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ARTICLE

Inflation-linked bonds in incomplete markets with sunspots[†]

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Abstract

This paper investigates the impact of the introduction of inflation-linked bonds, which incur transaction costs when traded, on the monetary economy with sunspots. This paper shows that there always exists a certain range of positive transaction costs, where both monetary and bond markets are active. This implies that on top of governments, profit-seeking financial entrepreneurs also have the incentive to issue these bonds. This paper also displays how financial innovation on the indexed bond can be Kaldor improving, even if not necessarily Pareto improving. This finding indicates that together with lump-sum tax policies, the government can attain consensus among consumers on bond issuance.

Keywords: Inflation-indexed bonds; sunspots; inflation volatility; transaction costs; Kaldor improving

1. Introduction

Market incompletenesswith sunspots provides an explanation for the excess volatility of inflation in rational-expectations models (Cass (1989, 1992)). With inflation uncertainty driven by sunspots, nominal securities are risky and unsafe. Conversely, real securities are more attractive to risk-averse consumers. Thus, introducing real securities to an economy with inflation volatility would cause the monetary market to be less active. This is theoretically demonstrated in various studies that show how introducing real securities or inflation-linked bonds can render the economy immune to sunspots, due to the resulting complete shutdown of nominal security and monetary markets (Mas-Colell (1992), Goenka and Préchac (2006)).[1] With the presence of riskless financial assets, there is no incentive for consumers to take any unnecessary risks by trading in nominal assets in an economy with sunspots. However, in a real economy, indexed bonds only serve as a minor supplement for money, as opposed to being the main financial instrument in monetary markets. This problem can be resolved by introducing transaction costs for intermediating indexed bonds.

The transaction cost of indexed bonds can be interpreted as having high liquidity cost compared to other types of financial assets, including money. Empirical evidence indicates that the liquidity premium of U.S. Treasury Inflation-Protected Securities (TIPS) is estimated to range from 0.4% to 2% from 1997 to 2008, a figure that is significantly higher than that of conventional securities (see D'Amico et al. (2018) and Pflueger and Viceira (2011)). Given the short history of the indexed bond market, associated secondary markets have not been sufficiently developed,

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resulting in higher operation and liquidity costs.[2] Therefore, considering liquidity cost in the trading of indexed bonds is a more natural approach than the assumption of frictionless financial markets.

Specifically, this paper assumes that the transaction cost proportionally increases with the trading volume of indexed bonds. In the general equilibrium model, it is common to assume that the trading cost is linearly increasing in volume, mainly because it ensures the existence of equilibrium. If the cost is not proportional (e.g., the marginal cost is increasing), the budget constraint is not linear, so the existence of equilibrium would not be guaranteed. In addition, it is common in empirical studies to calculate the liquidity premium based on the transaction cost data (e.g., bid-ask spread and bond yields). For instance, D'Amico et al. (2018) and Pflueger and Viceira (2011) estimate the liquidity premium based on the TIP yields in the secondary market, which implies that the transaction cost in the market is an important proxy in computing the liquidity premium. Finally, although there is common agreement that the transaction cost (or liquidity cost) is increasing in trading volume, there is no clear agreement on whether the marginal cost is decreasing, constant, or increasing in trading volume.

The model in this paper is constructed based on the incomplete-markets sunspot literature of Cass (1983) and Goenka and Prechac (2006) where a continuum of inflation volatility (or price-level volatility) is endogenously generated. Even with the introduction of indexed bonds, this paper shows that there would still be continuum of equilibrium inflation volatility with positive transaction costs. To compare two different continuums of equilibria, this paper defines the measure of volatility as the relative standard deviation of the price level.

This paper shows that the introduction of indexed bonds cannot induce a complete shutdown of the monetary market given that transaction costs are positive. This result contrasts with Mas-Colell (1992) and Goenka and Préchac (2006), who stated that the introduction of indexed bonds would result in a complete shutdown of the monetary market. The indexed-bond market could be inactive if the transaction costs are high enough or the inflation volatility level is sufficiently low. This paper also shows that there exists a certain range of proportional transaction costs that enable both the indexed bond and monetary markets to be active. This result implies a possibility where both governments and financial entrepreneurs are issuing these bonds. Since an economy with inflation volatility always has an arbitrage in the values of risk-free securities, financial entrepreneurs can also enter the market if they have the technology to issue indexed bonds at a small cost. When the economy has a higher level of inflation volatility, there will be a larger gap in the lenders' and borrowers' valuations on risk-free assets. Thus, the indexed bonds market could provide more attractive financial products for profit-seeking financial entrepreneurs.

In light of the positive effects, it is important to note that the introduction of indexed bonds can lead to decreased welfare for borrowers.[3] On the other hand, Viard (1993) shows that the lender's and borrower's welfare together can be better off with the introduction of indexed bonds. The finding is in contrast to the result in this paper showing that some consumers can be even worse off with indexed bonds. This difference is due to Viard's (1993) use of the Taylor approximation, in which the third and higher derivatives of the utility function are ignored in the welfare analysis. In this case, the expected utility becomes equivalent to the mean-variance utility, which guarantees Pareto improvement through financial innovation on indexed bonds. However, this paper uses general utility functions and shows that the introduction of indexed bonds in a sunspot economy does not necessarily result in Pareto improvement. The introduction of a risk-free asset decreases consumption uncertainty, and thus reduces precautionary saving motives. Consequently, it lowers the demand for money and thus lowers the value of money. This devaluation has a negative impact on borrowers' utility as they must borrow money at the increased interest rate. This situation could be a matter of concern for governments that need to gain consensus on the introduction of indexed bonds, but it may not apply to profit-seeking financial entrepreneurs.

As the introduction of indexed bonds does not necessarily lead to Pareto improvement for the economy, this paper considers a compensation test based on lump-sum tax-transfer plans that are denominated in period-0 good. Specifically, it demonstrates that there exist balanced lump-sum tax-transfer plans that would enable the equilibrium allocation with indexed bonds to be Pareto superior to those without indexed bonds. This indicates that the financial innovation on indexed bonds is Kaldor improving and thus the government can gain consensus on the introduction of indexed bonds as it uses a tax-transfer policy together with financial innovation.

The outline of this paper is as follows. Section 2 introduces the economy with both indexed bonds and money markets. The existence of equilibria and the activeness of the monetary market are shown in Section 3. The impact of inflation volatility on the activeness of the indexed bond market is investigated in Section 4. A numerical example is shown in Section 5. Section 6 discusses the welfare implications of financial innovation on indexed bonds. Section 7 is a brief conclusion. Proofs are shown in the Appendix.

