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PROJECTIVE AND MULTIGRADED REPRESENTATIONS OF MONOMIAL AND MULTISIGNED GROUPS I. GRADED REPRESENTATIONS OF A TWISTED PRODUCT

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ABSTRACT Motivated by the problem of giving a functorial (or at least uniform) description of the projective representations for wreath products $G \wr S_n$ in terms of those for G, we study a certain binary operation $\overset{\alpha}{\curlyvee}$ on the class of "cyclic covering groups with parities" Along with setting up the basic machinery associated to representations graded by $(\mathbf{Z}/2)^{\ell}$, the main result is a description of the irreducibles for $A\overset{\alpha}{\curlyvee}B$ in terms of a (tensorlike) product of those for A and for B Finally we describe a programme for producing a PSH-algebra theory in this context, analogous to that of Zelevinsky for the case $\ell = 0$, and that of the author with Michael Bean (structure) and with John Humphreys (applications) for the case $\ell = 1$

Let H be a finite group, n a positive integer, and $H \wr S_n$ the monomial group, that is, the wreath product of H with the symmetric group S_n. For any even integer $2m$ which is divisible by the exponent of the Schur multiplier $M(H)$, all classes in $M(H \wr S_n)$ may be "realized" by $2m$-fold cyclic covers of $H \wr S_n$ (defined below). That is, the map

$$H^2(H \wr S_n; \mathbf{Z}/2m) \longrightarrow H^2(H \wr S_n; \mathbf{Q}/\mathbf{Z}) \cong H^2(H \wr S_n; \mathbf{C}^{\times}) = M(H \wr S_n),$$

induced by the inclusion $\mathbf{Z}/2m \longrightarrow \mathbf{Q}/\mathbf{Z}$, is surjective. [Equivalently $M(H \wr S_n)$ has exponent dividing $2m$; in fact, when $n \geq 4$, its exponent is the least common multiple of the exponent of $M(H)$ and the integer 2.] Abstractly, $M(H \wr S_n)$ is independent of n for $n \geq 4$, and each of its elements can be made to correspond to a sequence of covers $\{Y_n \longrightarrow H \wr S_n : n \geq 0\}$. See Section 10.

Our ultimate objective is to give the projective representations of $H \wr S_n$ as a functor, in some sense, of the projective representations of H. The projective representations of $H \wr S_n$ indexed by a given cocycle in $M(H \wr S_n)$ may be identified with those linear representations of a corresponding cover of $H \wr S_n$ for which "the" generator of the kernel of the covering projection acts as multiplication by a fixed $2m$-th root of unity.

For the linear representations, a functorial rendering can be given as follows. The Young subgroup embeddings

$$(H \wr S_i) \times (H \wr S_j) \longrightarrow H \wr S_{i+j}$$

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give a multiplication

$$R(H \wr S_i) \times R(H \wr S_j) \longrightarrow R(H \wr S_{i+j}),$$

where $R(G)$ is the free abelian group on the irreducible representations of G. Then there is

(1) an isomorphism of graded rings

$$\bigoplus_{n=0}^{\infty} R(H \wr S_n) \cong \mathbf{Z}[x_{n\alpha} : \alpha \in \mathrm{Irrep}(H), n \geq 1]$$

where $x_{n\alpha}$ is the class of the irreducible representation of $H \wr S_n$ obtained by taking the n-fold tensor power of the α-th irreducible representation of H; and

(2) alternating sum formulae for operators on this ring whose iteration on $1 \in R(H \wr S_0)$ yield all the irreducibles in $R(H \wr S_n)$ for all n, as well as explicit formulae for them in terms of monomials in the $x_{i\alpha}$, branching rules, Littlewood-Richardson rule, *etc.* This is relatively simple, since the above graded ring is a tensor power of "$|\,\mathrm{Irrep}\, H|$" copies of the ring when $H = 1$, the latter being a much studied object alternatively known as the ring of stable symmetric functions [M], the free λ-ring on one generator [A – T], the cohomology of BU, the atomic PSH-algebra [Z], ...

