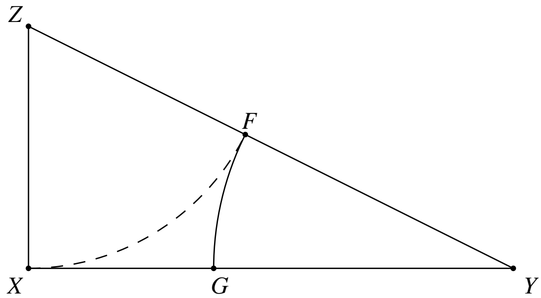

FIGURE 3

First locate Z with ZX perpendicular to XY and $XZ = \frac{1}{2}XY$ and then F on the line segment ZY with $ZF = ZX$. Lastly, let G on XY be such that $YG = YF$. This point G satisfies $\dfrac{GY}{GX} = \phi$.

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108.46 A generalisation of Fuss' theorem

Introduction

Fuss' theorem for bicentric quadrilaterals is a classic theorem of plane geometry that appeared in the 18th century in the works of Nikolai Fuss, an assistant of the great Leonhard Euler, see [1, 2, 3]. In [3], Juan Carlos Salazar gave a very simple and elegant solution to this theorem using only classical tools. This is an interesting idea, and we have exploited this idea to give a generalisation of Fuss' theorem. Here we shall propose a 'weaker' condition that only the inscribed quadrilateral is enough. The theorem is as follows:

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Theorem 1

Let $ABCD$ be a convex quadrilateral inscribed in a circle ω. Let O and R be the centre and radius of ω, respectively. Assume that bisectors of $\angle DAB$ and $\angle DCB$ meet at P lying inside $ABCD$. Let r_1 be the distance from P to the sides AB and AD. Let r_2 be the distance from P to the sides CB and CD. Let d be the distance between O and P. Let $\angle DAB = \alpha$ and $\angle DCB = \gamma$, (see Figure 1). Then,

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \sin\frac{\alpha - \gamma}{2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = 2\left(\frac{1}{(R - d)^2} + \frac{1}{(R + d)^2}\right).$$

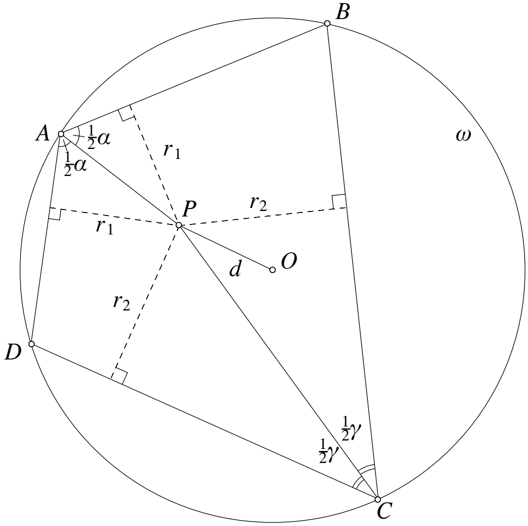

FIGURE 1: Illustration for Theorem 1

Remark: When $r_1 = r_2 = r$ then P is the incentre of the quadrilateral $ABCD$. In other words, $ABCD$ is a bicentric quadrilateral and we get Fuss' theorem

$$\frac{1}{r^2} = \frac{1}{(R - d)^2} + \frac{1}{(R + d)^2}.$$

Thus, Fuss' theorem is a particular case of Theorem 1.

Proof of Theorem 1

As mentioned above, in this solution we shall use Juan Carlos Salazar's idea in [3], but there are some differences as we have omitted the inscribed centre of the quadrilateral and replaced it with the intersection of the bisectors of two opposite angles.

