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SUBCOMPLEXES OF CERTAIN FREE RESOLUTIONS

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Abstract. We invoke the Bernstein–Gel′fand–Gel′fand (BGG) correspondence to study subcomplexes of free resolutions given by two well-known complexes, the Koszul and the Eagon–Northcott. This approach provides a complete characterization of the ranks of free modules in a subcomplex in the Koszul case and imposes numerical restrictions in the Eagon–Northcott case.

§1. Introduction

QUESTION 1.1. *What complexes can arise as subcomplexes of a minimal free resolution?*

We were first asked a special case of this question in relation to subcomplexes of linear complexes, which are relevant to the study of stability conditions for coherent sheaves on $\mathbb{P}^n$; see, for example, work of King [17] as well as ongoing work of Bertram [4], which relate Gieseker stability conditions [12] to an analysis of all subcomplexes of some d-regular free resolutions. One difficulty in this analysis stems from the challenge of "seeing" all of the possible subcomplexes of a given linear complex, leading to variants of the above question. In this paper, we handle this by providing explicit numerical criteria that can definitively rule out subcomplexes of certain types for free resolutions given by the Koszul and Eagon–Northcott complexes.

More broadly, we are also motivated by various uses of subcomplexes in the study of free resolutions. For instance, our main question is at the heart of recent work on virtual resolutions, where classifying and understanding subcomplexes with specific properties is the key to [2, Th. 3.1] as well as related results such as [16]. Furthermore, subcomplexes are fundamental objects of study in the world of free resolutions and syzygies; the linear strand, for example, plays an essential role in many results [11], [13]–[15]. Given the ubiquity of subcomplexes in the study of syzygies, we are hopeful that our methodology demonstrates the potential of the BGG correspondence in narrowing the search space and providing an alternate viewpoint.

The goal of this paper is to further understand the structure of subcomplexes–and restrictions on when a given complex may appear as a subcomplex of a minimal free resolution–from a numerical standpoint. In addition to the dependence of stability conditions on ranks rather than differentials, this numerical approach fits in with the well-established broader approach to understanding minimal free resolutions numerically (for instance, via the study of Betti tables or Poincaré series). What is more, the numerical realm is the natural place in which to explore our main question, since a change of basis introduces an infinite number of possible subcomplexes.

In order to precisely state the numerical version of our question, we must first introduce some terminology. For a free complex $\mathbf{F}$, we define the *rank sequence* of $\mathbf{F}$ to be the integer

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sequence $\operatorname{rs}(\mathbf{F}) = (r_0, r_1, \ldots)$, where r_i is the rank of the ith free module in $\mathbf{F}$ (we refrain from calling these "Betti numbers" since our subcomplexes may in general fail to be exact). For a complex $\mathbf{G}$, we will use $\operatorname{RS}(\mathbf{G})$ to denote the set of all integer sequences r, where $r = \operatorname{rs}(\mathbf{F})$ for some subcomplex $\mathbf{F}$ of $\mathbf{G}$. Now, we may ask the following question.

QUESTION 1.2. *Given a free resolution* $\mathbf{G}$ *of a module* M*, when is an integer sequence* r *in* $\operatorname{RS}(\mathbf{G})$*?*

In this paper, we provide answers to the above questions for two large classes of modules over the polynomial ring–complete intersections and quotients by some determinantal ideals–whose minimal free resolutions are given by the Koszul and Eagon–Northcott complexes, respectively.

Our first main result exactly characterizes the integer sequences that can arise as ranks of subcomplexes of a minimal free resolution of a complete intersection. Let $S = \Bbbk[x_1, \ldots, x_n]$ and $\mathbf{K}_{(m)}$ be the Koszul complex on a regular sequence $f_1, \ldots, f_m$.

THEOREM A (Theorem 4.1). *Let* $f_1, \ldots, f_m$ *be homogeneous polynomials forming a regular sequence in the polynomial ring* $S = \Bbbk[x_1, \ldots, x_n]$ *and* $r = (r_0, \ldots, r_m)$ *be a non-negative integer sequence. Then* $r \in \operatorname{RS}(\mathbf{K}_{(m)})$ *if and only if it is the zero sequence or* $r_0 = 1$ *and*

$$0 \le r_{i+1} \le r_i^{(i)} \text{ for } 1 \le i \le n-1,$$

where $r_i^{(i)}$ *is the shifted Macaulay expansion of* r_i *as in Definition 3.4.*

We then leverage this complete characterization to obtain meaningful restrictions on the integer sequences that can arise as ranks of subcomplexes of the Eagon–Northcott complex. Before we can state this result, we need to introduce some notation: given complexes $\mathbf{F}$ and $\mathbf{G}$, we write $\operatorname{RS}(\mathbf{F}) + \operatorname{RS}(\mathbf{G})$ for the set of sequences r that can be written as an entry-wise sum $r = r_1 + r_2$ for $r_1 \in \operatorname{RS}(\mathbf{F})$ and $r_2 \in \operatorname{RS}(\mathbf{G})$. For an integer a, the set $a\operatorname{RS}(\mathbf{F})$ is defined to be the a-fold sum $\operatorname{RS}(\mathbf{G}) + \cdots + \operatorname{RS}(\mathbf{G})$.

With this notation, we can state our second main result about resolutions of modules S/I that are minimally resolved by the Eagon–Northcott complex.

THEOREM B (Theorem 4.3). *Let* $S = \Bbbk[x_1, \ldots, x_n]$*, and let* ϕ *be a* $p \times q$ *matrix with* $p \le q$ *such that the maximal minors of* ϕ *generate an ideal* I *of codimension* $q - p + 1$*, where* S/I *is Cohen–Macaulay. Let* $\mathbf{F}$ *be the minimal free resolution of* S/I*, and let* r *be an integer sequence. If* $r \in \operatorname{RS}(\mathbf{F})$*, then* $r = (0, r_0, r_1, \ldots)$ *or* $r = (1, r_0, r_1, \ldots)$*, where the sequence* $(r_0, r_1, \ldots)$ *is in*

$$\sum_{j=0}^{q-p} \binom{q-j-1}{p-1} \operatorname{RS}(\mathbf{K}_{(q-p-j)}).$$

One might be tempted to approach this problem directly by trying to explicitly produce subcomplexes of free modules with prescribed ranks, but this raises certain subtleties even in small cases. We find that without some clear strategy for controlling subcomplexes, even this numerical question becomes hard.

For a concrete example, let $S = \Bbbk[x_1, x_2, x_3]$ and $\mathbf{G}$ be the minimal free resolution of the residue field which is given by the Koszul complex on x_1, x_2, x_3, and consider the question "Is there a subcomplex $\mathbf{F}$ of $\mathbf{G}$ with ranks $r = (1, 2, 2, 0)$?" That is, "Is $(1, 2, 2, 0)$ in $\operatorname{RS}(\mathbf{G})$?"

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To answer this question directly requires a linear algebra analysis of the maps required to fill in a diagram like the one below, thus ultimately producing $\mathbf{F}$ in its entirety:

$$\begin{array}{ccccccccccc}
\mathbf{F}:0 & \longrightarrow & 0 & \longrightarrow & S(-2)^2 & \xrightarrow{f_2} & S(-1)^2 & \xrightarrow{f_1} & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_0} & & \\
\mathbf{G}:0 & \longrightarrow & S(-3) & \longrightarrow & S(-2)^3 & \longrightarrow & S(-1)^3 & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

A key takeaway here is that, even though we have asked a purely numerical question, this direct constructive approach still results in a complete description of the subcomplex $\mathbf{F}$, maps and all. Thus, without an alternate approach, merely restricting to the numerical question does not give a commensurate improvement in the tractability of the problem.

This alternate approach uses the Bernstein–Gel'fand–Gel'fand (BGG) correspondence, which gives an equivalence of categories between graded linear complexes of free modules over a polynomial ring on the one hand and graded modules over an exterior algebra on the other. The question of subcomplexes thus becomes a question of submodules, where the restriction to possible ranks is translated to a question of possible Hilbert functions. Here, we make use of existing results in a way that makes broad restrictions possible without constructing entire complexes.

The novelty of our results is thus twofold: besides providing constraints on permissible subcomplexes, we demonstrate the efficacy of the BGG correspondence in tackling an, otherwise, intractable problem and provide insight on how similar results might be obtained for other free resolutions. In fact, since the BGG correspondence is an instance of Koszul duality, these techniques could extend to characterizing subcomplexes of linear complexes over general Koszul algebras, for example, the Priddy complex.