2. The model

There are two periods, today and tomorrow, labeled as subscripts $t = 0, 1$. At date 1, there are two states, $s = \alpha, \beta$ with positive probabilities $0 < \pi_\alpha < 1$ and $\pi_\beta = 1 - \pi_\alpha$, respectively. There are two consumers, labeled as superscripts $i \in I = \{B, S\}$ where B and S represent the asset buyer and the seller, respectively. Consumer i's consumption allocation is $x^i = (x_0^i, x_\alpha^i, x_\beta^i) \in X = \mathbb{R}^3_{++}$ corresponding to prices $p = (p_0, p_\alpha, p_\beta) \gg 0$. His endowment is $e^i = (e_0^i, e_\alpha^i, e_\beta^i) \in X$ where $e_\alpha^i = e_\beta^i = e_1^i$. Consumer i's utility functions, $v^i(x^i)$, are twice differentiable, strictly increasing, strictly concave, having the closure of indifference curves contained in $\mathbb{R}^2_{++}$ and satisfying the von Neumann–Morgenstern expected utility hypothesis. Consumer i's preferences are

$$u^i(x^i) = u^i\left(x_0^i, x_\alpha^i, x_\beta^i\right) = \pi_\alpha v^i\left(x_0^i, x_\alpha^i\right) + \pi_\beta v^i\left(x_0^i, x_\beta^i\right).$$

Throughout this paper, we assume that there is an incentive to trade:

$$\frac{\partial v^B\left(e_0^B, e_1^B\right)}{\partial x_0} \Big/ \frac{\partial v^B\left(e_0^B, e_1^B\right)}{\partial x_1} < \frac{\partial v^S\left(e_0^S, e_1^S\right)}{\partial x_0} \Big/ \frac{\partial v^S\left(e_0^S, e_1^S\right)}{\partial x_1}.$$

This condition implies that the initial endowment is not Pareto efficient. Since money can be considered a risky asset with sunspots, a lender and a borrower can be translated into an asset buyer and an asset seller.

In a monetary market, there is only one financial instrument (money) where m^i denotes consumer i's money holdings. Equilibrium in the monetary market is defined as follows: There are some positive spot prices $(p_0, p_\alpha, p_\beta) \gg 0$ and associated money holdings m such that each household solves the following maximization problem:

$$\max \quad u^i\left(x_0^i, x_\alpha^i, x_\beta^i\right)$$

$$\text{subject to} \quad \begin{cases} p_0 x_0^i + m^i \le p_0 e_0^i \\ p_\alpha x_\alpha^i \le p_\alpha e_1^i + m^i \\ p_\beta x_\beta^i \le p_\beta e_1^i + m^i \end{cases}$$

$$\text{and} \quad x^i \in X,$$

and each market clears:

$$\sum_i x_0^i = \sum_i e_0^i, \quad \sum_i x_\alpha^i = \sum_i x_\beta^i = \sum_i e_1^i$$

$$\text{and} \quad \sum_i m^i = 0.$$

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We now introduce inflation-indexed bonds with a relative transaction cost θ. With the existence of transaction costs, the selling price of the bond is not the same as the purchasing price. The cost is incurred at date 1, implying that the intermediaries exchange the transaction fees with the first period goods. Therefore, the market clearing condition for the first period is dependent on the amount of the indexed bonds traded in the market. We label a monetary market with indexed bonds as IB. The equilibrium in IB is defined as

Given prices $(p_0, p_\alpha, p_\beta, p)$, $(m^i, n^i, x_0^i, x_\alpha^i, x_\beta^i)_{h=1}^H$ solve the following maximization problem (P-IB)

$$\max u^i\left(x_0^i, x_\alpha^i, x_\beta^i\right)$$

subject to

$$p_0 x_0^i + m^i + p\,(1+\theta)\max(n^i, 0) + p\min(n^i, 0) \le p_0 e_0^i, \tag{1}$$

$$p_\alpha x_\alpha^i \le p_\alpha e_1^i + p_\alpha n^i + m^i, \tag{2}$$

$$p_\beta x_\beta^i \le p_\beta e_1^i + p_\beta n^i + m^i. \tag{3}$$

Market clear conditions are

$$\sum\nolimits_i m^i = 0, \sum\nolimits_i n^i = 0,$$

$$\sum\nolimits_i x_0^i + \frac{p}{p_0}\theta \sum\nolimits_i \max(n^i, 0) = \sum\nolimits_i e_0^i,$$

$$\text{and } \sum\nolimits_i x_\alpha^i = \sum\nolimits_i x_\beta^i = \sum\nolimits_i e_1^i,$$

where n^i represents the amount of indexed-bond holdings of consumer i. $p\,(1+\theta)$ and p in equation (1) represent the purchasing and selling prices of the indexed bonds, respectively. The nominal payoffs of the indexed bonds are p_α and p_β in state α and β, respectively, as shown in equations (2)–(3).

The budget constraint set with indexed bonds in equations (1)–(3) is still convex even with the transaction cost, which is necessary for proof of the existence of single-valued demand functions. In the next section, this paper investigates the existence and regularity of the equilibria in a economy with indexed bonds.

3. Equilibrium

The degree of inflation volatility in this paper is defined as a relative standard deviation of inflation:

$$\sigma^R = \frac{\sigma(\widetilde{p}_1/p_0)}{E(\widetilde{p}_1/p_0)}, \tag{4}$$

where $\widetilde{p}_1$ is the random variable whose values are p_α with probabilities π_α and p_β with probability π_β, respectively. $\sigma(\widetilde{p}_1/p_0)$ and $E(\widetilde{p}_1/p_0)$ are the standard deviation and the expected value of the level of inflation, respectively.[4] Kang (2015) indicates that generically in endowment, for any given $\sigma^R \ge 0$, there exists a finite number of equilibria in a monetary economy with sunspots. This section shows that with the addition of the indexed bonds, generically in endowments, a finite number of equilibria do still exist for any given σ^R.

As fixing the degree of inflation volatility level (σ^R) as constant, the real returns of money can be derived. The following equation shows that there is a one-to-one relationship between σ^R and

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price ratio $\mathcal{P}$ $(=p_\beta/p_\alpha)$:

$$\sigma^R = \frac{\sigma(\widetilde{p}_1/p_0)}{E(\widetilde{p}_1/p_0)} = \frac{\sqrt{\pi_\alpha\pi_\beta}\,|1-\mathcal{P}|}{\pi_\alpha+\pi_\beta\mathcal{P}}.$$

For a given price ratio $\mathcal{P}$, the ratio of the two states' real returns of money is determined by the following relationship:

$$\mathcal{P} = \frac{p_\beta}{p_\alpha} = \frac{1/p_\alpha}{1/p_\beta}$$

where $(\frac{1}{p_\alpha}, \frac{1}{p_\beta})$ is the real return of money in states α and β.

