case: one can express β_{tot} as an integral and then apply Proposition 4.3.

PROPOSITION 4.4. *Let σ be a tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms with a finite number of associated fixed points in $\mathbb{C}P(\mathbf{q})$. For every $a \in \mathbb{R}$, every integer $n \in \mathbb{N}^*$ and every field \mathbb{F} whose characteristic is either zero or prime to any of the weights q_j ,*

$$\beta_{\text{tot}}(\sigma; \mathbb{F}) = \frac{1}{2} \left(\int_0^1 \dim G_*^{(a+t, a+t+n)}(\sigma; \mathbb{F}) dt - n|\mathbf{q}| \right).$$

COROLLARY 4.5. *For every tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms with a finite number of associated fixed points in $\mathbb{C}P(\mathbf{q})$, for every prime number p that is prime with any of the weights,*

$$\beta_{\text{tot}}(\sigma^p; \mathbb{F}_p) \geq p\beta_{\text{tot}}(\sigma; \mathbb{F}_p).$$

Another easy consequence of the integral formula is the following proposition (a direct extension of [All22a, Proposition 6.4]).

PROPOSITION 4.6. *For every tuple of small $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms σ with a finite number of associated fixed points in $\mathbb{C}P(\mathbf{q})$, there exists an integer $N \in \mathbb{N}$ such that for all prime number $p \geq N$,*

$$\beta_{\text{tot}}(\sigma; \mathbb{F}_p) = \beta_{\text{tot}}(\sigma; \mathbb{Q}).$$

5. Proof of the main theorem

The proof of Theorem 1.2 is essentially identical to that given in [All22a]: we reproduce it with slight modifications for the reader’s convenience. The proof of Theorem 1.1 was given after Theorem 3.4.

Proof of Theorem 1.2. Let σ be any tuple of $\mathbb{R}_+^* \times S^1$ -equivariant Hamiltonian diffeomorphisms associated with φ , so that $N(\sigma; \mathbb{F}) = N(\varphi; \mathbb{F})$. Let us denote by $K(\sigma; \mathbb{F})$ the number of \mathbb{Z} -orbits of finite bars of the barcode associated with σ over the field \mathbb{F} . According to the universal coefficient theorem, one can assume that $\mathbb{F} = \mathbb{Q}$ if \mathbb{F} has characteristic zero and $\mathbb{F} = \mathbb{F}_p$ if it has characteristic $p \neq 0$.

Let us assume that $\mathbb{F} = \mathbb{Q}$. According to Proposition 3.8, $N(\sigma; \mathbb{Q}) > |\mathbf{q}|$ implies that $K(\sigma; \mathbb{Q}) > 0$ so the maximal length of a finite bar $\beta(\sigma; \mathbb{Q}) > 0$. According to Corollary 4.5, for all prime number p prime with any of the weights,

$$K(\sigma^p; \mathbb{F}_p)\beta(\sigma^p; \mathbb{F}_p) \geq \beta_{\text{tot}}(\sigma^p; \mathbb{F}_p) \geq p\beta_{\text{tot}}(\sigma; \mathbb{F}_p).$$

Thus, by Proposition 4.6, for all sufficiently large prime p ,

$$K(\sigma^p; \mathbb{F}_p)\beta(\sigma^p; \mathbb{F}_p) \geq p\beta_{\text{tot}}(\sigma; \mathbb{Q}) \geq p\beta(\sigma; \mathbb{Q}),$$

