BULL. AUSTRAL. MATH. SOC. 54C35, 54E99
VOL. 49 (1994) [91–100]

A NOTE ON FRAGMENTABLE TOPOLOGICAL SPACES

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We extend the results of N.K. Ribarska and A.V. Arhangel'skiĭ to the class of strongly countably complete spaces. And we show another characterisation of Eberlein and Radon-Nikodým compact spaces.

1. INTRODUCTION

All spaces considered in this paper are taken to be completely regular spaces. Jayne and Rogers in [5] and Ribarska in [10, 11] introduced the following notation.

DEFINITION 1.1: Let (X, τ) be a space with topology τ and let ρ be a metric on X. (The metric topology induced by ρ need not be related to τ in any way.) For each positive ε, (X,τ) is said to be *ε-fragmented by ρ* if, for each non-empty subset A of X, there exists a τ-open subset U of X such that $U \cap A \neq \emptyset$ and ρ-$\operatorname{diam}(U \cap A) \leqslant \varepsilon$.

We say that the space (X, τ) is *fragmented by ρ* (or *ρ-fragmented*) if (X, τ) is ε-fragmented by ρ for each positive ε.

The space X is said to be *fragmentable* if there exists a metric on X which fragments X.

In [10], Ribarska gave a necessary and sufficient condition for a space to be a fragmentable one (Theorem 2.8), and proved that a fragmentable compact Hausdorff space is fragmented by some complete metric.

A compact Hausdorff space is said to be *Eberlein compact* (for short EC) if it is homeomorphic to a weakly compact subset of a Banach space. A compact Hausdorrff space is said to be *Radon-Nikodým compact* (for short RNC) if it is homeomorphic to a weak* compact subset of a dual Banach space with the Radon-Nikodým property. We know some properties and characterisations of EC and RNC [7, 8].

Arhangel'skiĭ proved that a functionally complete compact Hausdorff space is an Eberlein compact space [2]. All Eberlein compact spaces are fragmented by a lower semi-continuous metric [8].

This paper consists of four sections and the introduction. In section 2 and 3, we extend the results of Arhangel'skiĭ and Ribarska to the class of the strongly countably complete spaces (Theorem 2.9 and Corollary 3.10). And we give a necessary and sufficient condition for a space to be a Namioka space (Theorem 3.5).

Received 16th February, 1993

In section 4 and 5, we show an another characterisation of EC and RNC.

In this paper, we use the following notations. $\mathbb{R}$ denotes the space of real numbers with the usual topology and $\mathbb{N} = \{1, 2, \cdots\}$. Let X be a space, then $C_p(X)$ denotes the space of all real valued continuous functions on X with the pointwise convergent topology. If X is a compact Hausdorff space, then $C(X)$ (respectively, $C(X)^*$) denotes the space of all real valued continuous functions on X (resptively, dual Banach space of $C(X)$) with the norm topology. In particular for any subset V of $C(X)^*$, (V, w^*) denotes a subspace of $C(X)^*$ with the weak* topology.

2. Complete metric case

In this section, we use the definitions and the notations in [3] and [10].

Definition 2.1: A well-ordered family $\mathbf{U} = \{U_\xi \mid 0 \leqslant \xi < \xi_0\}$ of subsets of the space X is said to be a *relatively open partitioning of* X, if

(1) $U_0 = \phi$;

(2) U_ξ is contained in $X \setminus \left(\bigcup_{\eta<\xi} U_\eta\right)$ and is relatively open in it for every ξ, $0 < \xi < \xi_0$;

(3) $X = \bigcup_{\xi<\xi_0} U_\xi$.

Definition 2.2: Let $\mathbf{W} = \{W_\xi \mid 0 < \xi \leqslant \xi_0\}$ be a well-ordered family of open subsets of X. We say that $\mathbf{W}$ *increases regularly*, if the following conditions hold:

(1) $W_1 = \emptyset$;

(2) $W_\xi \subset W_\eta$ whenever $\xi < \eta$;

(3) $W_\xi = \bigcup_{\eta<\xi} W_\eta$ for every limit ordinal ξ;

(4) $W_{\xi_0} = X$.