To proceed analogously for the projective representations, it is necessary to find embeddings

$$Y_i \overset{\alpha}{\mathsf{Y}} Y_j \hookrightarrow Y_{i+j}$$

for each of the sequences $\{Y_n \longrightarrow H \wr S_n\}$ of $2m$-fold cyclic covers mentioned above, using some suitable operation $\overset{\alpha}{\mathsf{Y}}$, and then to give a determination of the representations of $Y \overset{\alpha}{\mathsf{Y}} W$ vis-a-vis those of Y and of W. It turns out that $\overset{\alpha}{\mathsf{Y}}$ may be defined by twisting the usual multiplication, using a number of "sign" homomorphisms $Y \longrightarrow \mathbf{Z}/2$. That number (and so $\overset{\alpha}{\mathsf{Y}}$) depends not just on H but on which sequence of covers one is using. The number (apparently) needs to be arbitrarily large, by choosing a matched sequence of sufficiently subtle cocycles for $H \wr S_n$ ($n = 0, \ldots, \infty$) for a group H with sufficiently many homomorphisms into $\mathbf{Z}/2$.

In this first part we give a theory for treating the question immediately above, leaving the application to $H \wr S_n$ for later. Not unnaturally, since $\overset{\alpha}{\mathsf{Y}}$ depends on some homomorphisms from Y to $\mathbf{Z}/2$, it is necessary to consider representations also of the kernels of these homomorphisms, and their various intersections. To obtain an operation $\overset{\alpha}{\bowtie}$ something akin to the natural isomorphism

$$R(G_1) \otimes_{\mathbf{Z}} R(G_2) \longrightarrow R(G_1 \times G_2)$$

(which uses the tensor product), it seems to be essential to use representations graded by $(\mathbf{Z}/2)^{\ell}$ where ℓ is the number of "sign" homomorphisms involved. In order to obtain sanitary formulae, we use 2^A in place of $(\mathbf{Z}/2)^{\ell}$, where A is a fixed set of cardinality ℓ, and 2^A is the group of all subsets of A under symmetric difference. It turns out that, when $\ell > 1$, the map analogous to the above tensor product isomorphism is not an isomorphism. But it does determine all the representations of $Y \overset{\alpha}{\mathrm{Y}} W$ and its attached subgroups in a sense which is sufficient for the applications. The case of ordinary ungraded representations and the tensor product can be considered to be the case $\ell = 0$ of this theory. The main result is Theorem 8.2.

In the case $\ell = 1$ and $m = 1$, the theory is equivalent to that given by Hoffman-Humphreys [H-H1] [H-H3; Appendix 8]. An improvement here, which is crucial to the cases $\ell > 1$, is as follows. It is easy to see that when the multisign $Y \xrightarrow{\sigma} 2^A$ is surjective, the 2^A-graded representations of Y are essentially the same as representations of $\operatorname{Ker}\sigma$. If E is a subset of A of cardinality $\ell - k$, we find it essential to view representations of the kernel of $Y \longrightarrow 2^A/2^E$, not as representations of Y graded over $(\mathbf{Z}/2)^k$ (*i.e.* 2^{A-E}-graded), but rather as 2^A-graded representations of Y with extra structure. Without this, the definition of $\overset{\alpha}{\bowtie}$ and proofs of its properties become unmanageable. Several years ago Michael Bean (as an undergraduate research assistant) succeeded in the somewhat formidable task of producing a version of $\overset{\alpha}{\bowtie}$ for the case $\ell = 2$, $m = 1$, at a time when the crucial idea above was lacking [B]. The present work depends heavily for motivation on my previous work with Bean and with Humphreys; [B-H] [H-H1] [H-H2], [H-H3; Appendix 8].

Besides the application to $H \wr S_n$, a motivation for the theory below (at least for the author) is to clarify the status of the work with Humphreys for $\ell = 1 = m$. The operation $\boxtimes^-$ had a certain odour of the ad hoc about it whereas the operation here seems to arise more naturally out of the given data.