Given price ratio $\mathcal{P}$, it is possible to interpret nominal security (money) as a security with a real return (R_α, R_β) satisfying that $\mathcal{P} = R_\alpha/R_\beta$. Then, the equivalent economy with two types of real securities can be defined.[5] The equivalent maximization problem based on two types of real securities is defined as (E-IB):

$$\max\ u^i\left(e_0^i + z_0^i, e_1^i + n^i + R_\alpha b^i, e_1^i + n^i + R_\beta b^i\right) \tag{5}$$

$$\text{subject to}\quad q_0 z_0^i + q b^i + p\,(1+\theta)\max(n^i, 0) + p\min(n^i, 0) \le 0 \tag{6}$$

$$\text{where}\quad \frac{R_\beta}{R_\alpha} = \frac{1}{\mathcal{P}},$$

$$e_0^i + z_0^i = x_0^i,\ \ e_1^i + n^i + R_\alpha b^i = x_\alpha^i,\ e_1^i + n^i + R_\beta b^i = \ x_\beta^i,\ b^i = m^i/q,$$

and b^i represents the amount of risky-asset holdings of consumer i. z_0^i represents the excess demand for period-0 goods. q and q_0 represent the price of the real security and the price of the period-0 good, respectively.

The following lemma indicates that the maximization problem with risk assets (b^i) is equivalent to that with money (m^i):

Lemma 1 *Given $\mathcal{P} = p_\beta/p_\alpha$, P-IB is equivalent to E-IB where*

$$x_0^i = e_0^i + z_0^i, \qquad x_\alpha^i = e_1^i + n^i + R_\alpha b^i, \quad x_\beta^i = e_1^i + n^i + R_\beta b^i,$$
$$p_0 = q_0\,(=1)\,, \quad p_\alpha = \frac{q}{R_\alpha}, \quad \textit{and} \quad p_\beta = \frac{q}{R_\beta},$$

Proof.

(i) (P-IB⇒E-IB) P-IB is defined with the variables $\left(p_0, p_\alpha, p_\beta, p, \mathcal{P}\right)$ while the variables are $(q_0, q, R_\alpha, R_\beta, p, \mathcal{P})$ in E-IB. Defining p_0 and m^i as

$$p_0 \equiv q_0 \ \text{ and } \ \ m^i \equiv q b^i,$$

it is clear that the budget constraint in equation (6) in E-IB can be derived from that in P-IB. Next, defining p_α and p_β as

$$p_\alpha \equiv \frac{q}{R_\alpha},\ \ p_\beta \equiv \frac{q}{R_\beta},$$

x_α^i and x_β^i can be expressed as

$$x_\alpha^i \le e_1^i + n^i + R_\alpha b^i \text{ and } x_\beta^i \le e_1^i + n^i + R_\beta b^i.$$

Because utility functions are strictly increasing, $\le$ can be replaced with $=$, by the equation $x_0^i = e_0^i + z_0^i$, the utility function (equation (5)) in E-IB is the same as the one in P-IB.

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(ii) (E-IB⇒P-IB) defining q_0 and b^i as

$$q_0 \equiv p_0 \ \text{ and } \ b^i \equiv \frac{m^i}{q},$$

the period-0 budget constraint in equation (1) in P-IB is equivalent to that in equation (6) of E-IB. Defining x^i_α and x^i_β as

$$x^i_\alpha \equiv e^i_1 + n^i + R_\alpha b^i \text{ and } x^i_\beta \equiv e^i_1 + n^i + R_\beta b^i,$$

x^i_α and x^i_β can be expressed as

$$p_\alpha x^i_\alpha \leq p_\alpha e^i_1 + p_\alpha n^i + m^i$$
$$p_\beta x^i_\beta \leq p_\beta e^i_1 + p_\beta n^i + m^i,$$

where

$$R_\alpha = \frac{q}{p_\alpha}, \ \ R_\beta = \frac{q}{p_\beta},$$

which are the same as equations (2) and (3), respectively. Because $x^i_0 \equiv e^i_0 + z^i_0$, the utility function in P-IB is equivalent to that in E-IB. □

With competitive equilibrium, the mean of the real return (R_α, R_β) does not affect the equilibrium allocations since the value of q is adjusted according to the value of the mean. For the convenience of proofs and computation, we assume that the mean value of the real return is fixed as one, that is, $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$. So far, the analyses are based on the equivalent maximization problem (E-IB).

Lemma 2 *Generically in endowments, there exists a finite number of equilibria in the economy with indexed bonds for any given* $\left(\sigma^R, \theta\right) \geq 0$.

Proof. In the equivalent maximization problem, the budget sets are convex and the utility function is strictly concave for all $(q_0, R_\alpha, R_\beta, q, \mathcal{P}, p) \gg 0$. Therefore, the excess demand is single valued. (The proof can be done by defining a compact subset of the budget set.)

For the proof of a regular and determinate economy, we need to define the aggregate excess demand and check if it satisfies sufficient conditions for the existence of regular economies. The market aggregate demand function is defined as

$$Z(Q) = \sum\nolimits_i \begin{pmatrix} z^i_0(Q) + \theta \frac{p}{q_0} \max\left(n^i\left(Q\right), 0\right) \\ b^i\left(Q\right) \\ n^i\left(Q\right) \end{pmatrix},$$

$$\text{where } Q = \left(q_0, q, p\right).$$

This market aggregate demand function is not the same as a conventional one because of the additional term $\theta \frac{p}{p_0} \max\left(n^i(Q), 0\right)$ which is the amount of first-period goods the intermediaries charge as transaction fees.

We can show that $Z\left(Q\right)$ satisfies the five properties:

(i) continuous.

(ii) homogeneous of degree zero.

(iii) (Walras' law) $Q \cdot Z(Q) = 0$.

(iv) There is an $s > 0$ such that $Z(Q) > (-s, -s, -s)$ for all P.

(v) If $Q^n \to Q$, where $Q \neq 0$ and $p = 0$, then $\sum_i n^i(Q) \to \infty$.