Let $\mathbf{U} = \{U_\xi \mid 0 \leqslant \xi < \xi_0\}$ be a relatively open partitioning of X. Let us set $W_\xi = \bigcup_{\eta<\xi} U_\eta$ for every ordinal number ξ, $0 < \xi \leqslant \xi_0$. Then $\mathbf{W}(\mathbf{U}) = \{W_\xi \mid 0 < \xi \leqslant \xi_0\}$ is a regularly increasing family of open subsets of X.

Definition 2.3: Let X be a space and $\mathbf{U}$ and $\mathbf{V}$ two relatively open partitionings of X. We say that $\mathbf{V}$ is a *refinement of* $\mathbf{U}$ if the regularly increasing family $\mathbf{W}(\mathbf{U})$ corresponding to $\mathbf{U}$ is contained in the regularly increasing family $\mathbf{W}(\mathbf{V})$ corresponding to $\mathbf{V}$.

We say that $\mathbf{V}$ is a *strong refinement of* $\mathbf{U}$ if $\mathbf{V}$ is a refinement of $\mathbf{U}$ and for every element V of $\mathbf{V}$ there exists an element U of $\mathbf{U}$ with $\overline{V} \subset U$.

Proposition 2.4. *Let X be a space and $\mathbf{U} = \{U_\xi \mid 0 \leqslant \xi < \xi_0\}$ a relatively open partitioning of X. If $\mathbf{A}$ is an open covering of X, then there exists a relatively* *open partitioning* $\mathbf{V}$ *of* X *such that* $\mathbf{V}$ *is a strong refinement of* $\mathbf{U}$ *and for each* $V \in \mathbf{V}$, *there is* $A \in \mathbf{A}$ *such that* $\overline{V} \subset A$.

PROOF: Let us fix $\xi < \xi_0$. For each $x \in U_\xi$ there is $A_x \in \mathbf{A}$ such that $x \in A_x$. Then there exists an open set V_x such that $x \in V_x$ and $\overline{V_x} \subset \left(\bigcup_{\eta \leqslant \xi} U_\eta\right) \cap A_x$.

Let us well-order the set $U_\xi = \{x_\eta \mid 0 \leqslant \eta < \eta_\xi\}$. Then $\mathbf{U}^\xi = \{U_{\xi\eta} \mid 0 \leqslant \eta < \eta_\xi\}$ where

$$U_{\xi\eta} = V_{x_\eta} \Big\backslash \Big[\Big(\bigcup_{\zeta<\xi} U_\zeta\Big) \cup \Big(\bigcup_{\zeta<\eta} V_{x_\zeta}\Big)\Big]$$

is a relatively open partitioning of U_ξ, for which

$$\overline{U_{\xi\eta}} \subset \overline{V_{x_\eta}} \subset \Big(\bigcup_{\zeta\leqslant\xi} U_\zeta\Big) \cap A_{x_\eta}$$

and

$$\overline{U_{\xi\eta}} \subset X \Big\backslash \Big(\bigcup_{\zeta<\xi} U_\zeta\Big).$$

Hence $\overline{U_{\xi\eta}}$ is contained in $U_\xi \cap A_{x_\eta}$. Let $\mathbf{V} = \{U_{\xi\eta} \mid 0 \leqslant \xi < \xi_0,\ 0 \leqslant \eta < \eta_\xi\}$. Then $\mathbf{V}$ is a relatively open partitioning of X which is as required. ∎

DEFINITION 2.5: A family $\mathbf{U}$ of subsets of the space X is said to be a *σ-relatively open partitioning of* X if $\mathbf{U} = \bigcup_{n=1}^{\infty} \mathbf{U}^n$ where $\mathbf{U}^n$, $n \in \mathbb{N}$, are relatively open partitionings of X.

$\mathbf{U}$ is said to *separate the points of* X if whenever x and y are two different points of X there exists n such that x and y belong to different elements of the partitioning $\mathbf{U}^n$.