that is to say that $K(\sigma^p; \mathbb{F}_p)\beta(\sigma^p; \mathbb{F}_p)$ grows at least linearly with prime numbers p . According to Theorem 3.9, $\beta(\sigma^p; \mathbb{F}_p) \leq 1$ so $K(\sigma^p; \mathbb{F}_p)$ must diverge to $+\infty$ with prime numbers p and so must $N(\sigma^p; \mathbb{F}_p)$ by Proposition 3.8. Let $z_1, \dots, z_n \in \mathbb{C}P(\mathbf{q})$ be the fixed points of φ . According to Corollary 3.2, there exists $B > 0$ such that $\dim C_*(\sigma^p; z_k, t_j(z_k); \mathbb{F}_p) < B$ for all k, j , and any prime p that does not divide any of the weights. Let $A \in \mathbb{N}$ be such that for any prime $p \geq A$, $N(\sigma^p; \mathbb{F}_p) > nq_0q_1 \cdots q_d B$. Then, for any prime $p \geq A$, there must be at least one fixed point of φ^p that is not one of the z_k , that is, there must be at least one p -periodic point that is not a fixed point. Hence, the conclusion for the case \mathbb{F} of characteristic zero.

Let us assume that $\mathbb{F} = \mathbb{F}_p$ for some prime number p . By contradiction, let us assume that φ has only finitely many periodic points of period belonging to $\{p^k \mid k \in \mathbb{N}\}$. According to Corollary 4.5,

$$\beta_{\text{tot}}(\sigma^{p^k}; \mathbb{F}_p) \geq p^k \beta_{\text{tot}}(\sigma; \mathbb{F}_p), \quad \forall k \in \mathbb{N},$$

in particular, $N(\sigma^{p^k}; \mathbb{F}_p) > |\mathbf{q}|$ for all $k \in \mathbb{N}$. Thus, by taking a sufficiently large p^k -iterate of φ , one can assume that every periodic point of φ of period p^k for some k is an admissible fixed point of φ (see the text before Proposition 3.1 for the definition of an admissible fixed point). According to Proposition 3.1, it implies that $N(\sigma^{p^k}; \mathbb{F}_p) = N(\sigma; \mathbb{F}_p)$ for all $k \in \mathbb{N}$. But Corollary 4.5 together with Proposition 3.8 imply that the left-hand side of this equation must diverge to $+\infty$ as k grows, a contradiction. \square

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Appendix A. Homology properties of orbibundles

In this appendix, we show a version of the Thom isomorphism and the existence of the Gysin morphism for orbibundles.

We refer to [Sat57, § 3] for a precise definition of the orbifold structure of a smooth orbibundle (orbibundles are called V -bundles there). We only use the underlying topological spaces at stake and work with the following topological version of orbifold containing the underlying topological maps of the smooth definition. A continuous map $\pi : E \rightarrow B$ is an F -orbibundle, F being a topological space or pair, if B is covered by open sets (V_α) satisfying the following trivialization property. For all α , there exist a topological space U_α , a finite group Γ_α acting continuously on U_α , a continuous action of Γ_α on $U_\alpha \times F$ of the form

$$\gamma \cdot (x, y) := (\gamma \cdot x, f_\alpha(\gamma, x, y)), \quad \forall (x, y) \in U_\alpha \times F, \forall \gamma \in \Gamma_\alpha,$$

for some map f_α , there also exist Γ_α -invariant continuous map $U_\alpha \rightarrow V_\alpha$ and $U_\alpha \times F \rightarrow \pi^{-1}(V_\alpha)$ inducing the respective homeomorphisms $\varphi_\alpha : U_\alpha/\Gamma_\alpha \rightarrow V_\alpha$ and $\chi_\alpha : (U_\alpha \times F)/\Gamma_\alpha \rightarrow \pi^{-1}(V_\alpha)$ and making the following diagram commute:

$$\begin{array}{ccccc}
 \pi^{-1}(V_\alpha) & \xleftarrow[\chi_\alpha]{\simeq} & (U_\alpha \times F)/\Gamma_\alpha & \xleftarrow{\quad} & U_\alpha \times F \\
 \downarrow \pi & & \downarrow \text{pr}_1 & & \downarrow \text{pr}_1 \\
 V_\alpha & \xleftarrow[\varphi_\alpha]{\simeq} & U_\alpha/\Gamma_\alpha & \xleftarrow{\quad} & U_\alpha
 \end{array}$$

where the pr_1 denote the projection on the first factor and the unlabeled maps are quotient maps.