Therefore, the economy is regular and determinate. □

The existence of equilibria does not necessarily imply that either (both) an indexed bond market or (and) a monetary market are active. Thus, this paper proves that the monetary market is always active even with the introduction of indexed bonds if the transaction cost is not zero as shown in the following proposition:

Proposition 1 *The monetary market is active, that is, $m^i \neq 0$ for all $i \in I$ if $\theta > 0$.*[6]

Proof. (by contradiction) Let's assume that $m^i = 0$ for $i \in \{S, B\}$, labeling the asset buyer ($m^i > 0$) as $i = B$ and the asset seller ($m^i < 0$) as $i = S$. Assuming that $n^B = n^S = 0$, the monetary market should be active since the initial endowment is not Pareto efficient. Next, we need to check the case where $n^B \neq 0$ and $n^S \neq 0$. For this case, we can compute each individual's value of the risky-asset (money). The value is the ratio between the marginal utility of the risk asset and that of the period-0 good. Let us assume that the risky asset's return $\left(R_\alpha, R_\beta\right)$ satisfies that $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$. Then, the buyer's value of the risky asset is computed as

$$
\begin{aligned}
q_R^B &= \frac{\pi_\alpha \frac{\partial v^B(x_0^B, e_1^B + R_\alpha b^B)}{\partial b^B} + \pi_\beta R_\beta \frac{\partial v^B(x_0^B, e_1^B + R_\beta b^B)}{\partial b^B}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_0}} \\
&= \frac{\pi_\alpha R_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_0}}
\end{aligned} \tag{7}
$$

Since $m^B = 0$ (i.e., $b^B = 0$), we have $x_\alpha^B = x_\beta^B$. Also, it is true that $\pi_\alpha R_\alpha + \pi_\beta R_\beta = \pi_\alpha + \pi_\beta$ because we assume $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$. Therefore, equation (7) can be expressed as

$$
q_R^B = \frac{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_0} + \pi_\beta \frac{\partial v^B\left(x_0^i, x_\beta^i\right)}{\partial x_0}}. \tag{8}
$$

The right-hand side of equation (8) is the same as the purchasing price of the indexed bond. Thus, we have

$$
q_R^B = p(1 + \theta). \tag{9}
$$

In the same way, we can show that the asset seller's value of a risky-asset is the same as the selling price of the indexed bond:

$$
q_R^S = p. \tag{10}
$$

Because we know $q_R^B > q_R^S$ from equations (9)–(10), there is a price $q \in \left(q_R^S, q_R^B\right)$ in which the two consumers have an incentive to trade with money. This contradicts that $m^i = 0$ for $i = S, B$. □

Proposition 1 shows that the market for money is still operating at any level of inflation volatility if $\theta > 0$. However, without a transaction cost, that is, $\theta = 0$, the existence of inflation volatility makes the market for money completely shut down, as shown in Goenka and Préchac (2006).

4. Indexed bonds and inflation volatility

This section analyzes how transaction cost and inflation volatility, respectively, affect the indexed bond market. With the transaction cost, the budget set is not smoothly defined so it is difficult to directly analyze how the market is affected by the transaction cost. Therefore, this section first investigates the economy without the indexed bond. Even if inflation-indexed bonds are not issued in a monetary market, the lender's and borrower's values of those securities can be computed at equilibrium. If the gap between the buyer's and seller's values for indexed bonds in the pure monetary economy is greater than the transaction cost, the consumer would have an incentive to trade on indexed bonds when the bond is introduced. If the gap is not large enough, the indexed bond market will not be active. In this case, the equilibrium allocation in a pure monetary market would be identical to that with indexed bonds.

In a market with indexed bonds, let us define p_F^i as the value of a risk-free asset for consumer $i \in \{B, S\}$:

$$p_F^i = \frac{\pi_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_1} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_0} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_0}}, \tag{11}$$

which is the ratio between the marginal utility of risk-free assets and that of the period-0 good. The following lemma shows that there is an arbitrage between two values, p_F^B and p_F^S, where "B" and "S" represent the asset buyer and the asset seller, respectively.

Lemma 3 *If $\sigma^R > 0$ in a monetary market without the indexed bond, the asset buyer's value of the risk-free asset is greater than the asset seller's, that is,*[7]

$$p_F^S < p_F^B.$$

Proof. q^+ represents the equilibrium price of the risky asset in a market with the indexed bond:

$$q^+ = \frac{\pi_\alpha R_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_1}}{\pi_\alpha \frac{\partial v^i(x_0^i, x_\alpha^i)}{\partial x_0} + \pi_\beta \frac{\partial v^i\left(x_0^i, x_\beta^i\right)}{\partial x_0}}, \quad \text{where} \quad \begin{pmatrix} \pi_\alpha R_\alpha + \pi_\beta R_\beta = 1 \\ \mathcal{P} = R_\alpha / R_\beta \end{pmatrix} \tag{12}$$

Let us label the asset buyer ($b^i > 0$) as $i = B$ and the asset seller ($b^i < 0$) as $i = S$. Since the initial endowment allocation is not Pareto efficient, they will trade and therefore $b^B > 0$ and $b^S < 0$.

First, we want to show that p_F^B is higher than q^+. From equations (11) and (12), p_F^B / q^+ is

$$\frac{p_F^B}{q^+} = \frac{\pi_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}{\pi_\alpha R_\alpha \frac{\partial v^B(x_0^B, x_\alpha^B)}{\partial x_1} + \pi_\beta R_\beta \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}}. \tag{13}$$

Since the state β is inflationary (i.e., $\mathcal{P} > 1$), we have $R_\alpha > R_\beta$ and $x_\alpha^B > x_\beta^B$. Because v^i is strictly concave and $x_\alpha^B > x_\beta^B$, we have

$$\frac{\partial v^B\left(x_0^B, x_\alpha^B\right)}{\partial x_1} < \frac{\partial v^B\left(x_0^B, x_\beta^B\right)}{\partial x_1}. \tag{14}$$

Since $\pi_\alpha R_\alpha + \pi_\beta R_\beta = 1$, $\pi_\alpha + \pi_\beta = 1$ and $R_\alpha > R_\beta$, we have $\pi_\alpha R_\alpha > \pi_\alpha$, $\pi_\beta R_\beta < \pi_\beta$. Thus, from equations (13)–(14) we have:

$$\frac{p_F^B}{q^+} > 1. \tag{15}$$

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In the same way, we can prove the following:

$$\frac{p_F^S}{q^+} < 1. \tag{16}$$

From the two inequalities, we know that $p_F^S < p_F^B$. □

The arbitrage in the values of the risk-free securities between two consumers provides the incentive for them to accept the real securities as the financial instrument. However, the existence of the arbitrage does not necessarily imply that the indexed bond market is active. The magnitude of the arbitrage should be large enough, compared to the transaction cost, for the indexed bonds to be traded in the financial market. The following propositions show that the indexed bond markets can be active or inactive depending on the level of inflation volatility and the level of the transaction cost.

Proposition 2 *If $\sigma^R > 0$, there exists $\overline{\theta} > 0$ such that the indexed bond market is active (inactive) if $\theta < \overline{\theta}$ $\left(\theta \geq \overline{\theta}\right)$ where*

$$\overline{\theta} = \frac{p_F^B - p_F^S}{p_F^S}. \tag{17}$$

Proof.