In this case we say that X admits a *separating σ-relatively open partitioning.*

PROPOSITION 2.6. *Let* X *be a space and* $\mathbf{U} = \bigcup_{n=1}^{\infty} \mathbf{U}^n$ *a separating σ-relatively open partitioning of* X.

If $\{\mathbf{A}_n \mid n \in \mathbb{N}\}$ *is a sequence of open coverings of* X, *then there exists a separating σ-relatively open partitioning* $\mathbf{V} = \bigcup_{n=1}^{\infty} \mathbf{V}^n$ *such that* $\mathbf{V}^{n+1}$ *is a strong refinement of* $\mathbf{V}^n$ *for each* $n \in \mathbb{N}$ *and, moreover for each* $n \in \mathbb{N}$ *and for each* $V \in \mathbf{V}^n$ *there exists* $A \in \mathbf{A}_n$ *such that* $\overline{V} \subset A$.

PROOF: By Proposition 2.4, for $\mathbf{U}^1$ and $\mathbf{A}_1$ there exists a relatively open partitioning $\mathbf{V}^1$ of X such that $\mathbf{V}^1$ is a strong refinement of $\mathbf{U}^1$ and for each $V \in \mathbf{V}^1$ there is $A \in \mathbf{A}_1$ such that $\overline{V} \subset A$. Suppose $\mathbf{V}^n = \{V_\xi^n \mid 0 \leqslant \xi < \xi_n\}$, $n \geqslant 1$, is already constructed. Then $\mathbf{W}^{n\xi} = \{V_\xi^n \cap U_\eta^{n+1} \mid U_\eta^{n+1} \in \mathbf{U}^{n+1},\ 0 \leqslant \eta < \eta_{n+1}\}$ is a relatively open partitioning of V_ξ^n. By Proposition 1.8 in [10], $\mathbf{W}^{n+1} = \{V_\xi^n \cap U_\eta^{n+1} \mid 0 \leqslant \xi <$

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$\xi_n, 0 \leqslant \eta < \eta_{n+1}\}$ is a relatively open partitioning of X and $\mathbf{W}^{n+1}$ is a refinement of $\mathbf{V}^n$. Then for $\mathbf{W}^{n+1}$ and $\mathbb{A}_{n+1}$, by Proposition 2.4, there exists a relatively open partitioning $\mathbf{V}^{n+1}$ of X such that $\mathbf{V}^{n+1}$ is a strong refinement of $\mathbf{V}^n$ and, moreover, for each $V \in \mathbf{V}^{n+1}$ there is $A \in \mathbb{A}_{n+1}$ such that $\overline{V} \subset A$. Since $\mathbf{U}$ is separating, $\mathbf{V} = \bigcup_{n=1}^{\infty} \mathbf{V}^n$ is so. ∎

DEFINITION 2.7: Let $\{\mathbb{A}_n \mid n \in \mathbb{N}\}$ be a sequence of open families in the space X. The sequence $\{\mathbb{A}_n \mid n \in \mathbb{N}\}$ is said to be *strongly countably complete* if a decreasing sequence $\{F_n \mid n \in \mathbb{N}\}$ of closed subsets of X has a nonempty intersection provided that each F_n is contained in some $A_n \in \mathbb{A}_n$.

A space X is said to be *strongly countably complete* if there exists a strongly countably complete sequence of open coverings of X.

Ribarska [10] proved the following theorem.

THEOREM 2.8. *The space X admits a separating σ-relatively open partitioning if and only if there exists a metric which fragments X.*

In [10], Ribarska also proved that the compact fragmentable space X is fragmented by some complete metric ρ and the topology generated by ρ is stronger than the original topology on X.

Now we extend this result to the class of the strongly countably complete spaces.

THEOREM 2.9. *Let X be a strongly countably complete space. If X is fragmentable, then there exists a complete metric ρ on X such that X is ρ-fragmented and the topology generated by ρ is stronger than the original topology on X.*

PROOF: By Theorem 2.8, X admits a separating σ-relatively open partitioning of X. Since X is a strongly countably complete space, there exists a strongly countably complete sequence $\{\mathbb{A}_n \mid n \in \mathbb{N}\}$ of open coverings of X.