To see how our numerical results allows us to characterize subcomplexes without needing to construct them explicitly, consider the following example.

Example 1.3. Let $S=\Bbbk[x_1,x_2,x_3]$, $\phi=\begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ 0 & x_1 & x_2 & x_3 \end{bmatrix}$, and I be the ideal of 2×2 minors of ϕ. The minimal free resolution of S/I is an Eagon–Northcott complex of the form

$$\mathbf{G}: \quad 0 \to S(-4)^3 \to S(-3)^8 \to S(-2)^6 \to S,$$

with maps as shown in Example 3.8. We can find subcomplexes of $\mathbf{G}$ of the form

$$0 \to S(-4)^2 \xrightarrow{f_3} S(-3)^6 \xrightarrow{f_2} S(-2)^5 \xrightarrow{f_1} S$$

and

$$0 \to S(-4)^1 \to S(-3)^3 \to S(-2)^3 \to S,$$

but combining Theorem B with the characterization of subcomplexes of the Koszul complex given in Theorem A rules out a subcomplex of $\mathbf{G}$ of the form

$$0 \to S(-4)^3 \to S(-3)^5 \to S(-2)^5 \to S.$$

This is because $(1,5,5,3)$ is not of the form $(1,r)$, where r is in the set

$$R = 3\,\mathrm{RS}(\mathbf{K}_{(2)}) + 2\,\mathrm{RS}(\mathbf{K}_{(1)}) + \mathrm{RS}(\mathbf{K}_{(0)}).$$

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By Theorem A, $\mathrm{RS}(\mathbf{K}_{(0)}) = \{(1,0,0),(0,0,0)\}$, $\mathrm{RS}(\mathbf{K}_{(1)}) = \{(1,1,0),(1,0,0),(0,0,0)\}$, and $\mathrm{RS}(\mathbf{K}_{(2)}) = \{(1,2,1),(1,2,0),(1,1,0),(1,0,0),(0,0,0)\}$. If a sequence in R has a 3 in the last spot, we must use the sequence $(1,2,1)$ from $\mathrm{RS}(\mathbf{K}_{(2)})$ thrice. However, three times $(1,2,1)$ gives a 6 in the middle position, so any sequence in R with a 3 in the last spot has at least a 6 in the middle. Therefore $(5,5,3) \notin R$ and thus $(1,5,5,3) \notin \mathrm{RS}(\mathbf{G})$.

As seen in the example, the common theme throughout our results is that subtle numerics govern whether a given sequence of graded free modules and maps between them has any hope of being a complex. This situation is reminiscent of numerical conditions that tell us when a given complex can be exact–a far more well-studied question. Many results are concerned with precisely characterizing exactness, while understanding when a sequence of maps is a complex is taken for granted. [6] asks "What makes a complex exact?", while [18] asks "What makes a complex a virtual resolution?" In the course of studying subcomplexes, we must step back even further and confront the question: "What makes a graded complex a complex?"

§2. Background

For the sake of clarity, we settle on a formal definition of "subcomplex."

Definition 2.1. Let $\mathbf{F} = (F_i, f_i)$ and $\mathbf{G} = (G_i, g_i)$ be two complexes of free modules over the same ring. We say $\mathbf{F}$ is a *subcomplex* of $\mathbf{G}$ if there are split injective maps $\varphi_i : F_i \to G_i$ so that $\varphi_i \circ f_{i+1} = g_{i+1} \circ \varphi_{i+1}$, that is, each square of the following diagram commutes.

$$\begin{array}{ccccccccc} \mathbf{F}: \cdots & \longrightarrow & F_{i+1} & \xrightarrow{f_{i+1}} & F_i & \xrightarrow{f_i} & F_{i-1} & \xrightarrow{f_{i-1}} & \cdots \\ & & \downarrow{\scriptstyle \varphi_{i+1}} & & \downarrow{\scriptstyle \varphi_i} & & \downarrow{\scriptstyle \varphi_{i-1}} & & \\ \mathbf{G}: \cdots & \longrightarrow & G_{i+1} & \xrightarrow[g_{i+1}]{} & G_i & \xrightarrow[g_i]{} & G_{i-1} & \xrightarrow[g_{i-1}]{} & \cdots \end{array}$$

In particular, we exclude injective maps like $\varphi_i : G_i(-1) \xrightarrow{\cdot x} G_i$. In the cases we are interested in, these φ_i can be represented by matrices with full column rank and entries from the ground field $\Bbbk$.

Given a free complex $\mathbf{G}$, our goal will be to classify the ranks of free modules appearing in subcomplexes of $\mathbf{G}$. We introduce some notation that will be used throughout.

Definition 2.2. Given a free complex $\mathbf{F} = \cdots \to F_1 \to F_0$, the *rank sequence* of $\mathbf{F}$ is

$$\mathrm{rs}(\mathbf{F}) = (r_0, r_1, \ldots),$$

where r_i is the rank of the free module F_i. For a complex $\mathbf{G}$, we use $\mathrm{RS}(\mathbf{G})$ to denote the set of all possible rank sequences of subcomplexes of $\mathbf{G}$.

Notation. Given two sets of rank sequences, say $A = \mathrm{RS}(\mathbf{F})$ and $B = \mathrm{RS}(\mathbf{G})$, we will write $A+B$ to refer to the set of sequences that may be expressed as a sum of a sequence in A and a sequence in B. Similarly, we will write nA to refer to the set $A + A + \cdots + A$, where the sum has n terms.

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2.1 The BGG correspondence

The key tool for our results is the Bernstein–Gel'fand–Gel'fand correspondence [3], which allows us to translate questions about linear free complexes of modules over a symmetric algebra into questions about modules over an exterior algebra. We will cherry-pick what we need of this rich subject; for further detail, see [9, §7B] and [10].

Let $\mathbb{k}$ be a field, let V be a $\mathbb{k}$-vector space with basis $x_1,\dots,x_n$, and let W be the dual vector space of V with basis $e_1,\dots,e_n$. Let $E = \mathbb{k}\langle e_1,\dots,e_n\rangle$ denote the exterior algebra on W. Let S denote the symmetric algebra $\mathrm{Sym}(V)$, and identify S with the polynomial ring $\mathbb{k}[x_1,\dots,x_n]$. We will assume that the x_i are graded in degree 1, and the e_i are graded in degree -1. Unless otherwise stated, all tensor products are assumed to be over the ground field $\mathbb{k}$.

We define a pair of functors $\mathbb{L}$ and $\mathbb{R}$ as follows:

$$\mathbb{L}: \{\text{Graded } E\text{-modules}\} \to \{\text{Linear complexes of free } S\text{-modules}\}$$
$$N \longmapsto (\dots \to S\otimes N_d \xrightarrow{\partial_d} S\otimes N_{d-1} \to \dots)$$

with differential ∂_d defined by linearly extending

$$1\otimes f \mapsto \sum_{i=1}^{n} x_i \otimes f e_i$$

and

$$\mathbb{R}: \{\text{Graded } S\text{-modules}\} \to \{\text{Linear complexes of free } E\text{-modules}\}$$
$$M \longmapsto (\dots \to E\otimes M_d \xrightarrow{\partial_d} E\otimes M_{d+1} \to \dots)$$

with differential ∂_d defined by linearly extending

$$1\otimes g \mapsto \sum_{i=1}^{n} e_i \otimes g x_i.$$

EXAMPLE 2.3. Consider the module $N = \langle e_1, e_2e_3\rangle E$, where $E = \mathbb{k}\langle e_1,e_2,e_3,e_4\rangle$. We will use the following $\mathbb{k}$-bases for the graded pieces of N:

$$\begin{aligned}
&\text{degree } -1 &&: e_1.\\
&\text{degree } -2 &&: e_1e_2,\ e_1e_3,\ e_1e_4,\ e_2e_3.\\
&\text{degree } -3 &&: e_1e_2e_3,\ e_1e_2e_4,\ e_1e_3e_4,\ e_2e_3e_4.\\
&\text{degree } -4 &&: e_1e_2e_3e_4.
\end{aligned}$$

Tracing through the definition of $\mathbb{L}$ we can see, for example, that

$$\partial_{-2}(1\otimes e_1e_2) = \sum_{i=1}^{4} x_i\otimes e_1e_2e_i = x_3\otimes e_1e_2e_3 + x_4\otimes e_1e_2e_4.$$