A brief comment on the motivation for graded representations may be in order before we start; see also [D], [S]. If H is a subgroup of G, one can think of the process of inducing a representation V of H to a representation Y of G as a kind of "information losing" process. If H is normal in G, the process can be factored in the form

$$V \rightsquigarrow W \rightsquigarrow Y,$$

where W is a graded representation of G, graded over G/H. The first step $V \rightsquigarrow W$ is reversible, so the "information loss", which consists of forgetting the grading, is concentrated in the second step. We shall be dealing with a much specialized situation, where G/H is isomorphic to a product of cyclic groups of order 2. Because of this, quite a bit of extra structure exists.

What follows is the first part of a three part paper. Part II deals with a classification problem arising from Part I, and Part III with the application to monomial groups. More detailed introductions will be attached to those two parts.

1. **The categories $\mathcal{G}(\Lambda, m)$ and $\mathcal{T}_\Lambda^\Gamma(G, y, \sigma)$.** Let Λ be an abelian group whose elements will be denoted $B, C, \ldots$, with identity element $\emptyset$, and operation denoted $(B, C) \longmapsto B \triangle C$, for later ease of transition to the main example. Let m be a positive integer.

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DEFINITION. Let $\mathcal{G}(\Lambda, m)$ be the category whose objects are triples (G, y, σ) for which G is a group, y is a central element in G of order $2m$, and $\sigma: G \longrightarrow \Lambda$ is a homomorphism such that $\sigma(y) = \emptyset$, the identity element of Λ. A morphism $\theta \in \mathrm{Map}_{\mathcal{G}(\Lambda,m)}[(G_1, y_1, \sigma_1), (G_2, y_2, \sigma_2)]$ is a group homomorphism $\theta: G_1 \longrightarrow G_2$ for which $\theta(y_1) = y_2$ and $\sigma_2 \circ \theta = \sigma_1$. It is easily checked that $\mathcal{G}(\Lambda, m)$ is a category under ordinary composition of functions.

DEFINITION. For each (G, y, σ) in $\mathcal{G}(\Lambda, m)$, define a category "$\mathcal{T}(G)$" $= \mathcal{T}_\Lambda(G, y, \sigma)$ as follows. An object in $\mathcal{T}(G)$ is a pair consisting of:

(i) a G-module V (that is, a finite dimensional $\mathbf{C}$-vector space V together with a linear action of G on V) ; and

(ii) a Λ-grading on V (that is, a collection $\{V_B : B \in \Lambda\}$ of subspaces of V) ;

This data must satisfy the axioms:

(a) V is the (internal) direct sum $\sum \oplus_B V_B$;

(b) for all $g \in G$ and $B \in \Lambda$, we have

$$g \cdot V_B = V_{B \triangle \sigma g};$$

(c) y acts as multiplication by $e^{\pi\imath/m}$ on V.

A morphism $\varphi \in \mathrm{Map}_{\mathcal{T}G}(V, W)$ is a linear map $\varphi: V \longrightarrow W$ which commutes with the action [*i.e.* $\varphi(g \cdot v) = g \cdot \varphi(v)$], and such that $\varphi(V_B) \subset W_B$ for all $B \in \Lambda$. It is straightforward to see that $\mathcal{T}(G)$ is a category under composition.

DEFINITION. Let $C \in \Lambda$. For an object V in $\mathcal{T}G$, let $\rho_C V$ be V as a G-module, but with grading $(\rho_C V)_B = V_{B \triangle C}$. It is obvious that $\rho_C V$ is also an object of $\mathcal{T}G$. For a morphism $\varphi \in \mathrm{Map}_{\mathcal{T}G}(V, W)$ let $\rho_C \varphi = \varphi \in \mathrm{Map}_{\mathcal{T}(G)}(\rho_C V, \rho_C W)$. [There is a slight conflict with the usual conventions for categories here in that the sets of morphisms for different (domain, codomain) pairs ought to be disjoint. This will not lead to ambiguity, and so it will be ignored.]

