Adv. Appl. Prob. **39,** 1115–1116 (2007)
Printed in Northern Ireland

CORRECTION

CLINE, D. B. H. (2007). Stability of nonlinear stochastic recursions with application to nonlinear AR-GARCH models. *Adv. Appl. Prob.* **39,** 462–491.

A subtle error in the above paper has resulted in an incorrect proof of the critical Lemma 4.1. The result may be salvaged with the addition of another (quite reasonable) assumption.

First, we take note of Assumption 1.4(i), which we restate here.

Assumption 1.4. (i) *There exists a set $\Theta_\#$, open in $\Theta = \{x \in \mathbb{X}\colon \|x\| = 1\}$, such that $\{B(\cdot, u)\}_{|u|\le M}$ is equicontinuous on $\Theta_\#$ for all finite M.*

For the intended applications such as threshold models, $\Theta_\# = \bigcup_{i=1}^{N} \Theta_i$, where $\Theta_1, \dots, \Theta_N$ are open, connected, and disjoint. They usually have, however, common boundaries. An interpretation of the assumption was given as follows.

Assumption 1.4. (i) (Continued.) *For each $\varepsilon > 0$ and $M < \infty$, there exists $\delta > 0$ such that $\|\theta - \theta'\| < \delta$, $\theta, \theta' \in \Theta_\#$, implies that $\|B(\theta, u) - B(\theta', u)\| < \varepsilon$ for all $|u| \le M$.*

The interpretation was meant to treat $\|\theta - \theta'\|$ as a norm on $\Theta_\#$ (not on Θ), but this may be confusing. To be correct, we define a metric $d(\theta, \theta')$ on $\Theta_\#$ that reflects the intended meaning:

$$d(\theta, \theta') = \begin{cases} \|\theta - \theta'\| & \text{if there is a connected subset of } \Theta_\# \text{ containing both } \theta \text{ and } \theta', \\ \|\theta - \theta'\| + 1 & \text{otherwise.} \end{cases}$$

The equicontinuity of Assumption 1.4(i) above, therefore, is reinterpreted as follows.

Assumption 1.4. (i) (Revised.) *For each $\varepsilon > 0$ and $M < \infty$, there exists $\delta > 0$ such that $d(\theta, \theta') < \delta$, $\theta, \theta' \in \Theta_\#$, implies that $\|B(\theta, u) - B(\theta', u)\| < \varepsilon$ for all $|u| \le M$.*

The point is that not only $\theta, \theta' \in \Theta_\#$ is required but also (in the threshold model context) θ and θ' must be *in the same* Θ_i. In other words, if θ' and θ are close in Θ but on opposite sides of a boundary, they are not close in $\Theta_\#$.

Assumption 1.4(ii) accordingly should be rephrased as well, as follows.

Assumption 1.4. (ii) (Revised.) *For each $\varepsilon > 0$, there exists $L < \infty$ such that*

$$\mathrm{P}(\tilde\theta_1 \in \Theta_\#,\ \tilde\theta_1^* \in \Theta_\#,\ d(\tilde\theta_1, \tilde\theta_1^*) < 1 \mid X_0 = x) > 1 - \varepsilon$$

for all $x \in \mathbb{X}$ with $x/\|x\| \in \Theta_\#$ and $\|x\| > L$.

That is, it is highly likely that $\tilde\theta_1$ and $\tilde\theta_1^*$ are in $\Theta_\#$ and on the same side of all boundaries, when $\|x\|$ is large.

This confusion over norms and metrics also occurs in the proof of Lemma 4.1. Recalling the definition $\eta(\theta, u) = B(\theta, u)/\|B(\theta, u)\|$, we observe that, by the revised version of Assumption 1.4(i), for each $\upsilon > 0$, there exists δ such that $d(\theta, \theta') < \delta$, $|e_1| \le M$ implies that $\|\eta(\theta, e_1) - \eta(\theta', e_1)\| < \upsilon$. But this does not preclude the possibility that $d(\eta(\theta, e_1), \eta(\theta', e_1)) \ge 1$. As discussed below, this causes the existing proof of Lemma 4.1 to fail.

Before providing a resolution, we require an additional assumption, namely that $\Theta_\#$ may be chosen in such a way that the following holds.

Assumption 1.5. *For each $\varepsilon > 0$ and $M < \infty$, there exists $\delta > 0$ such that $d(\theta, \theta') < \delta$, $\theta, \theta' \in \Theta_\#$, implies that* $\mathrm{P}(d(\eta(\theta, e_1),\ \eta(\theta', e_1)) \geq 1,\ |e_1| \leq M) < \varepsilon$.