By Proposition 2.6, there exists a separating σ-relatively open partitioning $\mathbf{U} = \bigcup_{n=1}^{\infty} \mathbf{U}^n$ of X such that $\mathbf{U}^{n+1}$ is a strong refinement of $\mathbf{U}^n$ for $n \in \mathbb{N}$, and moreover, for each $n \in \mathbb{N}$ and each $U \in \mathbf{U}^n$ there is $A \in \mathbb{A}_n$ such that $\overline{U} \subset A$.

Then the metric ρ defined in the proof of Corollary 1.9 in [10] is a complete one. Indeed, let $\{x_n\}_{n=1}^{\infty}$ be a ρ-Cauchy sequence in X. Then for each $m \in \mathbb{N}$ there exists a positive integer $n(m)$ with $\rho(x_k, x_l) \leqslant 1/(m+1)$ whenever $k \geqslant n(m)$ and $l \geqslant n(m)$. Then the set $\{x_n \mid n \geqslant n(m)\}$ is contained in an element U^m of $\mathbf{U}^m$. Since $\mathbf{U}^{m+1}$ is a strong refinement of $\mathbf{U}^m$ and $U^{m+1} \cap U^m \neq \emptyset$, we have $\overline{U^{m+1}} \subset U^m$. And by the construction of $\mathbf{U}^m$, there exists an element A_m of $\mathbb{A}_m$ such that $\overline{U^m} \subset A_m$. Since $\{\mathbb{A}_n \mid n \in \mathbb{N}\}$ is a strongly countably complete sequence, there exists a point $x_0 \in \bigcap_{m=1}^{\infty} \overline{U^m}$. It is easy to see that $\{x_n\}_{n=1}^{\infty}$ is convergent to x_0. ∎

CLAIM. Let $x \in X$ and K be a closed subset of X which does not contain x. Then there exists $n \in \mathbb{N}$ and $U \in \mathrm{U}^n$ such that $x \in U$, $K \cap U = \emptyset$.

PROOF OF THE CLAIM:

If we assume that for every n the element $U^n_{\xi^n_x}$ of U^n, which contains x, intersects K, then $\{K \cap \overline{U^n_{\xi^n_x}} \mid n \in \mathbb{N}\}$ is a decreasing sequence of nonempty closed subsets of X. Now for every n, there exists $A_n \in \mathrm{A}_n$ such that $K \cap \overline{U^n_{\xi^n_x}} \subset \overline{U^n_{\xi^n_x}} \subset A_n$. Since $\{\mathrm{A}_n \mid n \in \mathbb{N}\}$ is a strongly countably complete sequence, therefore there exists a point $y \in \bigcap_{n=1}^{\infty} \left(K \cap \overline{U^n_{\xi^n_x}}\right)$. But $x \notin K$ and so $x \neq y$. Hence there exists a positive integer m with $y \notin U^m_{\xi^m_x}$ which contradicts $y \in \overline{U^{m+1}_{\xi^{m+1}_x}} \subset U^m_{\xi^m_x}$.

BACK TO THE PROOF:

By the claim, it is easy to see that every open subset of X is ρ-open. Hence the topology generated by ρ is stronger than the original topology on X. $\square$

3. LOWER SEMICONTINUOUS METRIC CASE

Let X, Y and Z be spaces. A function $f : X \times Y \to Z$ is said to be *separately continuous* if f_x is continuous for each $x \in X$ and f^y is continuous for each $y \in Y$, where f_x (respectively, f^y) is a function on Y (respectively, on X) given by $f_x(y) = f(x,y)$ (respectively, $f^y(x) = f(x,y)$).

DEFINITION 3.1: A space X is said to be a *Namioka space* if the following condition is satisfied for any compact space Y:

(*) for any separately continuous function $f : X \times Y \to \mathbb{R}$, there exists a dense G_δ subset A of X such that f is jointly continuous at each point of $A \times Y$.