PROPOSITION 1.1. *For each C, the above ρ_C is a functor. We have*

$$\rho_\emptyset = \mathrm{id}; \quad \rho_C \rho_D = \rho_{C \triangle D} = \rho_D \rho_C; \quad \rho_C \rho_C = \mathrm{id}.$$

DEFINITION OF $\mathcal{T}_\Lambda^\Gamma(G, y, \sigma)$. For each subgroup Γ of Λ, this category [whose name will be abbreviated to $\mathcal{T}^\Gamma(G)$] has objects $\mathcal{V} = (V, \beta)$, where V is an object in $\mathcal{T}(G)$, and $\beta = \{\beta(B) \mid B \in \Gamma\}$ is such that

(i) $\beta(B): V \longrightarrow \rho_B V$ is an isomorphism in $\mathcal{T}(G)$;

(ii) $\beta(B)\beta(C) = \beta(B \triangle C)$ for all B and C in Γ.

A morphism $\varphi \in \mathrm{Map}_{\mathcal{T}^\Gamma(G)}[(V, \beta), (W, \gamma)]$ is a morphism in $\mathcal{T}(G)$ such that $\varphi \circ \beta(B) = \gamma(B) \circ \varphi$ for all $B \in \Gamma$. It is evident that $\mathcal{T}^\Gamma(G)$ is a category under composition.

Note that $\mathcal{T}^{\{\emptyset\}}(G)$ may be identified with $\mathcal{T}(G)$, and that, due to (ii), axiom (i) may be altered to require only that $\beta(B)$ be a morphism in $\mathcal{T}(G)$, with the added requirement that $\beta(\emptyset)$ be the identity [as follows from (i) and (ii).] Several examples will be checked in this way. Note also that a morphism in $\mathcal{T}^\Gamma(G)$ which is bijective is an isomorphism in $\mathcal{T}^\Gamma(G)$.

DEFINITION OF $\oplus$ ON $\mathcal{T}^\Gamma(G)$. On objects, define $(V,\beta)\oplus(W,\gamma)=(V\oplus W,\beta\oplus\gamma)$. On morphisms $\varphi\oplus\psi$ is also the usual map between ordered pairs from linear algebra.

PROPOSITION 1.2. *The above operation is well defined, yielding a functor*

$$\oplus\colon \mathcal{T}^\Gamma(G)\times\mathcal{T}^\Gamma(G)\rightsquigarrow\mathcal{T}^\Gamma(G).$$

The standard formulae from linear algebra yield natural isomorphisms

$$\mathcal{V}\oplus\mathcal{W}\cong\mathcal{W}\oplus\mathcal{V}$$
$$(\mathcal{V}_1\oplus\mathcal{V}_2)\oplus\mathcal{V}_3\cong\mathcal{V}_1\oplus(\mathcal{V}_2\oplus\mathcal{V}_3)$$
$$\mathcal{V}\oplus 0\cong\mathcal{V}\text{ for the zero object }0.$$

COROLLARY 1.3. *There is a Grothendieck group "$T^\Gamma(G)$" $=T^\Gamma_\Lambda(G,y,\sigma)$ generated by the objects of $\mathcal{T}^\Gamma(G)$ with respect to $\oplus$.*

With the obvious definition of irreducible, the analogous results to those of Maschke, Schur, *etc.* hold, so that, for finite G, $T^\Gamma(G)$ is the free abelian group on the irreducibles. The next section makes this unsurprising, but direct proofs are also easy. These will be delayed until Section 8 in order to emphasize the independence from decomposition into irreducibles of the constructions in the intervening sections.

2. **"Real world" interpretation of $\mathcal{T}^\Gamma(G)$.** Each of the following propositions asserts the existence of an equivalence between categories; that is, of a pair of functors which are inverse up to natural isomorphism. These functors in fact commute with direct sums.