(i) *For $n^i \neq 0$ where $\theta < \overline{\theta}$*: (Proof by contradiction) Let us assume that $n^i = 0$ for $i = S, B$. Then, the equilibrium allocations in the market with indexed bonds are the same as those in a pure monetary market. If $\theta < \overline{\theta}$, there exists $p > 0$ such that $p_F^S < p < p\,(1+\theta) < p_F^B$. (This can be shown as follows: Let $p = p_F^S + \varepsilon$ and $\theta = \left(p_F^B - p_F^S - \delta\right)/p_F^S > 0$. Then, for all $\delta \in \left(0, p_F^B - p_F^S\right)$, there exists ε, which satisfies the inequality $p_F^S \leq p < p\,(1+\theta) \leq p_F^B$). This implies that the two consumers have an incentive to trade with the indexed bonds. This contradicts that $n^i = 0$.

(ii) *For $n^i = 0$ where $\theta \geq \overline{\theta}$*: (Proof by contradiction) Let us assume that $n^i \neq 0$ for $i = S, B$. Assuming that $\theta \geq \overline{\theta}$, there does not exist $p > 0$ such that $p_F^S < p < p\,(1+\theta) < p_F^B$. This contradicts that $n^i \neq 0$ for $i = S, B$. □

The next question is how inflation volatility affects trading on the indexed bond. In the model, the decision about whether financial entrepreneurs should enter the market (or whether the government should issue indexed bonds) depends on the value of $\overline{\theta}$. For a smaller value of $\overline{\theta}$, there would be a smaller chance for financial entrepreneurs to survive in the market. The following proposition shows that $\overline{\theta}$ increases with the increased volatility of inflation:

Proposition 3 *If v^i is additively separable, that is, $v^i(x_0^i, x_1^i) = f^i(x_0^i) + g^i(x_1^i)$, $\overline{\theta}$ is strictly increasing in σ^R.*

Proof. See Appendix for the proof. □

Proposition 3 is the extended version of Lemma 3 indicating that $\overline{\theta}$ is strictly positive where $\sigma^R > 0$ and equals zero where $\sigma^R = 0$. The proof for the nonseparable utility functions remains open. Proposition 3 implies that as the economy has higher inflation volatility, financial entrepreneurs have a better chance of making a profit and, consequently, stronger incentive to enter the indexed bond market.

The proof in Proposition 3 assumes that the utility function is concave in x_1 regardless of the change in x_0 with increased volatility. With this assumption, this paper shows that as the market is exposed to higher inflation volatility (i.e., as $\mathcal{P}$ increases), the gap between the buyer's and seller's values of indexed bonds is increasing. With additively separable utility functions, the change of

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Table 1. Equilibrium outcomes in an economy without (and with) indexed bonds

	Without indexed bonds		With indexed bonds	
	Buyer	Seller	Buyer	Seller
Utility level	3.07	3.26	3.19	3.18
Money (m^i)	0.5	−0.5	0.212	−0.212
Bond (n^i)	–		0.276	−0.276
(p_α, p_β)	(0.820, 1.640)		(0.744, 1.488)	
$(p, p(1+\theta))$	–		(0.948, 1.043)	
q	1.093		0.992	

x_1 with increased volatility does not affect the concavity of the utility function, so we can easily show that the expected utility is concave in $\mathcal{P}$. However, with nonseparable utility functions, the utility function is not necessarily concave in x_1 with the change of $\mathcal{P}$.[8] Thus, this paper rules out the nonseparable utility-function cases in Proposition 3.

5. An example

This section presents a numerical example to help explain the main results in this paper. Assume that there are two consumers, an asset buyer (B) and asset seller (S) with endowments of (10,0) and (0,10), respectively. Their expected utility functions are the same as $\log(x_0) + \log(x_1)$. State probabilities are $\pi_\alpha = \pi_\beta = 0.5$. Assuming that the inflation level $\mathcal{P} = 2$, the corresponding price-level (inflation) volatility is $\sigma^R = 33.3\%$ by equation (4). In this case, it is computed that $\overline{\theta} = 23\%$ by equation (17). Assuming that the transaction cost is 10% ($\theta = 0.1$) in the market with indexed bond, the equilibrium outcomes are summarized in Table 1.

The price levels in the monetary market are higher than those in the market with the indexed bonds. Higher price levels imply a higher value of money in period 0. As explained in Section 4, the higher value of money is attributed to the asset buyer's higher demand for money. Consequently, the value of money in the economy with the indexed bonds is not as high as that in a pure monetary market. The change in money value after the introduction of indexed bonds causes the asset seller's utility to decrease from 3.26 to 3.18. (See Table 1.)

Figure 1 represents the equilibrium allocations in the space of excess demand. The figure should be three dimensional including the excess demand of x_0, but we can imagine the original three-dimensional figure is projected onto a two-dimensional figure. The buyer's allocation is located in the northeast quadrant, while the seller's allocation is in the southwest quadrant. The allocations are symmetric to (0,0) by market clearing conditions. "+" and "o" represent the equilibrium allocations in pure monetary and indexed-bond markets, respectively. From the figure, we know that the equilibrium allocations move closer to the 45 degree line in the market with indexed bonds although both markets have the same volatility level.

Figure 2 shows how the two consumers' utility changes in transaction costs. The asset seller's utility is increasing in transaction cost θ if $\theta > 10\%$. Higher transaction costs make the demand for money increase by the substitution effects. Therefore, the price of money goes up and, consequently, the asset buyer would have higher financial income from selling money (risky asset) at a higher price. If the transaction cost is higher than 23%, the indexed bond market would be inactive. In that case, the equilibrium allocations in the market with indexed bonds are identical to those in a pure monetary economy.

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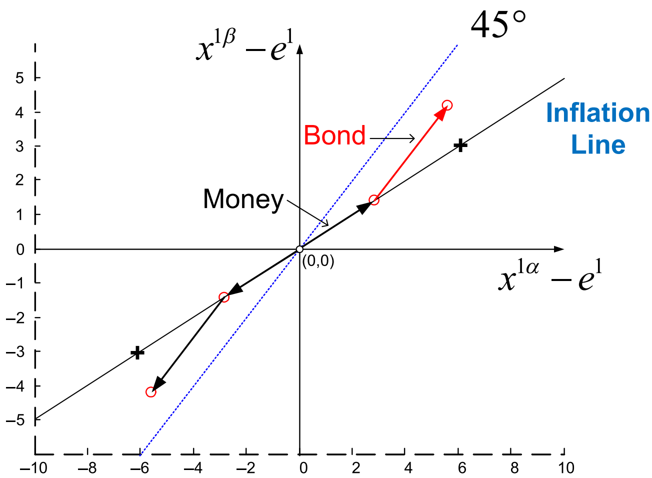

Figure 1. Equilibrium allocations in the space of excess demand.

