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A GENERAL TYCHONOFF THEOREM FOR MULTIFUNCTIONS

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1. **Introduction.** Let $\{X_a\}_{a\in A}$ be a non-empty family of compact spaces and let F be a subset of the m-product $P\{X_a:a\in A\}$. The Tychonoff problem is the determination of the sets F which are pointwise compact. It is known that $P\{X_a:a\in A\}$ is pointwise compact [5, p. 400]. Also the set F of all point-closed multifunctions of $P\{X_a:a\in A\}$ is pointwise compact. The proof for the point-closed case is given by Smithson under the hypothesis that the X_a be T_1-spaces [10, p. 42]; however, the T_1-hypothesis is superfluous, as shown in this paper. The point-closed case is important in Ascoli theory for multifunctions ([6], [9]). We may also consider the case where F is the set of all point-compact multifunctions of $P\{X_a:a\in A\}$, the important case in non-Hausdorff multifunction Ascoli theory [2]. This point-compact case motivates the general multifunction Tychonoff theorem of the paper, which contains the three above-mentioned cases.

The undefined terminology of the paper is that of Kelley [4].

2. **Multifunctions.** We review the established definitions for multifunctions ([1], [6], [10], [11]): Let X, Y be non-empty sets. A *multifunction* is a point to set correspondence, denoted $f:X\to Y$, such that, for all $x\in X$, fx is a non-empty subset of Y. For $B\subseteq Y$ it is customary to write $f^-(B)=\{x:x\in X \text{ and } fx\cap B\neq\varnothing\}$, $f^+(B)=\{x:x\in X \text{ and } fx\subseteq B\}$. If Y is a topological space, a multifunction $f:X\to Y$ is *point-closed* (*point-compact*) if fx is closed (compact) for each $x\in X$. If, further, X is a topological space, a multifunction $f:X\to Y$ is *continuous* if $f^-(U)$ and $f^+(U)$ are open in X whenever U is open in Y.

Henceforth, if X is a non-empty set, the symbol $\mathscr{F}(X)$ will denote a non-empty set of non-empty subsets of X. Let X be a topological space. The set $\mathscr{F}(X)$ may be topologized as follows: an open subbase for the topology τ_n on $\mathscr{F}(X)$ consists of the subsets of $\mathscr{F}(X)$ having one of the forms $\{A:A\cap U\neq\varnothing\}$, $\{A:A\subseteq U\}$, where U is open in X. We refer to τ_n as the *neighbourhood topology* on $\mathscr{F}(X)$, because, in the special case where $\mathscr{F}(X)$ is the set of non-empty closed subsets of X, it corresponds to the topology of the same name introduced by Frink [3, p. 576]. As was pointed out by Michael [7, p. 155], the neighbourhood topology is the same as the Vietoris or finite topology.

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Let X, Y be non-empty sets. A multifunction $f:X\to Y$ is $\mathscr{F}(Y)$-*valued* if $fx\in\mathscr{F}(Y)$ for all $x\in X$. An $\mathscr{F}(Y)$-valued multifunction $f:X\to Y$ determines the *induced function* $f^*:X\to\mathscr{F}(Y)$ defined by setting $f^*(x)=fx$, for all $x\in X$. The following lemma is known for the point-closed case [10, p. 36]:

2.1 LEMMA. *Let $f:X\to Y$ be an $\mathscr{F}(Y)$-valued multifunction on a topological space X to a topological space Y. Then f is continuous if and only if $f^*:X\to(\mathscr{F}(Y),\tau_n)$ is continuous.*

Proof. This follows from the equations:

$$f^{*-1}(\{A:A\subseteq U\})=f^+(U),\qquad f^{*-1}(\{A:A\cap U\neq\phi\})=f^-(U).$$

3. **Tychonoff theorem.** The following lemma is the dual of Kelley's version of the Alexander theorem [4, p. 139]:

3.1 LEMMA. *A topological space is compact if it possesses a closed subbase $\mathscr{S}$ such that every subfamily of $\mathscr{S}$, with the finite intersection property, has a non-empty intersection.*

The following lemma generalizes theorem 15(3) of Frink [3, p. 577]:

3.2 LEMMA. *If X is compact and $\mathscr{F}(X)$ contains the non-empty closed subsets of X, then $\mathscr{F}(X)$ is τ_n-compact.*