The entirety of $\mathbb{L}(N)$ is the complex:

$$0\to S\otimes N_{-1} \xrightarrow{\begin{bmatrix} x_2\\ x_3\\ x_4\\ 0\end{bmatrix}} S\otimes N_{-2} \xrightarrow{\begin{bmatrix} x_3 & -x_2 & 0 & x_1\\ x_4 & 0 & -x_2 & 0\\ 0 & x_4 & -x_3 & 0\\ 0 & 0 & 0 & x_4\end{bmatrix}} S\otimes N_{-3} \xrightarrow{\begin{bmatrix} x_4 & -x_3 & x_2 & -x_1\end{bmatrix}} S\otimes N_{-4}\to 0.$$

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The functors $\mathbb{L}$ and $\mathbb{R}$ as we have defined them may be extended to arbitrary complexes via a totalization procedure, thus yielding functors on the corresponding bounded derived categories. The BGG correspondence states that if we consider $\mathbb{L}$ and $\mathbb{R}$ as functors on the bounded derived categories then they are adjoint, implying that the derived categories of bounded linear complexes of finitely generated graded E-modules and S-modules are equivalent. This equivalence tells us that $\mathbb{L}$ and $\mathbb{R}$ are exact. What is more, the functor $\mathbb{L}$ gives a bijection on objects under which

1. Any linear complex $\mathbf{F}$ of S-modules may be expressed as $\mathbb{L}(N)$ for some E-module N [10].
2. Subcomplexes of $\mathbf{F}$ correspond to E-submodules of N.

REMARK 2.4. For $\mathbf{F} = \mathbb{L}(N)$, we can therefore relate the Hilbert function of N and the rank sequence $\mathrm{rs}(\mathbf{F})$. Take $r = (r_0, r_1, \ldots, r_n)$ and $h = (h_0, h_1, \ldots, h_n)$ to be two sequences of non-negative integers. Then $r = \mathrm{rs}(\mathbf{F})$ if and only if $h_i = r_{n-i}$ is the Hilbert function of N, that is, if $h_i = \dim_{\Bbbk}(N_{-i}) = r_{n-i}$. Note that we are still considering the Hilbert function h as a function from $\mathbb{N}$ to $\mathbb{N}$, despite the negative grading on E. We will occasionally commit the minor sin of conflating h as a function and an integer sequence, and thus write $h(N)$ for the sequence $(h_0, h_1, \ldots, h_n)$, where $h_i = \dim_{\Bbbk}(N_{-i})$.

REMARK 2.5. A quick check reveals that shifting the homological degree of a complex $\mathbf{F}$ corresponds with twisting an E-module by that same degree, that is, if $\mathbb{L}(N) = \mathbf{F}$, then $\mathbb{L}(N(i)) = \mathbf{F}[i]$, where $\mathbf{F}[i]_j = \mathbf{F}_{i+j}$.

We also make use of the following relationship between $\mathbb{L}$ and $\mathbb{R}$.

THEOREM 2.6. *(Reciprocity Theorem) [10, Th. 3.7] Let M be a graded S-module, and let N be a graded E-module. Then*

$$N \to \mathbb{R}(M)$$

is an injective resolution if and only if

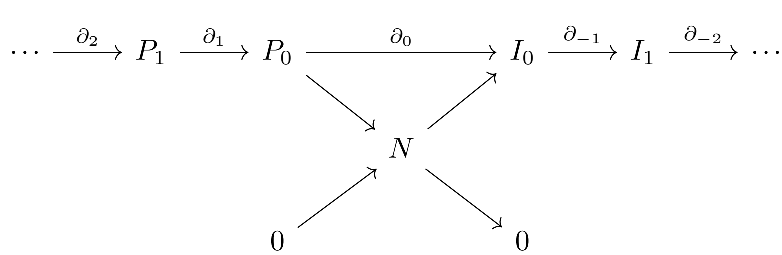
$$\mathbb{L}(N) \to M$$

is a free resolution.

2.2 Tate resolutions

The following construction, when considered in tandem with the BGG correspondence, will play a key role in Section 3.2.

DEFINITION 2.7. For any module N over any ring, we can combine a projective resolution $\mathbf{P}$ of N and an injective resolution $\mathbf{I}$ of N in the following way



to produce a *Tate resolution*.

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More detail about general Tate resolutions can be found in [10], but we are most interested in Tate resolutions of modules over E, where injective and projective modules are both free. In this case, we can take $\mathbf{P}$ to be a minimal free resolution of N and $\mathbf{I}$ to be the dual of the minimal free resolution of the dual of N to create a unique doubly infinite exact complex of free modules where the image of P_0 is isomorphic to N. We will call this doubly infinite complex *the* Tate resolution $\mathbf{T}(N)$.

Example 2.8. If $N = E/\langle e_1,\ldots,e_n\rangle \cong \Bbbk$, then the Cartan resolution $(\mathbf{C},\partial)$ is a projective resolution of N (cf. [9, Cor. 7.10]). The dual of $\mathbf{C}$ is an injective resolution of $\Bbbk$ (which is its own dual), so stitching the two together yields the Tate resolution of $\Bbbk$. Below is a snippet of $\mathbf{T}(\Bbbk)$ in the $n=3$ case:

$$\cdots \longrightarrow E^6(2) \xrightarrow{\partial_2} E^3(1) \xrightarrow{\partial_1} E \xrightarrow{\qquad\partial_0\qquad} E(-3) \xrightarrow{\partial_1^T} E^3(-4) \xrightarrow{\partial_2^T} E^6(-5) \longrightarrow \cdots$$

$$E \searrow \Bbbk \nearrow E(-3), \qquad 0 \nearrow \Bbbk \searrow 0$$

where $\partial_0 = \begin{bmatrix} e_1e_2e_3 \end{bmatrix}$, $\partial_1 = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}$, and $\partial_2 = \begin{bmatrix} e_1 & e_2 & 0 & e_3 & 0 & 0 \\ 0 & e_1 & e_2 & 0 & e_3 & 0 \\ 0 & 0 & 0 & e_1 & e_2 & e_3 \end{bmatrix}$.

In general, the differential ∂_s in the Cartan resolution can be computed by indexing the columns of ∂_s with the degree-s monomials in the x_i's and the rows by the degree $s-1$ monomials in the x_i's. Then, if column i is indexed by a monomial m and row j is indexed by a monomial m', the (i,j)th entry of ∂_s is e_k if $m/m' = x_k$ if $m' \mid m$ and 0 if $m \nmid m'$. In Example 2.8, the indexing monomials for the entries of the ∂_s are listed in graded reverse lexicographic order with $x_1 > x_2 > x_3$.

Remark 2.9. Because E is free over $\Bbbk\langle e_1,\ldots,e_m\rangle$ for $m \le n$, extending scalars from $\Bbbk\langle e_1,\ldots,e_m\rangle$ to E is faithfully flat. This means the Tate resolution $\mathbf{T}(E/\langle e_1,\ldots,e_m\rangle)$ has the same structure as the Tate resolution $\mathbf{T}(\Bbbk\langle e_1,\ldots,e_m\rangle/\langle e_1,\ldots,e_m\rangle)$ as a complex of $\Bbbk\langle e_1,\ldots,e_m\rangle$-modules. That is, the complex of $\Bbbk\langle e_1,\ldots,e_m\rangle$-modules $\mathbf{T}(\Bbbk)$ and the complex of E-modules $\mathbf{T}(\Bbbk\langle e_{m+1},\ldots,e_n\rangle)$ have modules of the same rank and twists, and differentials with the same entries, regardless of the ambient ring. For example, the Tate resolution $\mathbf{T}(\Bbbk\langle e_4\rangle)$ over $\Bbbk\langle e_1,\ldots,e_4\rangle$ will "look" the same as the one shown in Example 2.8, with all E's replaced by $\Bbbk\langle e_1,\ldots,e_4\rangle$.

§3. Resolutions of $\mathfrak{m}^d$

As before, let $S = \Bbbk[x_1,\ldots,x_n]$, where $\Bbbk$ is a field. Use $\mathfrak{m}$ to denote the homogeneous maximal ideal $\langle x_1,\ldots,x_n\rangle$. We begin by exploring the possible rank sequences of subcomplexes of resolutions of $\mathfrak{m}^d$, in particular, as they are presented by the Koszul complex in the $d=1$ case and the Eagon–Northcott complex in the $d \ge 2$ case.