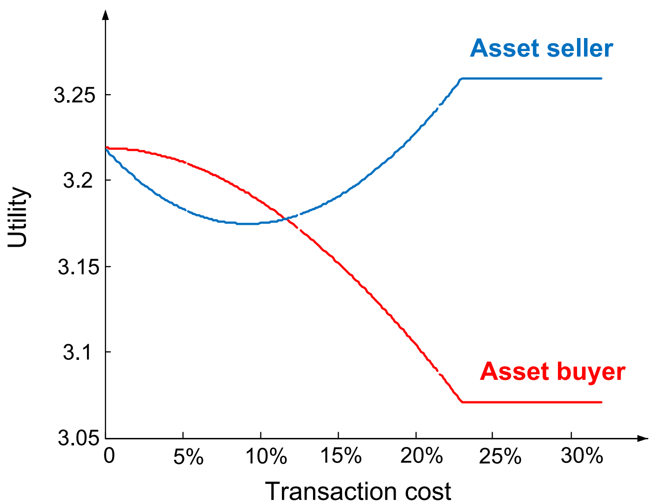

Figure 2. Consumers' utility changes in transaction costs.

6. Welfare analysis

The indexed bond market gives consumers more trading choices than a pure monetary market. Therefore, it can be expected that equilibrium with indexed bonds is Pareto superior to that without the bond. However, it has been shown that financial innovations to incomplete markets do not necessarily induce the economy to be Pareto improving (Elul (1995) and Cass and Citanna (1998)). Financial innovations are known to affect market prices under a general equilibrium setting. These price changes affect agents' real wealth and, consequently, their utility values. Specifically, the introduction of indexed bonds causes the value of money to decrease due to a precautionary savings effect, which has a negative impact on asset borrowers' wealth.

The example in Section 5 is the case where the introduction of indexed bonds make some consumers worse off. As the level of inflation volatility increases, the second consumer's (i.e., borrower) utility value can increase, as shown in Section 5. The intuition is as follows. With a high level of inflation volatility, the real return of assets becomes small in an inflationary state. If the

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inflationary state is realized, the first consumer (i.e., saver) will end up getting a small amount of the good at date 1 and, consequently, his utility will be particularly low. Therefore, the first consumer's demand for the asset will be higher to insure against the inflationary state, due to the precautionary savings effect. The high demand for savings results in a high value of money. Finally, the borrower obtains more income as the asset price becomes higher in a monetary market.

However, the introduction of new riskless securities could be a substitute for money and cause a decrease in money demand. Therefore, the value of money when indexed bond are available is smaller than that in a pure monetary market. A decrease in the money value negatively affects the borrowers's utility level. In this example, even though the asset seller has more trading choices with the indexed bond, his utility level is actually lower than that in a pure monetary market. (See the Appendix for more details about the example.)

Although some of consumers may be negatively affected, it is not possible for all of them to be worse off with the introduction of indexed bonds according to the revealed preferences hypothesis. The following proposition shows this.

Proposition 4 *For any given $\overline{\sigma}^R > 0$, the introduction of indexed bonds in the economy with sunspots (i) would always improve at least one consumers' welfare but (ii) can decrease some consumers' welfare.*

Proof.

(i) When the value of money, which is q in the equivalent maximization problem, is not the same in both markets, there must be at least one consumer who will benefit from the change in value. The consumer should be the asset buyer (seller) if the value decreases (increases). Since the consumer has more choices with the indexed bond, according to the revealed preference argument, the consumer must be better off.

(ii) See the example in Section 5. □

The introduction of the indexed bonds can be considered a sunspot-stabilizing policy in the sense that it makes the equilibrium allocations less volatile.[9,10] However, Proposition 4 indicates that the introduction of indexed bonds does not necessarily make all agents better off. This implies that the government can fail to gain consensus in adopting the financial innovation on indexed bonds. This section shows that in Proposition 5, if balanced lump-sum tax plans are allowed along with the financial innovation, consumer consensus on the policy can be achieved. Although there is no clear Pareto ranking between equilibrium allocation with and without the indexed bonds as shown in Proposition 4, this section shows that equilibrium with indexed bonds is Kaldor superior to that without the bond by considering balanced lump-sum tax plans. The space of the balanced lump-sum tax plan is defined as

$$T = \left\{(\tau^1, \tau^2) \in \mathbb{R}^2 | \tau^1 + \tau^2 = 0\right\}.$$

The tax-transfer plan (τ^1, τ^2) is applied to the market with indexed bonds. Then, the budget constraints at date 0 are modified to be

$$s.t \begin{cases} p_0 x_0^i + m^i + p\,(1+\theta)\max(n^i, 0) + p\min(n^i, 0) \le p_0 e_0^i - p_0 \tau^i \\ p_\alpha x_\alpha^i \le p_\alpha e_1^i + p_\alpha n^i + m^i \\ p_\beta x_\beta^i \le p_\beta e_1^i + p_\beta n^i + m^i \end{cases} \tag{P-CM}$$

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We define Kaldor improving as follows. The financial innovation on indexed bonds is Kaldor improving if there exist(s) $(\tau^1, \tau^2) \in T$ such that the equilibrium allocations in the economy with the indexed bonds is Pareto superior to those without indexed bonds. The following proposition shows that the market with indexed bonds is superior to a pure monetary market based on the compensation test:

Proposition 5 *Financial innovation on indexed bonds is Kaldor improving.*

Proof. See the Appendix. □

7. Concluding remarks

Government inflation-linked bonds have become available in a number of countries including the USA and UK, and they have attracted enormous research interest. However, most research papers are written based on partial-equilibrium standards. In partial equilibrium, it is trivial that the agents facing inflation uncertainty can increase welfare by trading in indexed bond markets. However, in general equilibrium, the introduction of indexed bonds can change other assets prices so it is not easy to answer whether (1) both lenders and borrowers are willing to trade in the indexed bonds market and (2) introduction of indexed bonds improves welfare. This paper answers both questions by incorporating transaction costs in trading on the indexed bonds with sunspots.[11]

This paper shows that inflation-linked bonds can be more actively traded in an economy with higher inflation volatility. In the same way, we show that the uncertainty of a nominal interest rate can provide room for inflation-linked bonds. Using a two-period general equilibrium model, it may be possible to show that monetary authorities' imperfect process of controlling the money supply can cause the economy to prefer indexed bonds. While inflation volatility is based on market psychology, volatility in the nominal interest rate is caused by an inconsistent monetary policy.

This paper shows that financial innovation in indexed bonds changes all price levels in a continuum of equilibria. Specifically, the introduction of indexed bonds endogenously changes the continuum of equilibrium price levels, which makes comparative statics difficult to conduct. Thus, this paper fixes the volatility level ($\mathcal{P}$) so it can be interpreted as an "exogenous" variable. It is beyond the scope of this paper to show how the exogenous variable ($\mathcal{P}$) is made endogenously, which is also nontrivial issue in the literature of incomplete markets.