Proof. It is clear that the family $\mathscr{S}$ of all subsets of $\mathscr{F}(X)$ of the form $K(F,F')=\{A:A\cap F\neq\varnothing \text{ and } A\subseteq F'\}$, where F, F' are closed in X, constitutes a closed subbase for τ_n. Let $\{K(F_i,F_i')\}_{i\in I}$ be a subfamily of $\mathscr{S}$ with the finite intersection property. Then $\{F_i'\}_{i\in I}$ has the finite intersection property, so that $F'=\bigcap_{i\in I}F_i'\neq\varnothing$, and, by the hypothesis, $F'\in\mathscr{F}(X)$. Because of 3.1, it will suffice to show that $F'\in K(F_i,F_i')$ for all $i\in I$. Let $i\in I$ be arbitrary. Since $F'\subseteq F_i'$, it remains to show that $F'\cap F_i\neq\varnothing$. This will follow if we show that the family $\{F_j'\cap F_i\}_{j\in I}$ has the finite intersection property. If $\{F_{j_k}'\cap F_i\}_{1\leq k\leq m}$ is a finite subfamily of $\{F_j'\cap F_i\}_{j\in I}$, there exists $A\in\mathscr{F}(X)$ such that $A\in\bigcap_{k=1}^m K(F_{j_k},F_{j_k}')\cap K(F_i,F_i')$, and therefore $\bigcap_{k=1}^m(F_{j_k}'\cap F_i)\supseteq A\cap F_i\neq\varnothing$.

Let $\{X_a\}_{a\in A}$ be a non-empty family of non-empty sets. The *m-product* of the sets X_a, written $P\{X_a:a\in A\}$, is the set of all multifunctions $f:A\to\bigcup_{a\in A}X_a$ such that $fa\subseteq X_a$ for all $a\in A$. For $a\in A$, the multifunction $pr_a:P\{X_a:a\in A\}\to X_a$, defined by $pr_a(f)=fa$, is the *a-projection* ([5], [8]). Suppose a set $\mathscr{F}(X_a)$ assigned to each $a\in A$. Let $Q_{\mathscr{F}}$ be the set of all $f\in P\{X_a:a\in A\}$ such that $fa\in\mathscr{F}(X_a)$ for all $a\in A$. It is clear that each restriction $pr_a\mid Q_{\mathscr{F}}$ is $\mathscr{F}(X_a)$-valued. Identifying $f\in Q_{\mathscr{F}}$ with its induced function $f^*\in\prod_{a\in A}\mathscr{F}(X_a)(f^*(a)=fa$ for $a\in A)$, we have $Q_{\mathscr{F}}=\prod_{a\in A}\mathscr{F}(X_a)$.

Now let the X_a be topological spaces. The family of all subsets of $Q_{\mathscr{F}}$ of the forms $(pr_a\mid Q_{\mathscr{F}})^-(U_a)$, $(pr_a\mid Q_{\mathscr{F}})^+(U_a)$, where U_a is open in X_a, $a\in A$, is an open subbase for the *pointwise topology* τ_p on $Q_{\mathscr{F}}$. Thus, τ_p is the smallest topology on $Q_{\mathscr{F}}$

rendering the $pr_a \mid Q_{\mathscr{F}}$ continuous ([5], [8]). Consequently, if τ_n^a is the neighbourhood topology on $\mathscr{F}(X_a)$, it follows from 2.1 that $(Q_{\mathscr{F}}, \tau_p) = \prod_{a\in A} (\mathscr{F}(X_a), \tau_n^a)$.

3.3 THEOREM. *Let $\{X_a\}_{a\in A}$ be a non-empty family of compact-spaces. If, for each $a \in A$, $\mathscr{F}(X_a)$ contains the non-empty closed subsets of X_a, then $Q_{\mathscr{F}}$ is τ_p-compact.*

Proof. Since $(Q_{\mathscr{F}}, \tau_p) = \prod_{a\in A}(\mathscr{F}(X_a), \tau_n^a)$, the conclusion follows from 3.2 and the classical Tychonoff theorem.

COROLLARY 1. ([5, p. 400]). *The m-product of a non-empty family of compact spaces is τ_p-compact.*

COROLLARY 2. (*Generalization of* [10, *proposition* 4.2].) *Let $\{X_a\}_{a\in A}$ be a non-empty family of compact spaces. The set of all point-closed multifunctions of $P\{X_a: a \in A\}$ is τ_p-compact.*

COROLLARY 3. *Let $\{X_a\}_{a\in A}$ be a non-empty family of compact spaces. The set of all point-compact multifunctions of $P\{X_a: a \in A\}$ is τ_p-compact.*

Proof. Let $\mathscr{F}(X_a)$ be the set of all non-empty compact subsets of $X_a (a \in A)$. Since X_a is compact, $\mathscr{F}(X_a)$ contains the non-empty closed subsets of X_a. The corollary now follows from the theorem, because $Q_{\mathscr{F}}$ is the set of all point-compact members of $P\{X_a: a \in A\}$.

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