3.1 The Koszul complex

Definition 3.1. The Koszul complex $\mathbf{K}(x_1,\ldots,x_m)$ is the graded exact complex

$$\mathbf{K}(x_1,\ldots,x_m): 0 \to S(-m) \xrightarrow{\partial_m} \cdots \xrightarrow{\partial_3} S(-2)^{\binom{m}{2}} \xrightarrow{\partial_2} S(-1)^m \xrightarrow{\partial_1} S^1 \to 0,$$

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where we index basis elements of $K_d := S(-d)^{\binom{m}{d}}$ by size d subsets of m. For $T = \{i_1,\ldots,i_d\}$, the differential ∂_d acts on e_T by $\partial_d(e_T) = \sum_{j=1}^{d}(-1)^j x_{i_j} e_{T\setminus i_j}$.

EXAMPLE 3.2. The Koszul complex $\mathbf{K}(x_1,x_2,x_3)$ is given by

$$\mathbf{K}(x_1,x_2,x_3): 0 \to S(-3) \xrightarrow{\partial_3} S(-2)^3 \xrightarrow{\partial_2} S(-1)^3 \xrightarrow{\partial_1} S \to 0,$$

where

$$\partial_1 = \begin{bmatrix} \boxed{x_1 \quad x_2} & x_3 \end{bmatrix} \quad \partial_2 = \begin{bmatrix} \boxed{\begin{matrix} -x_2 \\ x_1 \end{matrix}} & \begin{matrix} -x_3 \\ 0 \end{matrix} & \begin{matrix} 0 \\ -x_3 \end{matrix} \\ 0 & x_1 & x_2 \end{bmatrix} \quad \text{and} \quad \partial_3 = \begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}.$$

Looking at the above example, we can immediately identify some subcomplexes of the Koszul complex. If $m \leq n$, the complex $\mathbf{K}(x_1,\ldots,x_m)$ is a subcomplex of $\mathbf{K}(x_1,\ldots,x_n)$ – one can see the Koszul complex $\mathbf{K}(x_1,x_2)$ boxed in red in Example 3.2. One can also simply truncate $\mathbf{K}(x_1,x_2,x_3)$ after two modules and omit the $S(-3)$ module at the end, or even omit $S(-3)$ and some summands of the $S(-2)^3$ in the next spot. This observation yields certain sequences that we can be sure must occur as rank sequences of subcomplexes of $\mathbf{K}(x_1,\ldots,x_m)$ but to obtain a more complete classification we can peer through the BGG lens and, in particular, use the following key fact.

FACT 3.3 (See Example 7.6, [9]). The linear complex $\mathbb{L}(E(-n))$ is (isomorphic to) the Koszul complex $\mathbf{K}(x_1,\ldots,x_n)$.

The BGG correspondence thus tells us that subcomplexes of the Koszul complex are in correspondence with submodules of the exterior algebra E twisted by $(-n)$, so our question about the possible rank sequences of subcomplexes of $\mathbf{K}$ is transformed into a question about the possible Hilbert functions of submodules of E itself (after the appropriate twist). This perspective immediately reveals that subcomplexes of $\mathbf{K}$ are less restricted than one might guess from the $n = 3$ case. Indeed, Example 2.3 shows that we can obtain a subcomplex of $\mathbf{K}(x_1,\ldots,x_4)$ whose rank sequence is $(1,4,4,1,0)$, which is not the rank sequence of a smaller Koszul complex and furthermore cannot be obtained by truncating free summands from the tail of K.

This observation also underscores the complexity of the structural question of classifying all subcomplexes in the case of the Koszul complex. Such a task would be equivalent to classifying all ideals in E. Though the feasibility of such classification is yet unknown, it is worth noting that the parallel question of classifying ideals in S is impossible by Vakil's Murphy's Law [19].

By work of Aramova–Herzog–Hibi [1, Th. 4.1], possible Hilbert sequences of submodules of the exterior algebra are exactly those corresponding to f-vectors of simplicial complexes as described by the Kruskal–Katona theorem. We can use these results to characterize the possible rank sequences for a subcomplex of the Koszul complex with the following notation.

DEFINITION 3.4. If a is a positive integer, then, for every positive integer i, a has a unique *Macaulay expansion*

$$a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j},$$

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where $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$. Define

$$a^{(i)} := \binom{a_i}{i+1} + \binom{a_{i-1}}{i} + \cdots + \binom{a_j}{j+1}.$$

THEOREM 3.5. *A non-negative integer sequence $(r_0, r_1, \ldots, r_n)$ is in $\mathrm{RS}(\mathbf{K}(x_1, \ldots, x_n))$ if and only if it is the all zeros sequence or if $r_0 = 1$ and r satisfies*

$$0 \leq r_{i+1} \leq r_i^{(i)} \text{ for } 1 \leq i \leq n-1.$$

Proof. If r is the sequence of all zeros, it is the rank sequence of the zero complex, which is a subcomplex of any complex.

Using Corollary 5.3 from [1] and Remark 2.4, we see that $h(E/(0:I)) = \mathrm{rs}(\mathbb{L}(I))$. Because every ideal in E satisfies $0:(0:I) = I$ and can therefore be recognized as an annihilator, classifying Hilbert sequences $h(E/I)$ is equivalent to classifying Hilbert sequences $h(E/(0:I))$. Combining these two sentences, we see that classifying rank sequences $\mathrm{rs}(\mathbb{L}(I))$ is equivalent to classifying rank sequences $h(E/I)$.

By [1, Th. 4.1], a non-negative integer sequence $h = (1, h_1, \ldots, h_n)$ is the Hilbert sequence of a module E/I if and only if $0 \leq h_{i+1} \leq h_i^{(i)}$ for all $1 \leq i \leq n-1$. This translates directly to the set $\mathrm{RS}(\mathbf{K}(x_1, \ldots, x_n))$, and our theorem is proven. □

REMARK 3.6. In an analogous way, Macaulay's theorem (cf. [5, Th. 4.2.14]) characterizes the ranks of subcomplexes of the Cartan resolution of $\Bbbk$ over E.

3.2 The Eagon–Northcott complex

The Eagon–Northcott complex [7] plays the same role for determinantal ideals that a Koszul complex plays for a sequence of ring elements. We provide a brief presentation here that describes the complex for S-modules; more details can be found in [9, Appendix A2H]. Throughout, we will choose bases for our free modules so that we can represent these maps as matrices.

DEFINITION 3.7. Let $F = S^f$ and $G = S^g$, with $g \leq f$, and $\alpha : F \to G$ a map represented by a $g \times f$ matrix A with respect to bases $\{e_1, \ldots, e_f\}$ of F and $\{\varepsilon_1, \ldots, \varepsilon_g\}$ of G. Then the Eagon–Northcott complex of the map α is the complex

$$\mathbf{EN}(\alpha) : 0 \to EN_{f-g+1} \xrightarrow{d_{f-g+1}} EN_{f-g} \xrightarrow{d_{f-g}} \cdots \xrightarrow{d_3} EN_2 \xrightarrow{d_2} EN_1 \xrightarrow{\Lambda^g \alpha} \Lambda^g G,$$

where $EN_{k+1} = (\mathrm{Sym}_k G)^* \otimes \Lambda^{g+k} F$ and $d_{k+1} : (\mathrm{Sym}_k G)^* \otimes \Lambda^{g+k} F \to (\mathrm{Sym}_{k-1} G)^* \otimes \Lambda^{g+k-1} F$ is the map

$$(\varepsilon_1^{p_1} \ldots \varepsilon_g^{p_g})^* \otimes e_{s_1} \wedge \cdots \wedge e_{s_{g+k}} \mapsto \sum_{i=1}^{g+k} (-1)^{i-1} \left[\sum_{j=1}^{g} A_{j,s_i} (\varepsilon_1^{p_1} \ldots \varepsilon_j^{p_j - 1} \ldots \varepsilon_g^{p_g})^* \right] \otimes e_{s_1} \wedge \cdots \wedge \widehat{e_{s_i}} \wedge \cdots \wedge e_{s_{g+k}}$$

for $k \geq 1$, where $p_1 + \cdots + p_g = k$ and we adopt the convention that $\varepsilon_j^p = 0$ if $p < 0$.

Note that, if we represent α by the matrix A, then $A_{j,s} = \varepsilon_j^*(\alpha(e_s))$, and that using a different basis to express A will give an isomorphic complex.

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EXAMPLE 3.8. For this example, let $S = \Bbbk[x_1, x_2, x_3]$. Consider the example $\alpha : S^4 \to S^2$ represented by the matrix $A = \begin{bmatrix} x_1 & x_2 & x_3 & 0 \\ 0 & x_1 & x_2 & x_3 \end{bmatrix}$.