DEFINITION 3.2: Let X be a space and H a subset of $C(X)$. H is said to be *equicontinuous at a point* x_0 *of* X if, for each $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $|f(x_0) - f(x)| \leqslant \varepsilon$ for all $f \in H$ and all $x \in U$.

If H is equicontinuous at each point of X, then H is said to be *equicontinuous*.

The following lemma is very important.

LEMMA 3.3. [6] *Let X be a space, $x_0 \in X$ and let Y a compact space. Suppose that the function $f : X \times Y \to \mathbb{R}$ is jointly continuous at each point of $\{x_0\} \times Y$. Then for each $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that $|f(x,y) - f(x_0,y)| \leqslant \varepsilon$ for all $x \in U$ and all $y \in Y$.*

This lemma means that if Y is a compact space and $f : X \times Y \to \mathbb{R}$ is jointly continuous at each point of $\{x\} \times Y$, then the function $F : X \to C(Y)$ defined by $F(x)(y) = f(x,y)$ for all $x \in X$ and all $y \in Y$, is continuous at x, and then, by definition 3.2, $\{f^y \mid y \in Y\}$ is equicontinuous at x.

Let X be a space and Y a compact subset of $C_p(X)$. The function $f:X\times Y\to\mathbb{R}$ given by $f(x,y)=y(x)$ is separately continuous. Hence by Lemma 3.3, if a function f is jointly continuous at each point of $\{x_0\}\times Y$, then Y is equicontinuous at x_0.

Namioka [6] proved the following theorem.

THEOREM 3.4. *Let X be a strongly countably complete regular space. If H is a compact subset of $C_p(X)$, then H is equicontinuous at each point of a dense G_δ set in X.*

Now we prove that the above condition is a necessary and sufficient condition for a space to be a Namioka space.

THEOREM 3.5. *Let X be a space. X is a Namioka space if and only if for any compact subset Y of $C_p(X)$ there exists a dense G_δ subset $A(Y)$ of X such that Y is equicontinuous at each point of $A(Y)$.*

PROOF: Let X be a Namioka space and Y a compact subset of $C_p(X)$. The function $f:X\times Y\to\mathbb{R}$ given by $f(x,y)=y(x)$ is obviously separately continuous. Then there exists a dense G_δ subset $A(Y)$ of X such that f is jointly continuous at each point of $A(Y)\times Y$. By Lemma 3.3 and by the definition of equicontinuity, Y is equicontinuous at each point of $A(Y)$.

Conversely, let Y be a compact space and $f:X\times Y\to\mathbb{R}$ a separately continuous function. Since the function $\varphi:Y\to C_p(X)$ given by $\varphi(y)=f^y$ is continuous, then Y_f is a compact subset of $C_p(X)$ (where $Y_f=\{f^y\mid y\in Y\}$). By the assumption of the theorem, there exists a dense G_δ subset $A(Y_f)$ of X such that Y_f is equicontinuous at each point of $A(Y_f)$. Then f is jointly continuous at each point (x_0,y_0) of $A(Y_f)\times Y$. Because, for each $\varepsilon>0$ there exists a neighbourhood U of x_0 such that $|f^y(x_0)-f^y(x)|<\varepsilon/2$ for all $x\in U$ and all $f^y\in Y_f$, since Y_f is equicontinuous at each point of $A(Y_f)$. Since f_{x_0} is continuous on Y, there exists a neighbourhood V of y_0 such that $|f_{x_0}(y_0)-f_{x_0}(y)|<\varepsilon/2$ for all $y\in V$. Thus we have that $|f(x_0,y_0)-f(x,y)|<\varepsilon$ for each point (x,y) of $U\times V$. Hence X is a Namioka space. ∎

The following proposition is easy to prove.

PROPOSITION 3.6. *Let X be a compact space. Then an equicontinuous family Y of $C_p(X)$ is fragmented by the metric induced by the norm on $C(X)$.*

DEFINITION 3.7: [2] Let X be a space. A subset F of $C(X)$ is said to be a *X-separating* if whenever x and y are two different points of X there exists f of F such that $f(x)\neq f(y)$.