PROPOSITION 2.1. *Letting σ_Γ be the composite $G\xrightarrow{\sigma}\Lambda\longrightarrow\Lambda/\Gamma$,*

$$\mathcal{T}^\Gamma_\Lambda(G,y,\sigma)\cong\mathcal{T}_{\Lambda/\Gamma}(G,y,\sigma_\Gamma)$$

for any (G,y,σ) in $\mathcal{G}(\Lambda,m)$ and any subgroup Γ of Λ.

PROPOSITION 2.2. *Factoring σ as $G\xrightarrow{\sigma_1}\operatorname{Im}\sigma\hookrightarrow\Lambda$,*

$$\mathcal{T}_\Lambda(G,y,\sigma)\cong[\mathcal{T}_{\operatorname{Im}\sigma}(G,y,\sigma_1)]^{|\operatorname{Coker}\sigma|}.$$

PROPOSITION 2.3. *If σ is surjective, then*

$$\mathcal{T}_\Lambda(G,y,\sigma)\cong\mathcal{R}^{(m)}(\operatorname{Ker}\sigma,y)$$

where $\mathcal{R}^{(m)}(H,y)$, for central y of order $2m$ in H, is the category of H-modules on which y acts as $e^{i\pi/m}$.

Combining 2.1 and 2.3, if σ is surjective, the category $\mathcal{T}^\Gamma(G)$ is just a complicated substitute for the category of those representations of $\sigma^{-1}\Gamma$ for which the element y multiplies by a fixed primitive $2m$-th root of unity. The motivation for considering $\mathcal{T}^\Gamma$ is that the operations κ_F, and particularly $\overset{\alpha}{\bowtie}$ and $\mathcal{H}$, of Sections 3, 5 and 7, are very awkward to deal with in $\mathcal{R}^{(m)}$. If σ is not surjective, one uses also 2.2 to see that $\mathcal{T}^\Gamma(G)$ is really just a power of a category $\mathcal{R}^{(m)}(H)$ for suitable H.

PROOF OF 2.2. This is almost immediate, since for V in $\mathcal{T}_\Lambda(G)$ and each $\pi \in \operatorname{Coker}\sigma$, the subspace $V_{(\pi)} = \sum_{B\in\pi} V_B$ is invariant. The required functor

$$\mathcal{T}_\Lambda(G, y, \sigma) \rightsquigarrow [\mathcal{T}_{\operatorname{Im}\sigma}(G, y, \sigma_1)]^{\operatorname{Coker}\sigma}$$

takes V to the function sending π to $V_{(\pi)}$, where $V_{(\pi)}$ is graded by

$$(V_{(\pi)})_C = \begin{cases} V_C & \text{if } C \in \pi; \\ 0 & \text{if not.} \end{cases}$$

PROOF OF 2.3. Define a functor $\theta: \mathcal{T}_\Lambda(G) \rightsquigarrow \mathcal{R}^{(m)}(\ker\sigma)$ by $\theta(V) = V_\emptyset$ and $\theta(\varphi) = \varphi|_{V_\emptyset}$. Let $s: \Lambda \longrightarrow G$ be a cross section of σ [a function s with $\sigma \circ s = \mathrm{id}$] such that $s(\emptyset) = 1_G$. Define a functor $\mu: \mathcal{R}^{(m)}(\ker\sigma) \rightsquigarrow \mathcal{T}_\Lambda(G)$ by $\mu(W) = W^\Lambda = \{\eta: \Lambda \longrightarrow W\}$ as a vector space, with grading defined by

$$[\mu(W)]_C = \{\eta \in \mu(W) \mid \eta(B) = 0 \text{ for all } B \neq C\},$$

and with action defined by

$$(g \cdot \eta)(B) = [s(B)^{-1} g s(\sigma g^{-1} \triangle B)] \cdot \eta(\sigma g^{-1} \triangle B).$$