Making the appropriate identifications for each module, we can see that $\mathbf{EN}(\alpha)$ is the resulting graded complex

$$\mathbf{EN}(\alpha) : 0 \to S(-4)^3 \xrightarrow{d_3} S(-3)^8 \xrightarrow{d_2} S(-2)^6 \xrightarrow{d_1} S.$$

Note that the ideal of maximal minors in Example 3.8 is the ideal $\langle x_1, x_2, x_3 \rangle^2$. In general, the Eagon–Northcott complex minimally resolves any power of the maximal ideal $\mathfrak{m}^d$ by constructing the complex for the $d \times (n+d-1)$ matrix (e.g., per [8, Exer. A2.17d])

$$M^{n,d} = \begin{bmatrix} x_1 & x_2 & \dots & x_n & 0 & \dots & \dots & 0 \\ 0 & x_1 & x_2 & \dots & x_n & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & x_1 & x_2 & \dots & x_n & 0 \\ 0 & \dots & \dots & 0 & x_1 & x_2 & \dots & x_n \end{bmatrix}.$$

With this notation, the matrix A in Example 3.8 is $M^{3,2}$.

Our eventual goal is to understand, up to rank sequence, the possible subcomplexes of the Eagon–Northcott complex in a similar way as we did with the Koszul complex. We will see that these sequences are much more difficult to classify completely, but that we can still find restrictions that narrow down the possibilities. To do this, we will once again leverage the BGG correspondence in order to understand subcomplexes of Eagon–Northcott complexes. In order to do this, we must restrict ourselves to the linear maps–and therefore the degree d strand–of the Eagon–Northcott complex. Note, however, that with the exception of the first map, the Eagon–Northcott complex is always linear, and any subcomplex of the degree d strand will extend to a subcomplex of the entire Eagon–Northcott complex.

We define the complex $L_{n,d}$ to be the resolution of the ideal $\langle x_1, \dots, x_n \rangle^d$ as an S-module as presented by $\mathbf{EN}(M^{n,d})$ and note that $L_{n,d}$ is linear. Indeed, for $d \geq 2$, $L_{n,d}$ is the degree–d strand of $\mathbf{EN}(M^{n,d})$ shifted by one homological degree, while for $d = 1$ the entire complex is linear, so $L_{n,1}$ is not the entire linear strand since it is missing the first map. This complex corresponds to a specific E-module $N_{n,d}$ under the BGG correspondence, that is, $L_{n,d} = \mathbb{L}(N_{n,d})[-n+1]$. Therefore, classifying subcomplexes of $L_{n,d}$ corresponds to understanding the E-submodules of $N_{n,d}(-n+1)$.

REMARK 3.9. Given any subcomplex $\mathbf{F}$ of $L_{n,d}$, we can always extend to a subcomplex of $\mathbf{EN}(M^{n,d})$. In fact, since the first term of $\mathbf{EN}(M^{n,d})$ is S^1, we can extend $\mathbf{F}$ by either 0 or by S^1 to obtain a subcomplex of $\mathbf{EN}(M^{n,d})$. At the level of rank sequences, this means that $r \in \mathrm{RS}(\mathbf{EN}(M^{n,d}))$ has the form $(0, r')$ or $(1, r')$ for $r' \in \mathrm{RS}(L_{n,d})$.

PROPOSITION 3.10. *The module* $N_{n,d}$ *is the cokernel of* ∂_{d-1}^T *in the Tate resolution* $\mathbf{T}(\Bbbk)$.

Proof. To obtain a presentation for $N_{n,d}$, we can appeal to the Reciprocity Theorem (Theorem 2.6) and the Tate resolution $\mathbf{T}(\Bbbk)$. Because $\mathbb{L}(N_{n,d}) \to \mathfrak{m}^d$ is a free resolution, $N_{n,d} \to \mathbb{R}(\mathfrak{m}^d)$ is an injective resolution, so $N_{n,d}$ is the kernel of $\mathbb{R}(\mathfrak{m}^d)$.

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We start by finding $\mathbb{R}(\mathfrak{m})$. Observe by Theorem 2.6 that $N_{n,1} \to \mathbb{R}(\mathfrak{m})$ is an injective resolution if and only if $\mathbb{L}(N_{n,1}) \to \mathfrak{m}$ is a projective resolution. The resolution of $\mathfrak{m}$ comes from the Koszul complex without the 0th free module, which corresponds to E without its degree-0 piece under the BGG correspondence, so $N_{n,1}$ is the ideal $\langle e_1,\dots,e_n\rangle$. Therefore, $\mathbb{R}(\mathfrak{m})$ is an injective resolution of $\langle e_1,\dots,e_n\rangle$.

Let $\mathbf{I}: 0 \to \Bbbk \to I_0 \xrightarrow{\partial_1^T} I_1 \xrightarrow{\partial_2^T} \dots$ be the injective resolution of $\Bbbk$ as an E-module, obtained by taking the dual of the Cartan complex as in Example 2.8. The map ∂_1^T is $\begin{bmatrix} e_1 & \dots & e_n \end{bmatrix}^T$, meaning that $0 \to I_1 \xrightarrow{\partial_2^T} I_2 \xrightarrow{\partial_3^T} \dots$ is an injective resolution of $\langle e_1,\dots,e_n\rangle$, and is thus $\mathbb{R}(\mathfrak{m})$.

To obtain $\mathbb{R}(\mathfrak{m}^d)$ from $\mathbb{R}(\mathfrak{m})$, observe that, by the way that the functor $\mathbb{R}$ is defined, truncations of modules to certain graded degrees corresponds to truncations of complexes to certain homological degrees. In particular, $\mathbb{R}(\mathfrak{m}^d)$ is the truncation of $\mathbb{R}(\mathfrak{m})$ to $0 \to I_d \xrightarrow{\partial_{d+1}^T} I_{d+1} \xrightarrow{\partial_{d+2}^T} \dots$. This means that $N_{n,d} = \ker \partial_{d+1}^T$, which we can rewrite by the Tate resolution as $\operatorname{coker} \partial_{d-1}^T$, since the Tate resolution is the unique exact way to extend $\mathbb{R}(\mathfrak{m}^d)$ to the left. ∎

Example 3.11. We expound upon the case shown in Example 2.8, which considers the matrix $M^{3,2}$ whose minors give the ideal $\langle x_1,x_2,x_3\rangle^2 \subseteq S = \Bbbk[x_1,x_2,x_3]$. The complex $L_{3,2}$ corresponds to the E-module $N_{3,2} = \operatorname{coker} \partial_1^T$, where ∂_1^T is the map in the Tate resolution given in Example 2.8.

In this way, our search for possible rank sequences of subcomplexes of $L_{n,d}$ is translated to a search for possible Hilbert functions of submodules of $N_{n,d}$ (again with appropriate twist). We denote this set of the possible Hilbert functions of submodules of $N_{n,d}$ by $\operatorname{HF}(N_{n,d})$ and prove this set satisfies certain constraints. Some persnickety bookkeeping is necessary proceeding.

Remark 3.12. The module $N_{m,d}$ has the same presentation matrix when viewed as a $\Bbbk\langle e_1,\dots,e_m\rangle$-module and as a $\Bbbk\langle e_1,\dots,e_n\rangle$-module. This follows from combining the argument for the presentation of $N_{n,d}$ and Remark 2.9. However, the Hilbert function is *not* the same when we consider $N_{m,d}$ as a $\Bbbk\langle e_1,\dots,e_m\rangle$-module and as a $\Bbbk\langle e_1,\dots,e_n\rangle$-module. For example, as a $\Bbbk\langle e_1,e_2\rangle$-module, $N_{1,1} = \operatorname{coker}\begin{bmatrix} e_1 \end{bmatrix}$ has Hilbert function $(1,1,0)$. This differs from the Hilbert function of $\operatorname{coker}\begin{bmatrix} e_1 \end{bmatrix}$ when viewed as a $\Bbbk\langle e_1\rangle$-module, which is simply $(1,0,0)$. Therefore, in general, the set $\operatorname{HF}(N_{m,d})$ will vary, depending on the ambient exterior algebra, so we must introduce more precise notation. We will continue to let $E = \Bbbk\langle e_1,\dots,e_n\rangle$ and $N_{n,d}$ for the module where $L_{n,d} = \mathbb{L}(N_{n,d}(-n+1))$. If we are considering the module $N_{m,d}$ as an E-module, we will do so via the extension of scalars along the inclusion $\Bbbk\langle e_1,\dots,e_m\rangle \hookrightarrow E$ and will use the notation $\overline{N_{m,d}}$. Note that $\overline{N_{m,d}} = N_{m,d} \otimes \Bbbk\langle e_{m+1},\dots,e_n\rangle$.