X is said to be *functionally complete* if there exists a compact subset F of $C_p(X)$ which is a X-separating.

We introduce the next notation.

DEFINITION 3.8: Let X be a space. If every closed subspace of X is a Namioka space, then X is said to be a *closed-hereditary Namioka space*.

A metric ρ on the space X is said to be a *lower semi-continuous* (l.s.c.) if for each $\varepsilon > 0$, $\{(x,y) \mid \rho(x,y) \leqslant \varepsilon\}$ is closed in $X \times X$.

THEOREM 3.9. *Let X be a space. If X is a functionally complete closed-hereditary Namioka space, then X is fragmented by a l.s.c. metric.*

PROOF: Let X be a functionally complete closed-hereditary Namioka space. Since X is functionally complete, there exists a compact subset F of $C_p(X)$ which is a X-separating. Let ρ be the metric on X defined by $\rho(x,y) = sup\{|f(x) - f(y)| \ : f \in F\}$ for all x and y in X. Clearly, ρ is a l.s.c. metric.

Now for any nonempty closed subset K of X, the function $\varphi : K \times F \to \mathbb{R}$ defined by $\varphi(x,f) = f(x)$ is separately continuous. Since K is a Namioka space, there exists a dense G_δ subset A of K such that φ is jointly continuous at each point of $A \times F$. Since F is compact, by Lemma 3.3 for every $x \in A$ and every $\varepsilon > 0$, there exists a neighbourhood U of x in X such that $|\varphi(x,f) - \varphi(x',f)| < \varepsilon/2$ for all $f \in F$ and all $x' \in U \cap K$. This implies that $sup\{|f(x') - f(x'')| \ : x', x'' \in U \cap K\} < \varepsilon$ for all $f \in F$, that is, ρ- diam $(U \cap K) \leqslant \varepsilon$.

Hence X is ρ-fragmented. The proof is complete. ∎

A strongly countably complete space is a Namioka space [6, 9]. Since every closed subspace of a strongly countably complete space is also strongly countably complete, every strongly countably complete space is a closed-hereditary Namioka. Hence, by Theorem 3.9 we have the following corollary which is extention of the result of Arhagel'skiĭ to the class of the strongly countably complete spaces.

COROLLARY 3.10. *Every strongly countably complete, functionally complete space is fragmented by a l.s.c. metric.*

4. EBERLEIN COMPACT SPACES

DEFINITION 4.1: [1] A Banach space E is said to be *weakly compactly generated* (W.C.G.) if there is a weakly compact subset K of E such that $\overline{sp}K = E$.

THEOREM 4.2. [1, 2, 7] *Let X be a compact space. The following statements are equivalent:*

(a) X is EC;
(b) X is functionally complete;
(c) $C(X)$ is a W.C.G. Banach space;
(d) (V, w^*) is EC, where V is a unit ball of $C(X)^*$.

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DEFINITION 4.3: We say that the space (X, τ) is *σ-fragmented by ρ* if, for each $\varepsilon > 0$,

$$X = \bigcup_{n=1}^{\infty} X_n^{\varepsilon}$$

where each X_n^{ε} is ε-fragmented by ρ.

THEOREM 4.4. [4] *Let X be a Čech-analytic space (particularly, a σ-compact space) and ρ a l.s.c. metric on X. Then X is σ-fragmented by ρ if and only if, for each compact subset K of X, K is fragmented by ρ.*

LEMMA 4.5. *If E is a W.C.G. Banach space, then (E^*, w^*) is functionally complete.*

PROOF: Let E is a W.C.G. Banach space. Then there exists a weakly compact subset K of E such that $\overline{sp}K = E$. The function $\varphi : (E^*, w^*) \to C_p(K)$ defined by $\varphi(f)(x) = f(x)$ is a continuous injection. Hence (E^*, w^*) is functionally complete [2]. ∎

Now we get the following theorem.