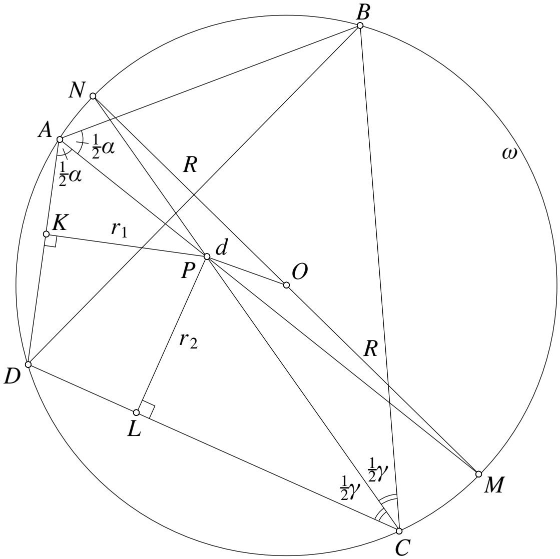

FIGURE 2: Illustration for proof of Theorem 1

Proof (see Figure 2): Let M and N be the second intersections of lines PA and PC with ω, respectively. Since, AP and CP are the bisectors of $\angle DAB$ and $\angle DCB$, respectively, M and N lie on the perpendicular bisector of BD. It is also true that MN is a diameter of ω so that O is the midpoint of MN. By Apollonius's theorem for the median in a triangle, we have

$$PM^2 + PN^2 = 2PO^2 + \tfrac{1}{2}MN^2 = 2\left(d^2 + R^2\right). \qquad (1)$$

Let K and L be the orthogonal projections of P on the sides AD and CD, respectively. We note that $\angle PAK = \frac{1}{2}\angle DAB = \frac{1}{2}\alpha$ and $\angle PCL = \frac{1}{2}\angle DCB = \frac{1}{2}\gamma$, and obtain

$$\frac{\sin\frac{1}{2}\alpha}{r_1} = \frac{1}{PA} \text{ and } \frac{\sin\frac{1}{2}\gamma}{r_2} = \frac{1}{PC}.$$

It follows that

$$\frac{\sin^2\frac{1}{2}\alpha}{r_1^2} + \frac{\sin^2\frac{1}{2}\gamma}{r_1^2} = \frac{1}{PA^2} + \frac{1}{PC^2}. \qquad (2)$$

By the power theorem and since the two chords AM and CN of ω meet at P,

$$PA \times PM = PC \times PN = R^2 - OP^2 = R^2 - d^2.$$

This leads to

$$\frac{1}{PA^2} + \frac{1}{PC^2} = \frac{PM^2 + PN^2}{\left(R^2 - d^2\right)^2}. \qquad (3)$$

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From (1), (2), and (3), we get that

$$\frac{\sin^2 \frac{1}{2}\alpha}{r_1^2} + \frac{\sin^2 \frac{1}{2}\gamma}{r_2^2} = \frac{2\left(R^2 + d^2\right)}{(R^2 - d^2)^2} = \frac{1}{(R - d)^2} + \frac{1}{(R + d)^2}. \tag{4}$$

We notice that α and γ are the measures of two opposite angles of the cyclic quadrilateral $ABCD$, so $\alpha + \gamma = 180°$ or $\frac{1}{2}\alpha + \frac{1}{2}\gamma = 90°$ which results in $\sin^2 \frac{1}{2}\alpha + \sin^2 \frac{1}{2}\gamma = 1$. So we have

$$\frac{\sin^2 \frac{1}{2}\alpha}{r_1^2} + \frac{\sin^2 \frac{1}{2}\gamma}{r_2^2} = \frac{\sin^2 \frac{1}{2}\alpha}{r_1^2} + \frac{1 - \sin^2 \frac{1}{2}\alpha}{r_2^2} = \frac{1}{r_2^2} + \left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)\sin^2 \frac{1}{2}\alpha. \tag{5}$$