Theorem 3.13. *The set of possible Hilbert functions of submodules of $N_{n,d}$ is restricted by the following containment:*

$$\operatorname{HF}(N_{n,d}) \subseteq \operatorname{HF}(N_{n,d-1}) + \operatorname{HF}(\overline{N_{n-1,d}}).$$

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Proof. First, we prove that

$$0 \to N_{n,d-1} \to N_{n,d} \to \overline{N_{n-1,d}} \to 0$$

is a short exact sequence. Note that $\mathfrak{m}^d$ is the S-module $S_{\geq d}$, from which we get the exact sequence of S-modules

$$0 \to S_{\geq d-1}(-1) \to S_{\geq d} \to S'_{\geq d} \to 0,$$

where $S' = \Bbbk[x_1, \ldots, x_{n-1}]$ is an S-module in the usual way: $x_n \cdot f = 0$ for any $f \in S'$.

Because $\mathbb{L}$ preserves exactness, this means that

$$0 \to \mathbb{R}(S_{\geq d-1}(-1)) \to \mathbb{R}(S_{\geq d}) \to \mathbb{R}(S'_{\geq d}) \to 0$$

is a short exact complex of linear complexes of E-modules. In particular, we know what the kernel of each complex is: exactly the corresponding module N. Therefore,

$$0 \to N_{n,d-1} \to N_{n,d} \to \overline{N_{n-1,d}} \to 0$$

is indeed a short exact sequence of E-modules.

Now suppose that $N \subseteq N_{n,d}$ is a submodule with Hilbert function $h(N)$. The image of N in $\overline{N_{n-1,d}}$ is also a submodule, which we will denote N'. Let N'' be the kernel of the induced map $N \to N'$. It is a submodule of $N_{n,d-1}$, so we have a short exact sequence

$$0 \to N'' \to N \to N' \to 0.$$

Because Hilbert functions sum over short exact sequences, we have $h(N) = h(N') + h(N'')$, so the Hilbert function of $N \subseteq N_{n,d}$ is realized as a sum of Hilbert functions of submodules of $\overline{N_{n-1,d}}$ and $N_{n,d-1}$. Thus, we see that

$$\mathrm{HF}(N_{n,d}) \subset \mathrm{HF}(\overline{N_{n-1,d}}) + \mathrm{HF}(N_{n,d-1}).$$ ∎

Note that the containment in Theorem 3.13 is not an equality. We have shown that any Hilbert function of a submodule of $N_{n,d}$ may be realized as a sum of Hilbert functions of submodules of $\overline{N_{n-1,d}}$ and $N_{n,d-1}$, but there may be submodules of $\overline{N_{n-1,d}}$ and $N_{n,d-1}$ the sum of whose Hilbert functions is not the Hilbert function of a submodule of $N_{n,d}$. Indeed, the following example shows that the containment is strict in even a very small case.

EXAMPLE 3.14. For this example, we will use $E = \Bbbk\langle e_1, e_2\rangle$ and consider the E-module $N_{2,2}$. From above, we have

$$\mathrm{HF}(N_{2,2}) \subseteq \mathrm{HF}(\overline{N_{1,2}}) + \mathrm{HF}(N_{2,1}).$$

Recall that $N_{2,2} = \operatorname{coker}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$, $\overline{N_{1,2}} = \operatorname{coker}\begin{bmatrix} e_1 \end{bmatrix}$, and $N_{2,1} = \operatorname{coker}\begin{bmatrix} e_1 e_2 \end{bmatrix}$. The module $\overline{N_{1,2}}$ has Hilbert function $(1,1,0)$, while the submodule $0 \subset N_{2,1}$ has Hilbert function $(0,0,0)$, so we get the sum $(1,1,0) + (0,0,0)$ as a potential Hilbert function for a submodule of $N_{2,2}$. However, there is no submodule of $N_{2,2}$ with Hilbert function $(1,1,0)$ by the following argument.

The module $N_{2,2}$ has two generators in degree 0, which we will call α and β. In degree -1, we have $e_1\alpha, e_1\beta$, and $e_2\alpha$, with $e_1\alpha = -e_2\beta$. Suppose we have a submodule $N \subset N_{2,1}$ with one generator in degree 0. If we denote this degree 0 generator of N by ζ, then we have that $\zeta = a\alpha + b\beta$ for some $a, b \in \Bbbk$. This gives us 2 elements in degree -1: $e_1\zeta = ae_1\alpha + be_2\beta$

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and $e_2\zeta = ae_2\alpha + be_2\beta = (ae_2 - be_1)\alpha$. We can see that these are linearly independent since the only relation in N is the relation $e_1\alpha = -e_2\beta$, so the Hilbert function of N cannot be $(1,1,0)$.

Our goal now will be to restate Theorem 3.13 as a statement about $\mathrm{RS}(L_{n,d})$ in terms of complexes for which we have a complete characterization of possible rank sequences, namely Koszul complexes. We first introduce some helpful notation: for a nonnegative integer m, we will write $E_{(m)} = E/\langle e_{m+1},\dots,e_n\rangle$. Note that $E_{(m)} \cong \Bbbk$ when $m = 0$. Furthermore, we will write I_i to refer to the ideal generated by the single degree $-i$ monomial $e_1e_2\dots e_i$ and we will make the convention that I_0 is the unit ideal. We will write $\mathbf{K}_{(m)}$ to mean the Koszul complex on m variables for $m \le n$.

Remark 3.15. As a $\Bbbk$-module, $E_{(m)}$ is simply the exterior algebra on m variables, but since we are considering everything over E, we have that $\mathbb{L}(E_{(m)}) = \mathbf{K}_{(m)}[-n+m]$, that is to say, the ith free module in the complex $\mathbb{L}(E_{(m)})$ is the $(i-n+m)$th free module in the Koszul complex on m variables.

Lemma 3.16. *For $1 \le j \le n$, we have the equality of sets*

$$\mathrm{HF}(\overline{N_{1,d}}) = \mathrm{HF}(E_{(n-1)}).$$

Proof. We will show in fact that $\overline{N_{1,d}} \cong E_{(n-1)}$. First, observe that $N_{1,d} = \Bbbk\langle e_1\rangle/\langle e_1\rangle = \Bbbk$. Therefore, $\overline{N_{1,d}} = \Bbbk \otimes \Bbbk\langle e_2,\dots,e_n\rangle \cong E_{(n-1)}$. □

Lemma 3.17. *For $n,d \ge 2$, the Hilbert functions of submodules of $N_{n,d}$ are restricted by the following containment:*

$$\mathrm{HF}(N_{n,d}) \subseteq \sum_{i=1}^{n} \binom{n+d-2-i}{n-i} \mathrm{HF}(\overline{N_{i,1}}).$$

Proof. Theorem 3.13 gives the containment

$$\mathrm{HF}(N_{n,d}) \subset \mathrm{HF}(\overline{N_{n-1,d}}) + \mathrm{HF}(N_{n,d-1}).$$

We can then iterate until we have $\mathrm{HF}(N_{n,d})$ expressed completely in terms of $\mathrm{HF}(\overline{N_{1,j}})$ and $\mathrm{HF}(\overline{N_{i,1}})$ for $2 \le i,j \le n$. Ultimately, we reduce to

$$\mathrm{HF}(N_{n,d}) \subset \sum_{i=2}^{n} \alpha_{i,1}\,\mathrm{HF}(\overline{N_{i,1}}) + \sum_{j=2}^{d} \alpha_{1,j}\,\mathrm{HF}(\overline{N_{1,j}}),$$

where $\alpha_{i,j}$ counts the number of times that $N_{i,j}$ appears in the sum. This quantity $\alpha_{i,j}$ is the number of times that (i,j) appears as the result of repeatedly subtracting $(1,0)$ and $(0,1)$ from (n,d), with the caveat that, since $(1,i+1)$ is a base case, we never reach $(1,i)$ by subtracting $(0,1)$ from $(1,i+1)$, and similarly for $(1,j)$. One can thus interpret $\alpha_{i,1}$ as the number of integer lattice paths from $(i,2)$ to (n,d) and $\alpha_{1,j}$ as the number of integer lattice paths from $(2,j)$ to (n,d). The number of such lattice paths from (i,j) to (n,d) is given by $\binom{n-i+d-j}{n-i}$. This gives

$$\mathrm{HF}(N_{n,d}) \subset \sum_{i=2}^{n} \binom{n+d-2-i}{n-i} \mathrm{HF}(\overline{N_{i,1}}) + \sum_{j=2}^{d} \binom{n+d-2-j}{n-2} \mathrm{HF}(\overline{N_{1,j}}).$$