Given a ring of coefficients R and an R -oriented manifold F , we say that an F -orbibundle $\pi : E \rightarrow B$ is R -oriented if there are a preferred orientation of each fiber $\pi^{-1}(b)$ and covering (V_α) such that the χ_α and the Γ_α -actions respect the orientation fiberwise. Such a covering of π is called an R -oriented covering.

THEOREM A.1 Thom isomorphism. *Let $\pi : E \rightarrow B$ be an $(\mathbb{R}^n, 0)$ -orbibundle that is R -oriented (as an \mathbb{R}^n -orbibundle). Let us assume that the order of the finite linear groups Γ_α of an R -oriented covering of π are prime with the characteristic of R or that the characteristic of R is zero. Then there exists a natural class $\tau \in H^n(E, E_0; R)$ called the Thom class of π such that the respective morphisms $\alpha \mapsto \pi_*(\alpha \frown \tau)$ and $u \mapsto \tau \smile \pi^*u$ are isomorphisms*

$$H_*(E, E_0; R) \xrightarrow{\simeq} H_{*-n}(B; R) \quad \text{and} \quad H^*(B; R) \xrightarrow{\simeq} H^{*+n}(E, E_0; R),$$

where $E_0 \subset E$ denotes the total space of the associated $(\mathbb{R}^n \setminus 0)$ -orbibundle.

Proof. Let us define the restriction of the Thom class τ_α to $(E, E_0) \cap \pi^{-1}(V_\alpha)$ for any α . The key point of this generalization is that the quotient maps associated with U_α and $U_\alpha \times (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ induce isomorphism in cohomology (and homology) over R according to Lemma 2.2:

$$H^*((E, E_0) \cap \pi^{-1}(V_\alpha); R) \xrightarrow{\simeq} H^*(U_\alpha \times (\mathbb{R}^n, \mathbb{R}^n \setminus 0); R)^{\Gamma_\alpha}, \tag{A.1}$$

indeed, the characteristic of R is either zero or prime with the order of Γ_α . Let $\tau'_\alpha \in H^n(U_\alpha \times (\mathbb{R}^n, \mathbb{R}^n \setminus 0))$ be the Thom class of the trivial $(\mathbb{R}^n, 0)$ -bundle $U_\alpha \times (\mathbb{R}^n, 0) \rightarrow U_\alpha$. As the action of Γ_α on $U_\alpha \times \mathbb{R}^n$ sends oriented fiber to oriented fiber, it preserves τ'_α and the Thom isomorphism on this trivial bundle is Γ_α -equivariant. Therefore, τ_α can naturally be defined as the inverse image of τ'_α under (A.1). The proof now follows *verbatim* the proof of the usual Thom isomorphism theorem given in [MS74, § 10]. □

Similarly to the case of \mathbb{S}^n -bundles, one can canonically include any \mathbb{S}^n -orbibundle inside an $(\mathbb{R}^{n+1}, 0)$ -orbibundle $E \rightarrow B$ and consider the long exact sequence of pair (E, E_0) to obtain the following corollary.

COROLLARY A.2 The Gysin long exact sequence. *Let $\pi : E \rightarrow B$ be an \mathbb{S}^n -orbibundle that is R -oriented. Let us assume that the order of the finite linear groups Γ_α of an R -oriented covering of π are prime with the characteristic of R or that the characteristic of R is zero. Then there exists a natural long exact sequence in homology called the Gysin long exact sequence*

$$\cdots \rightarrow H_{*+1}(E; R) \xrightarrow{\pi_*} H_{*+1}(B; R) \xrightarrow{\frown e} H_{*-n}(B; R) \xrightarrow{\pi^*} H_*(E; R) \rightarrow \cdots,$$

where $e \in H^{n+1}(B; R)$ is called the Euler class of the orbibundle, it is the pullback of the Thom class of the associated $(\mathbb{R}^{n+1}, 0)$ -orbibundle, and the natural morphism $\pi^* : H_*(B; R) \rightarrow H_{*+n}(E; R)$ is called the Gysin morphism.

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