Similarly, we have

$$\frac{\sin^2 \frac{1}{2}\alpha}{r_1^2} + \frac{\sin^2 \frac{1}{2}\gamma}{r_2^2} = \frac{1 - \sin^2 \frac{1}{2}\gamma}{r_1^2} + \frac{\sin^2 \frac{1}{2}\gamma}{r_2^2} = \frac{1}{r_1^2} + \left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right)\sin^2 \frac{1}{2}\gamma. \tag{6}$$

By adding the two equations (5) and (6), we obtain

$$2\left(\frac{\sin^2 \frac{1}{2}\alpha}{r_1^2} + \frac{\sin^2 \frac{1}{2}\gamma}{r_2^2}\right) = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)\left(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\gamma\right). \tag{7}$$

The trigonometric transformation with the condition $\alpha + \gamma = 180°$ gives us

$$\begin{aligned}\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\gamma &= \frac{1 - \cos\alpha}{2} - \frac{1 - \cos\gamma}{2} \\ &= \frac{\cos\gamma - \cos\alpha}{2} = -\sin\frac{\gamma + \alpha}{2}\sin\frac{\gamma - \alpha}{2} \\ &= \sin\frac{\alpha - \gamma}{2}. \end{aligned} \tag{8}$$

From (7) and (8), we see that

$$2\left(\frac{\sin^2 \frac{1}{2}\alpha}{r_1^2} + \frac{\sin^2 \frac{1}{2}\gamma}{r_2^2}\right) = \frac{1}{r_1^2} + \frac{1}{r_2^2} + \sin\frac{\alpha - \gamma}{2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right). \tag{9}$$

Now from (4) and (9), we deduce that

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \sin\frac{\alpha - \gamma}{2}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right) = 2\left(\frac{1}{(R - d)^2} + \frac{1}{(R + d)^d}\right).$$

This completes our proof.

Acknowledgments

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References

1. H. Dorrie, Fuss' problem of the chord-tangent quadrilateral, §39 in 100 *Great Problems of Elementary Mathematics*, Dover (1965) pp. 188-193.
2. F. G-M., *Exercises de geometrie* (6th edn, 1920), J. Gabay reprint, Paris (1991) pp. 837-839.
3. J. C. Salazar, Fuss' theorem, *Math. Gaz.*, **90** (July 2006), pp. 306-307.

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108.47 Thoughts on the Fermat point of a triangle

Introduction

Much has been written about the Fermat point of a triangle, and here we provide an alternative arrangement of the existing material which, we suggest, has certain advantages over the usual developments. First, a little of the history. According to [1], in 1638 Descartes invited Fermat to investigate the locus of a point X such that, for a given set $\{A, B, C, D\}$ of distinct points, the sum $XA + XB + XC + XD$ of the four distances is constant. Later, in 1643, Fermat asked Torricelli for the point X which minimises the sum of the distances $XA + XB + XC$ to three given points A, B and C. Subsequently, Torricelli found several solutions to the problem, and then, in 1659, his pupil Viviani published a solution. Briefly, there is a unique point P (now called the *Fermat*, or *Fermat-Torricelli*, point of the triangle ΔABC) which minimizes $XA + XB + XC$ over all points X in the plane. In fact, P must lie inside, or on the boundary of, ΔABC for otherwise (by relabelling the triangle if necessary) it would lie on the opposite side of the line ℓ through A and B to the vertex C. Now let Q be the reflection of P in the line ℓ. Then ℓ is given by $\{X : XP = XQ\}$, and C lies on the same side of ℓ as Q does, namely in $\{X : XQ < XP\}$; thus $QC < PC$. Since A and B lie on ℓ, we have $QA = PA$, $QB = PB$, so that

$$QA + QB + QC < PA + PB + PC$$

which is a contradiction. Thus, as illustrated in Figure 1, P must lie in the closed triangle ΔABC. Further, a search through the literature shows that not only does the Fermat point P exist within the closed triangle ΔABC, it lies strictly inside this triangle if each angle of the triangle is less than 120°; otherwise, it lies at the vertex with the largest angle.