THEOREM 4.6. *Let X be a compact space. Then the following statements are equivalent:*

(a) *X is EC (that is, X is functionally complete).*
(b) *$(C(X)^*, w^*)$ is functionally complete.*

PROOF: By Theorem 4.2 and Lemma 4.5, (a) $\Rightarrow$ (b) is clear. (b) $\Rightarrow$ (a): As functionally completeness is hereditary and X is regarded as a w^*-compact subset of $C(X)^*$, X is functionally complete. ∎

PROPOSITION 4.7. *Let X be a space. If X is functionally complete, then there exists a l.s.c. metric ρ on X such that, for each compact subset K of X, K is fragmented by ρ.*

PROOF: Let X be functionally complete, then there exists a compact subset F of $C_p(X)$ which is a X-separating. As in the proof of Theorem 3.9, let ρ be the metric on X defined by $\rho(x, y) = sup\{|f(x) - f(y)| : f \in F\}$ for all x and y in X. Clearly ρ is a l.s.c. metric and for each compact subset K of X, K is fragmented by ρ. ∎

Since $(C(X)^*, w^*)$ is a σ-compact space, by Theorem 4.4, 4.6 and Proposition 4.7, we have the following corollary.

COROLLARY 4.8. *If X is EC, then $(C(X)^*, w^*)$ is σ-fragmented by a l.s.c. metric.*

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5. Radon-Nikodým compact spaces

Theorem 5.1. [7, 8] *Let X be a compact space and ρ a l.s.c. metric on X. Then the following statements are equivalent:*

(a) *X is RNC;*

(b) *X is fragmented by ρ;*

(c) *(V, w^*) is RNC, where V is a unit ball of $C(X)^*$.*

Every EC and scattered compact space is RNC. X is a scattered compact space if and only if $(C(X)^*, w^*)$ is σ-fragmented by the norm metric [7, 8]. We proved that if X is EC, then $(C(X)^*, w^*)$ is σ-fragmented by a l.s.c. metric (Corollary 4.8).

Conversely, if $(C(X)^*, w^*)$ is σ-fragmented by a l.s.c. metric then, by Theorem 4.4, X is RNC.

Now we show the following theorem.

Theorem 5.2. *Let X be a compact space. If X is RNC, then $(C(X)^*, w^*)$ is σ-fragmented by a l.s.c. metric.*

Proof: By Corollary 3.8 in [8], there is a norm bounded subset Γ of $C(X)$ which is a X-separating and, for each countable subset A of Γ, X is d_A-separable, where $d_A(x,y) = sup\{|f(x) - f(y)| : f \in A\}$.

As in the proof of Theorem 5.6 in [8], we put $\Psi = \bigcup\{n^{-1}\Gamma^n \mid n \in \mathbb{N}\}$. Then by the Stone-Weierstrass theorem, the linear span of Ψ is norm-dense in $C(X)$, and therefore Ψ separates points of $C(X)^*$. Let ρ be the metric on $C(X)^*$ defined by $\rho(u,v) = sup\{|u(f) - v(f)| : f \in \Psi\}$ for all u, v in $C(X)^*$. Then clearly ρ is a l.s.c. metric and (V, w^*) is fragmented by ρ where V is a unit ball of $C(X)^*$ [8]. So (nV, w^*) is fragmented by ρ for each $n \in \mathbb{N}$. Since $C(X)^* = \bigcup\{nV \mid n \in \mathbb{N}\}$, $(C(X)^*, w^*)$ is σ-compact, hence Čech-analytic.

For each w^*-compact subset K of $C(X)^*$, there exists $n \in \mathbb{N}$ such that $K \subset nV$, therefore (K, w^*) is ρ-fragmented. By Theorem 4.4, $(C(X)^*, w^*)$ is σ-fragmented by ρ. ∎

Now we get the following theorem.

Theorem 5.3. *Let X be a compact Hausdorff space. The following statements are equivalent.*

(a) *X is RNC (that is, X is fragmented by a l.s.c. metric).*

(b) *$(C(X)^*, w^*)$ is σ-fragmented by a l.s.c. metric.*

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