In other words, the probability that $\eta(\theta, e_1)$ and $\eta(\theta', e_1)$ fail to be in the same Θ_i is small, *uniformly* for $d(\theta, \theta') < \delta$.

Returning specifically to the model, (1.2), investigated in Section 5 of the paper, we define $\Theta_\#$ as was done in the proof of Theorem 5.3. Now suppose that δ is chosen to ensure that $d(\theta, \theta') < \delta$ implies that $\|\eta(\theta, e_1) - \eta(\theta', e_1)\| < \upsilon$ when $|e_1| \leq M$. Note that θ and θ' are in the same Θ_i. We may see that, under this scenario, the event

$$d(\eta(\theta, e_1), \eta(\theta', e_1)) \geq 1 \quad \text{and} \quad |e_1| \leq M$$

is contained in a finite union of events of the form $|r_i e_1 - s_i| < \upsilon$, with r_i and s_i chosen independently of θ and θ'. (The correct definition of $\Theta_\#$ is important here.) Therefore, since the density of e_1 is assumed to be bounded, it follows that

$$\mathrm{P}(d(\eta(\theta, e_1),\ \eta(\theta', e_1)) \geq 1, |e_1| \leq M) < L_1 \upsilon$$

for some constant L_1. Thus, Assumption 1.5 holds with the choice $\upsilon = \varepsilon / L_1$. Similarly, the probability in the revised version of Assumption 1.4(ii) differs from 1 by no more than $L_2/\|x\|$ for some other constant L_2, and so the revised version of Assumption 1.4(ii) holds as well with $L = L_2/\varepsilon$.

Now we will revise the proof of Lemma 4.1.

Proof of Lemma 4.1. (Revised.) Suppose that q is a bounded function, uniformly continuous (with respect to d) on $\Theta_\#$. The error in the proof of Lemma 4.1 occurs after (4.4), where it is claimed, incorrectly, that Assumption 1.4(i), above, 'implies $\{q(\eta(\theta, u))(w(\theta, u))^\zeta\}_{|u| \leq M}$ is likewise equicontinuous on $\Theta_\#$'. (Note that $w(\theta, u) = \|B(\theta, u)\|$.) This is because $\eta(\theta, u)$ may cross a boundary with even a small change in θ.

What we may say instead, based on the revised version of Assumption 1.4(i) and the uniform continuity of q, is that there exists sufficiently small δ and υ such that $d(\theta, \theta') < \delta$ implies that $\|\eta(\theta, u) - \eta(\theta', u)\| < \upsilon$ and

$$|q(\eta(\theta, u))(w(\theta, u))^\zeta - q(\eta(\theta', u))(w(\theta', u))^\zeta| < \frac{\varepsilon}{4} \tag{1.1}$$

if $|u| \leq M$ and $d(\eta(\theta, u), \eta(\theta', u)) < 1$. Additionally, let $K = \bar{b}(1 + M)^\zeta \sup_{\theta \in \Theta} q(\theta)$ (see Assumption 1.2(ii)). By Assumption 1.5, we actually may choose δ so that also

$$\mathrm{P}(d(\eta(\theta, e_1),\ \eta(\theta', e_1)) \geq 1,\ |e_1| \leq M) < \frac{\varepsilon}{8K}. \tag{1.2}$$

From (1.1) and (1.2), it follows, with little calculation, that

$$\begin{aligned}
&|\,\mathrm{E}(q(\theta_1^*)(W_1^*)^\zeta\, \mathbf{1}_{|e_1| \leq M} \mid \theta_0^* = \theta) - \mathrm{E}(q(\theta_1^*)(W_1^*)^\zeta\, \mathbf{1}_{|e_1| \leq M} \mid \theta_0^* = \theta')| \\
&\quad \leq \mathrm{E}(|q(\eta(\theta, e_1))(w(\theta, e_1))^\zeta - q(\eta(\theta', e_1))(w(\theta', e_1))^\zeta|\, \mathbf{1}_{|e_1| \leq M}) \\
&\quad < \frac{\varepsilon}{2},
\end{aligned}$$

which is (4.5) in the paper. The rest of the proof then proceeds as originally stated.

The remainder of the results and proofs may be allowed to stand essentially as they are, with the understanding that d is to be used as the metric on $\Theta_\#$. Note that the revised version of Assumption 1.4(ii) supports substituting the norm with d in the proofs of Lemma 4.3 and Theorem 3.4, in particular.