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Since $N_{1,j} = \Bbbk\langle e_1\rangle/\langle e_1\rangle = \Bbbk$, we see that $\overline{N_{1,j}} = \Bbbk \otimes \Bbbk\langle e_2,\dots,e_n\rangle \cong E_(n-1)$, so we can write the sum

$$\sum_{j=2}^{d} \binom{n+d-2-j}{n-2} \mathrm{HF}(\overline{N_{1,j}}) = \left(\sum_{j=2}^{d} \binom{n+d-2-j}{n-2}\right) \mathrm{HF}(E_{(n-1)}).$$

We can reindex and convert via the hockey stick identity to see that

$$\sum_{j=2}^{d} \binom{n+d-2-j}{n-2} = \sum_{k=n-2}^{n+d-4} \binom{k}{n-2} = \binom{n+d-3}{n-1}$$

so we have

$$\begin{aligned} \mathrm{HF}(N_{n,d}) &\subset \binom{n+d-3}{n-1} \mathrm{HF}(E_{(n-1)}) + \sum_{i=2}^{n} \binom{n+d-2-i}{n-i} \mathrm{HF}(\overline{N_{i,1}}) \\ &= \sum_{i=1}^{n} \binom{n+d-2-i}{n-i} \mathrm{HF}(\overline{N_{i,1}}), \end{aligned}$$

where the incorporation of the first term into the sum uses the fact that $E_{(n-1)} \cong \overline{N_{1,1}}$. ∎

LEMMA 3.18. *For $1 \le i \le n$, we have the containment of sets*

$$\mathrm{HF}(\overline{N_{i,1}}) \subseteq \sum_{j=0}^{i-1} \mathrm{HF}(E_{(n-j-1)}(j)).$$

Proof. When $d = 1$, the module $N_{i,1}$ is easily computable as $N_{i,1} = \operatorname{coker}\left[e_1 \dots e_i\right] = \Bbbk\langle e_1,\dots,e_i\rangle/I_i$, and so $\overline{N_{i,1}} = (E_{(i)}/I_i) \otimes \Bbbk\langle e_{i+1},\dots,e_n\rangle = E/I_i$.

Now, we can reduce using the short exact sequences of E/I_i-modules

$$0 \to \langle e_i\rangle E/I_i \to E/I_i \to E/(\langle e_i\rangle + I_i) \to 0.$$

But $\langle e_i\rangle E/I_i \cong E_{(n-1)}/I_{i-1}(1)$ and that $E/(\langle e_i\rangle + I_i) \cong E_{(n-1)}$, so by a similar argument as we have used previously in the proof of Theorem 3.13, we may now write

$$\mathrm{HF}(E/I_m) \subseteq \mathrm{HF}(E_{(n-1)}/I_{m-1}(1)) + \mathrm{HF}(E_{(n-1)}).$$

Now, we can split $\mathrm{HF}(E_{(n-1)}/I_{m-1}(1))$ and proceed inductively to get

$$\mathrm{HF}(E/I_i) \subseteq \sum_{j=0}^{i-1} \mathrm{HF}(E_{(n-j-1)}(j)).$$ ∎

THEOREM 3.19. *For $n, d \ge 2$, the rank sequence of any subcomplex of $L_{n,d}$ can be written as a positive integral sum of rank sequences of Koszul subcomplexes on fewer than n variables. In particular,*

$$\mathrm{RS}(L_{n,d}) \subseteq \sum_{j=0}^{n-1} \binom{n-j+d-2}{d-1} \mathrm{RS}(\mathbf{K}_{(n-j-1)}).$$

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Proof. Combining Lemmas 3.17 and 3.18, we can see that

$$\mathrm{HF}(N_{n,d}) \subseteq \sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{n+d-2-i}{d-2}\mathrm{HF}(E_{(n-j-1)}(j)).$$

Twisting each side of this equality by $(-n+1)$ gives

$$\mathrm{HF}(N_{n,d}(-n+1)) \subseteq \sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{n+d-2-i}{d-2}\mathrm{HF}(E_{(n-j-1)}(-n+j+1)),$$

which, fed through the functor $\mathbb{L}$, yields a containment of sets of rank sequences:

$$\mathrm{RS}(L_{n,d}) \subseteq \sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{n+d-2-i}{d-2}\mathrm{RS}(\mathbf{K}_{(n-j-1)}).$$

We can switch the order of the double sum, reindex, and apply the hockey stick identity once again to conclude the proof:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=0}^{i-1}\binom{n+d-2-i}{d-2}\mathrm{RS}(\mathbf{K}_{(n-j-1)}) &\subseteq \sum_{j=0}^{n-1}\sum_{i=j+1}^{n}\binom{n+d-2-i}{d-2}\mathrm{RS}(\mathbf{K}_{(n-j-1)})\\ &= \sum_{j=0}^{n-1}\left(\sum_{i=0}^{n-j-1}\binom{d-2+i}{d-2}\right)\mathrm{RS}(\mathbf{K}_{(n-j-1)})\\ &= \sum_{j=0}^{n-1}\binom{n-j+d-2}{d-1}\mathrm{RS}(\mathbf{K}_{(n-j-1)}).\end{aligned}$$ ∎

Example 3.20. Let $n=4, d=3$. The complex $L_{4,3}$ resolves the ideal of maximal minors of the matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & 0 & 0\\ 0 & x_1 & x_2 & x_3 & x_4 & 0\\ 0 & 0 & x_1 & x_2 & x_3 & x_4\end{bmatrix}$$

and has the form

$$0 \to S^{10} \to S^{36} \to S^{45} \to S^{20} \to 0.$$

We can use Theorem 3.19 to rule out some integer sequences as possible rank sequences for subcomplexes of $L_{4,3}$. The containment in Theorem 3.19 states that

$$\begin{aligned}\mathrm{RS}(L_{4,3}) &\subseteq \sum_{j=0}^{3}\binom{5-j}{2}\mathrm{RS}(\mathbf{K}_{(3-j)})\\ &= 10\,\mathrm{RS}(\mathbf{K}_{(3)}) + 6\,\mathrm{RS}(\mathbf{K}_{(2)}) + 3\,\mathrm{RS}(\mathbf{K}_{(1)}) + \mathrm{RS}(\mathbf{K}_{(0)}),\end{aligned}$$

that is, any rank sequence of a subcomplex must be expressible as a sum of 10 rank sequences of subcomplexes of the Koszul complex on 3 variables, 6 rank sequences of subcomplexes of the Koszul complex on 2 variables, 3 rank sequences of subcomplexes of the Koszul complex on 1 variable, and 1 rank sequence of a subcomplex of the Koszul complex on 0 variables.

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If we consider the Koszul complex on three variables, Theorem 3.5 tells us that any rank sequence (r_0, r_1, r_2, r_3) of a subcomplex of $\mathbf{K}_{(3)}$ with $r_3 = 1$ must have $r_2 = 3$. This means that for a rank sequence (r_0, r_1, r_2, r_3) of a subcomplex of $L_{4,3}$, we must have $r_2 \geq 3r_3$. For instance, we may say for certain that the sequence $(10, 16, 20, 8)$ is not a possible rank sequence of a subcomplex of $L_{4,3}$, since $20 < 3 \cdot 8$. In this way, we are able to use our complete characterization of rank sequence of Koszul subcomplexes to eliminate certain potential rank sequences from consideration in the Eagon–Northcott case.

§4. More general resolutions

With our previous results for Koszul and Eagon–Northcott complexes resolving powers of the maximal ideal in hand, we turn now to a more general setting. In particular, we are interested in other ideals I that specialize to powers of the maximal ideal in such a way that S/I is still resolved by the Koszul or Eagon–Northcott complex

4.1 The general Koszul complex

The Koszul complex can be defined more generally to give a minimal free resolution of a complete intersection. For $f_1, \ldots, f_m \in S$, a regular sequence of homogeneous elements, we replace the differential in definition 3.1 by $\partial_d(e_T) = \sum_{j=1}^{d} (-1)^j f_{i_j} e_{T-i_j}$, adjusting the twists accordingly.