Recently, a significant number of inflation-linked bonds have been issued by governments and private financial institutions. These bonds are taking up a more important position as widely accepted financial instruments. The findings of this study provide a better understanding of these bonds from a theoretical point of view.

Notes

1 Mas-Colell (1992) indicates that financial markets can be immune to sunspots by introducing as many real securities as the number of goods in each state. In addition, Goenka and Préchac (2006) show that the introduction of inflation-indexed bonds completely eliminates sunspot effects in incomplete markets. The introduction of these real securities results in a complete shutdown of nominal financial markets.

2 Specifically, the U.S. Treasury Inflation-Protected Securities (TIPS) were introduced only in 1997. For discussion on U.S. inflation-linked bond markets and the high liquidity cost, see Dudley et al. (2009), Christensen et al. (2010), Gükaynak et al. (2010), Fleming and Krishnan (2012), and Haubrich et al. (2012).

3 This can be explained by the substitution effect and precautionary motive shown in Goenka and Prechac (2006), Kajii (2006, 2009) and Kang (2019) in the model with zero transaction cost.

4 Price-level volatility and inflation volatility based on the relative standard deviation have the same value in a single-good economy. In a multi-good economy, the two volatility measures also have the same value if inflation is defined by the consumer price index (CPI)

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5 One is the equivalent real security to money and the other is the inflation-indexed bonds. In this paper, I call them a risky asset and a risk-free asset, respectively.
6 It is proven that where $\theta = 0$, money is not traded in the market and the sunspot effects will disappear. (See Goenka and Préchac0 (2006).)
7 An economy without inflation volatility ($\sigma^R = 0$) is the same as a certainty economy. Therefore, where $\sigma^R = 0$ there is no arbitrage, that is, $p_F^S = p_F^B$.
8 For example, assume that the utility function is $v(x_0, x_1) = \sqrt{x_0, x_1}$ and x_0 also changes when P changes. Denoting $\left(x_0(\mathcal{P}), x_\alpha(\mathcal{P}), x_\beta(\mathcal{P})\right)$ by the equilibrium consumption given $\mathcal{P}$, if we have $x_0(\mathcal{P}) = \mathcal{P}$ and $x_\alpha(\mathcal{P}) = \mathcal{P}$ in equilibrium, the function is no longer concave in inflation volatility ($\mathcal{P}$) in state α. This may be an extreme example that would not happen in equilibrium, but it could explain why it is difficult to prove this proposition with non-separable utility function.
9 More accurately, the relative standard deviation of the excess demand $\left\{x_\alpha - e_1, x_\beta - e_1; \pi_\alpha, \pi_\beta\right\}$ is lower with the introduction of the indexed bonds.
10 The financial innovation in this paper is different from conventional sunspot-stabilizing policies because it cannot completely eliminate the consumption volatility unless the transaction cost is zero. Several studies have suggested stabilizing policies to completely eliminate the effects of sunspots on incomplete markets. Three dominant policies have been shown: (1) the introduction of new types of nominal securities. (see Cass and Shell (1983, Proposition 3)); (2) the introduction of as many real securities as the number of goods in each state (see Mas-Colell (1992)); and (3) the introduction of options. (see Antinolfi and Keister (1998) and Kajii (1997).
11 In a general equilibrium model, several theoretical studies have attempted to understand the role of indexed bonds under inflation volatility, but explanations about the source of inflation volatility vary in the literature. For example, Magill and Quinzii (1997) assume that inflation volatility is from monetary shocks while Geanakoplos (2005) assumes that it is from intrinsic endowment shocks. In a general equilibrium model under unstable monetary policy, Kang (2020) showed that indexed bonds do not necessarily lead to Pareto-improvement.

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Appendix

A proof of proposition 3

Lemma 4 *x_α^B is increasing but x_β^B is decreasing in $\mathcal{P}$ where $\mathcal{P} > 1$. (Subscript "B" represents the asset buyer.)*

Proof. With increased volatility in inflation (i.e., with increased $\mathcal{P}$), there are four cases for the direction of changes in x_α^B and x_β^B:

Case 1 : both x_α^B and x_β^B increase in $\mathcal{P} > 1$.

Case 2 : x_α^B increases but x_β^B decreases in $\mathcal{P} > 1$.($*$)

Case 3 : x_α^B decreases but x_β^B increases in $\mathcal{P} > 1$.

Case 4 : both x_α^B and x_β^B decrease in $\mathcal{P} > 1$.

First, Case 3 is ruled out since x_β^i/x_α^i must increase in $\mathcal{P} > 1$. Then, we need to show that Cases 1 and 4 are not true.

Assuming that Case 1 is true: for $1 < \mathcal{P} < \mathcal{P}'$, $x_\alpha^B(\mathcal{P}) < x_\alpha^B(\mathcal{P}')$ and $x_\beta^B(\mathcal{P}) < x_\beta^B(\mathcal{P}')$. Market clearing conditions imply $x_\alpha^S(\mathcal{P}) > x_\alpha^S(\mathcal{P}')$ and $x_\beta^S(\mathcal{P}) > x_\beta^S(\mathcal{P}')$.

Where $q_0 = 1$, the price of a risky asset is given by

$$q^+ = \frac{\pi_\alpha R_\alpha g^{S\prime}\left(x_\alpha^S\right) + \pi_\beta R_\beta g^{S\prime}\left(x_\beta^S\right)}{f^{S\prime}\left(x_0^S\right)} = \frac{\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)}{f^{B\prime}\left(x_0^B\right)}. \tag{A1}$$

By $x_\alpha^S(\mathcal{P}) > x_\alpha^S(\mathcal{P}')$ and strictly concavity of g^i, $\pi_\alpha R_\alpha g^{S\prime}\left(x_\alpha^S\right) + \pi_\beta R_\beta g^{S\prime}\left(x_\beta^S\right)$ decreases in $\mathcal{P}$. In the same way, we can show that $\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)$ increases in $\mathcal{P}$. Therefore, by equation (A1), the market clearing condition, $x_0^S + x_0^B = e_0^S + e_0^B$ and strict concavity of f^i, the following inequalities must be satisfied:

$$x_0^B(\mathcal{P}) < x_0^B(\mathcal{P}') \text{ and } x_0^S(\mathcal{P}) > x_0^S(\mathcal{P}').$$

The asset buyer's budget constraints are

$$x_0^B(\mathcal{P}) + q^+(\mathcal{P})b^B(\mathcal{P}) = 0 \quad \text{and}$$

$$x_0^B(\mathcal{P}') + q(\mathcal{P}')b^B(\mathcal{P}') = 0$$

Since $x_0^B(\mathcal{P}) < x_0^B(\mathcal{P}')$ and $b^B(\mathcal{P}) < b^B(\mathcal{P}')$, we have

$$q^+(\mathcal{P}) > q^+(\mathcal{P}') \tag{A2}$$

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The asset seller's budget constraints are

$$x_0^S(\mathcal{P}) + q^+(\mathcal{P})b^S(\mathcal{P}) = 0 \text{ and}$$

$$x_0^S(\mathcal{P}') + q^+(\mathcal{P}')b^S(\mathcal{P}') = 0$$

Since $x_0^S(\mathcal{P}) > x_0^S(\mathcal{P}')$ and $b^S(\mathcal{P}) > b^S(\mathcal{P}')$,

$$q^+(\mathcal{P}) < q^+(\mathcal{P}') \tag{A3}$$

Inequalities (A2) and (A3) contradict each other.