THEOREM 4.1. *Let $f_1, \ldots, f_m$ be a regular sequence of homogeneous polynomials in S. An integer sequence $r = (r_0, \ldots, r_m)$ is in* $\mathrm{RS}(\mathbf{K}(f_1, \ldots, f_m))$ *if and only if it satisfies*

$$0 \leq r_{i+1} \leq r_i^{(i)} \text{ for } 1 \leq i \leq m-1.$$

Proof. We will show that $\mathrm{RS}(\mathbf{K}(f_1, \ldots, f_m)) = \mathrm{RS}(\mathbf{K}(x_1, \ldots, x_m))$, then apply Theorem 3.5.

Let $\mathbf{K}$ be the Koszul complex on the variables $x_1, \ldots, x_m$. For $\mathbf{K}'$ a general Koszul complex $\mathbf{K}(f_1, \ldots, f_m)$ over the ring $S' = \Bbbk[y_1, \ldots, y_n]$, there is a map $\mathbf{K} \to \mathbf{K}'$ induced by the map $S \to S'$ sending x_i to f_i.

If $\mathbf{F}$ is a subcomplex of $\mathbf{K}$, then the image of $\mathbf{F}$ under this map is a subcomplex of $\mathbf{K}$ with the same rank sequence. So any possible rank sequence of a subcomplex of $\mathbf{K}$ must also be possible for a subcomplex of $\mathbf{K}'$.

To see that the possible rank sequences for subcomplexes of $\mathbf{K}'$ are *exactly* those that are possible for subcomplexes of $\mathbf{K}$, we need to check that given a subcomplex $\mathbf{F}'$ of $\mathbf{K}'$, the differentials of $\mathbf{F}'$ are described by matrices over the subalgebra $R = \Bbbk[f_1, \ldots, f_m] \subseteq S'$.

For each i, we have

$$\begin{array}{ccc} F'_i & \xrightarrow{\partial} & F'_{i-1} \\ \downarrow & & \downarrow \\ K'_i & \xrightarrow{\partial'} & K'_{i-1} \end{array}$$

where the vertical maps are given by matrices over $\Bbbk$. So the differential ∂ is a matrix over R if and only if ∂' is. But entries of ∂' are linear in the f_i, so they are defined as matrices over R.

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Now given a subcomplex $\mathbf{F}'$ of $\mathbf{K}'$, we need only replace each F_i' by a free S-module of the same rank and each f_i in the differential by x_i to obtain a subcomplex $\mathbf{F}$ of $\mathbf{K}$ with the same rank sequence. ∎

4.2 More general Eagon–Northcott complexes.

Just as the Koszul complex can be generalized to give a minimal free resolution of a complete intersection, the Eagon–Northcott complex can be generalized to give a minimal free resolution of certain Cohen–Macaulay algebras of the form S/I, where I has the maximum possible codimension. We can relate the behavior of subcomplexes of the Eagon–Northcott complex resolving $\mathfrak{m}^d$ to the behavior of subcomplexes of a general Eagon–Northcott complex as follows. First, we consider a motivating example.

EXAMPLE 4.2. Let $Y = [y_{i,j}]$ be a $p \times q$ generic matrix with $p \le q$. Then there is a containment of sets

$$\mathrm{RS}(\mathbf{EN}(Y)) \subset \mathrm{RS}(\mathbf{EN}(M^{q-p+1,p})).$$

Let $n = pq$, so our matrix is a map $S^q \to S^p$ for $S = \Bbbk[y_{i,j}] \cong \Bbbk[x_1, \dots, x_n]$. This specializes to the matrix

$$M^{q-p+1,p} = \begin{bmatrix} x_1 & x_2 & \cdots & x_{q-p+1} & 0 & \cdots & 0 \\ 0 & x_1 & \cdots & x_{q-p} & x_{q-p+1} & \cdots & 0 \\ \vdots & & \ddots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & x_1 & \cdots & x_{q-p} & x_{q-p+1} \end{bmatrix}$$

under a map that we will call φ. The maximal minors of this matrix define the ideal $(x_1, \dots, x_{q-p+1})^p$.

The map φ gives us a map of complexes $\mathbf{EN}(Y) \to \mathbf{EN}(M^{q-p+1,p})$. What is more, under φ any subcomplex of $\mathbf{EN}(Y)$ gives a subcomplex of $\mathbf{EN}(M^{q-p+1,p})$.

This gives us a containment

$$\mathrm{RS}(\mathbf{EN}(Y)) \subset \mathrm{RS}(\mathbf{EN}(M^{q-p+1,p})). \tag{4.1}$$

Note that the generic nature of Y had no bearing on the argument in Example 4.2, so a more general statement relating general Eagon–Northcott complexes to the complex $\mathbf{EN}(M^{n,d})$ holds by the same reasoning.

THEOREM 4.3. *Let Z be a $p \times q$ matrix whose maximal minors define an ideal I whose codimension is $q-p+1$ and where S/I is Cohen–Macaulay (equivalently, I has grade $q-p+1$). Then there is a containment of sets*

$$\mathrm{RS}(\mathbf{EN}(Z)) \subset \mathrm{RS}(\mathbf{EN}(M^{q-p+1,p})).$$

Proof. With the hypotheses above, $\mathbf{EN}(Z)$ gives a minimal free resolution of S/I. Furthermore, the Artinian reduction of S/I is isomorphic to $S'/\mathfrak{m}^p$ for a polynomial ring $S' \cong \Bbbk[x_1, \dots, x_{q-p+1}]$. This specialization takes any subcomplex of $\mathbf{EN}(Z)$ to a subcomplex of $\mathbf{EN}(M^{q-p+1,p})$, so (4.1) holds for $\mathbf{EN}(Z)$ as it does in Example 4.2. ∎

While this theorem relates rank sequences of subcomplexes of the entire complexes $\mathbf{EN}(Z)$ and $\mathbf{EN}(M^{q-p+1,p})$ rather than their degree d strands, Remark 3.9 tells us that our restrictions on $\mathrm{RS}(L_{q-p+1,p})$, together with the above theorem, still give us valuable

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information about $\mathrm{RS}(\mathbf{EN}(Z))$. It should be noted, however, that the result in Theorem 4.3 is a strict containment, as demonstrated in the following example.

EXAMPLE 4.4. Let $S = \Bbbk[x,y,z,w]$. Consider the Eagon–Northcott complex on the matrix $\begin{bmatrix} 0 & y & z \\ y & z & 0 \end{bmatrix}$, which resolves the square of the maximal ideal in the subalgebra $\Bbbk[y,z] \subset S$. This is a specialization of the Eagon–Northcott complex on $\begin{bmatrix} x & y & z \\ y & z & w \end{bmatrix}$ obtained via the map $\varphi : S \to S$ defined by $x,w \mapsto 0$ and $y,z \mapsto y,z$, so we have the following map of complexes:

$$\begin{array}{ccccccc}
\mathbf{F}' : 0 \longrightarrow & S(-3)^2 & \xrightarrow{\left[\begin{smallmatrix} -z & -w \\ y & z \\ x & -y \end{smallmatrix}\right]} & S(-2)^3 & \xrightarrow{[-y^2+xz \ \ -yz+xw \ \ -z^2+yw]} & S^1 \\
& \Big\downarrow \varphi^* & & \Big\downarrow \varphi^* & & \Big\downarrow \varphi^* \\
\mathbf{F} : 0 \longrightarrow & S(-3)^2 & \xrightarrow[\left[\begin{smallmatrix} -z & 0 \\ y & z \\ 0 & -y \end{smallmatrix}\right]]{} & S(-2)^3 & \xrightarrow[{[-y^2 \ \ -yz \ \ -z^2]}]{} & S^1
\end{array}$$

Consider the following subcomplex of $\mathbf{F}$:

$$\mathbf{G} : 0 \longrightarrow S(-3) \xrightarrow{\left[\begin{smallmatrix} -z \\ y \end{smallmatrix}\right]} S(-2)^2 \xrightarrow{[-y^2 \ \ -yz]} S,$$

which has $\mathrm{rs}(\mathbf{G}) = (1,2,1)$. The subcomplex $\mathbf{G}$ is realized as the image of

$$\mathbf{G}' : 0 \longrightarrow S(-3) \xrightarrow{\left[\begin{smallmatrix} -z \\ y \end{smallmatrix}\right]} S(-2)^2 \xrightarrow{[-y^2+xz \ \ -yz+xw]} S$$

under φ^*. However, one can check that $\mathbf{G}'$ is not a subcomplex of $\mathbf{F}'$. Moreover, a straightforward linear algebra computation confirms that there is no subcomplex of $\mathbf{F}'$ with rank sequence $(1,2,1)$, so $\mathrm{RS}(\mathbf{F}) \subsetneq \mathrm{RS}(\mathbf{G})$.

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