We can show that Case 4 is not true in the same way. □

Continuing the proof of Proposition 3, we have

$$\frac{p_h^F}{q^+} = \frac{\pi_\alpha g^{i\prime}\left(x_\alpha^i\right) + \pi_\beta g^{i\prime}\left(x_\beta^i\right)}{\pi_\alpha R_\alpha g^{i\prime}\left(x_\alpha^i\right) + \pi_\beta R_\beta g^{i\prime}\left(x_\beta^i\right)} \text{ for } i = B, S.$$

We need to show that $\frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) > 0$ and $\frac{d}{d\mathcal{P}}\left(\frac{p_F^S}{q^+}\right) < 0$. There are four variables $R_\alpha(\mathcal{P})$, $R_\beta(\mathcal{P})$, $x_\alpha^i(\mathcal{P})$, and $x_\beta^i(\mathcal{P})$ which should be considered in the total derivative. The total derivative can be divided into two parts:

$$\begin{aligned}\frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) = & \frac{d}{d\mathcal{P}}\left(\frac{\pi_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta g^{B\prime}\left(x_\beta^B\right)}{\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)}\right)_{x_\alpha^i, x_\beta^i = \text{constant}} \\ & + \frac{d}{d\mathcal{P}}\left(\frac{\pi_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta g^{B\prime}\left(x_\beta^B\right)}{\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)}\right)_{R_\alpha, R_\beta = \text{constant}}.\end{aligned}$$

Since g^i is strictly concave and $x_\alpha^B > x_\beta^B$ for the asset buyer, $g^{B\prime}\left(x_\alpha^B\right) < g^{B\prime}\left(x_\beta^B\right)$. Since $\pi_\alpha R_\alpha\left(\mathcal{P}\right)$ and $\pi_\beta R_\beta\left(\mathcal{P}\right)$ increases and decreases in $\mathcal{P}$, respectively, and $\pi_\alpha R_\alpha\left(\mathcal{P}\right) + \pi_\beta R_\beta\left(\mathcal{P}\right)$ is constant in $\mathcal{P}$, we have

$$\frac{d}{d\mathcal{P}}\left(\pi_\alpha R_\alpha g^{B\prime}\left(x_\alpha^B\right) + \pi_\beta R_\beta g^{B\prime}\left(x_\beta^B\right)\right)_{R_\alpha, R_\beta = \text{constant}} < 0.$$

Therefore, the first term is strictly positive.

The second term can be expressed as

$$T(G) = \frac{\pi_\alpha + \pi_\beta G}{\pi_\alpha R_\alpha + \pi_\beta R_\beta G} \text{ where } G = \frac{g^{B\prime}\left(x_\beta^B\right)}{g^{B\prime}\left(x_\alpha^B\right)}$$

$T(G)$ is increasing in G since $\pi_\beta > \pi_\beta R_\beta$. $\frac{dT}{d\mathcal{P}}$ is positive since $\frac{dx_\alpha^B}{d\mathcal{P}} > 0$, $\frac{dx_\beta^B}{d\mathcal{P}} < 0$ (by Lemma 5) and strictly concavity of g'. Thus, the second term also increases in $\mathcal{P}$ and $\frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) > 0$.

We can prove that $\frac{d}{d\mathcal{P}}\left(\frac{p_F^S}{q^+}\right) < 0$ in the same way. Finally, we have

$$\frac{d}{d\mathcal{P}}\left(\frac{p_F^B - p_F^S}{p_F^S}\right) = \frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{p_F^S}\right) = \frac{d}{d\mathcal{P}}\left(\frac{p_F^B}{q^+}\right) / \frac{d}{d\mathcal{P}}\left(\frac{p_F^S}{q^+}\right) > 0.$$

End of Proof.

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B proof of proposition 5

Without loss of generality, we assume that $p_0 = 1$ in the proof. The budget set in a pure monetary economy is defined as

$$BS^i_+ \left(q\right) = \left\{\left(z^i, b^i\right) | z^i + qb^i \leq 0\right\}.$$

The budget set with indexed bonds is defined as

$$BS^i_* \left(q, p, \tau^i\right) = \left\{\left(z^i, b^i, n^i\right) | z^i + qb^i + p \max\left(n^i, 0\right) + p\left(1+\theta\right) \min\left(n^i, 0\right) \leq \tau^i\right\}.$$

We need to show that for any equilibrium of the money market $\left(z^i_+, b^i_+\right)$, there exists a lump-sum transfer plan $\sum_{i \in I} \tau^i = 0$ such that $\left(z^i_+, b^i_+, 0\right) \in B^i_* \left(q, p, \tau^i\right)$ for all $i \in I$ and any q (since we do not know about the equilibrium price q^* after a tax plan, we need to prove it for any price $q \in \mathbb{R}_{++}$). Then, the allocation $\left(z^i_+, b^i_+, 0\right)^I_{i=1}$ is also affordable in the combined market. This means that the combined market is at least weakly Pareto superior to the money market by the revealed preferences hypothesis.

Assuming that the equilibrium price and allocations in the monetary market are q_+ and $(z^i_+, b^i_+)^I_{i=1}$, the following equation is satisfied

$$z^i_+ + q_+ b^i_+ = 0 \text{ for all } i = 1, ..., I. \tag{B1}$$

In the combined market, we need to show the existence of τ^i for $i = 1, .., I$ such that $\sum_i \tau^i = 0$ and $\left(z^i_+, b^i_+, 0\right) \in B^i_* \left(q, p, \tau^i\right)$ for any q. Then,

$$z^i_+ + qb^i_+ = \tau^i \text{ for all } i = 1, ..., I \tag{B2}$$

subtracting equation (B2) with (B1), we get

$$\left(q - q_+\right) b^i_+ = \tau^i.$$

That means that if $\tau^i = \left(q - q_+\right) B^i_+$, $(z^i_+, b^i_+, 0) \in B^i_+(q, p, \tau^i)$. Also, we can prove that $\sum_i \tau^i = 0$ by market clearing:

$$\sum_i \tau^i = \sum_i \left(q - q_+\right) b^i_+ = \left(q - q_+\right) \sum_i b^i_+ = 0.$$

End of Proof.

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