It is straightforward to check that $\mu(W)$ is a well-defined object of $\mathcal{T}_\Lambda(G)$. Define μ on morphisms by $\big(\mu(\varphi)\big)(\eta) = \varphi \circ \eta$. Then it is easily seen that μ is a functor, once one checks that $\mu(\varphi)$ is a $\mathcal{T}_\Lambda(G)$-morphism. Define a map from $\theta\big(\mu(W)\big)$ to W by sending η to $\eta(\emptyset)$. This is an isomorphism in $\mathcal{T}_\Lambda(G)$ and yields a natural transformation from $\theta \circ \mu$ to the identity functor. Define a map from $\mu\big(\theta(V)\big)$ to V by sending η to $\sum_{B\in\Lambda} s(B) \cdot \eta(B)$. This is surjective and linear, and so bijective since $\dim \mu\big(\theta(V)\big) = \dim V$. It is a morphism in $\mathcal{R}^{(m)}(\ker\sigma)$, and gives a natural transformation from $\mu \circ \theta$ to the identity functor.

PROOF OF 2.1. Define a functor

$$\omega: \mathcal{T}_\Lambda^\Gamma(G) \rightsquigarrow \mathcal{T}_{\Lambda/\Gamma}(G)$$

by letting $\omega(V, \beta)$ be the set of all functions $\zeta \in V^\Lambda$ for which $\zeta(B) \in V_B$ for all $B \in \Lambda$, and $\zeta(B \triangle C) = \beta(C)\big(\zeta(B)\big)$ for all $B \in \Lambda$ and $C \in \Gamma$; with action $(g \cdot \zeta)(B) := g \cdot \big(\zeta(B \triangle \sigma g)\big)$; and grading

$$[\omega(V, \beta)]_{D\triangle\Gamma} := \{\zeta \in \omega(V, \beta) \mid \zeta(B) = 0 \text{ for } B \notin D\triangle\Gamma\}.$$

Define ω on morphisms by $\big(\omega(\varphi)\big)(\zeta) = \varphi \circ \zeta$. It may be checked that $\omega(V, \beta)$ and $\omega(\varphi)$ are in $\mathcal{T}_{\Lambda/\Gamma}(G)$ and that ω is a functor.

Define a functor

$$\nu: \mathcal{T}_{\Lambda/\Gamma}(G) \rightsquigarrow \mathcal{T}_\Lambda^\Gamma(G)$$

by $\nu(W) := (V, \beta)$ and $[\nu(\psi)](\eta) := \psi \circ \eta$; where

$$V := \{\eta \in W^\Lambda \mid \eta(B) \in W_{B\triangle\Gamma} \text{ for all } B \in \Lambda\};$$

with action $(g \cdot \eta)(B) := g \cdot \big(\eta(B \triangle \sigma g)\big)$; and grading

$$V_C := \{\eta \in V \mid \eta(D) = 0 \text{ for all } D \neq C\};$$

and with $\big(\beta(S)(\eta)\big)(B) := \eta(B \triangle S)$ for $S \in \Gamma$ and $B \in \Lambda$. It may be checked that $\nu(W)$ and $\nu(\psi)$ are in $\mathcal{T}_\Lambda^\Gamma(G)$, and that ν is a functor.

Define a function from $\psi\big(\nu(W)\big)$ to W by sending ζ to $|\Lambda|^{-1} \sum_{B \in \Lambda} \big(\zeta(B)\big)(B)$. This is a morphism in $\mathcal{T}_\Lambda^\Gamma(G)$, and is surjective and so bijective since $\dim \psi\nu(W) = \dim W$. It is a natural transformation from $\psi \circ \nu$ to the identity functor.

Define a morphism from $\nu\big(\omega(V, \beta)\big)$ to (V, β) by sending η to $\sum_{B \in \Lambda} \big(\eta(B)\big)(B)$, checking that this behaves with respect to grading and β-maps and that it commutes with the action. Counting dimensions, this is bijective. It is easily seen to be a natural transformation, completing the proof.

3. **The ring K and operations κ_F acting on $\mathcal{T}^E$.** For the remainder of the paper, A will denote a fixed finite set whose cardinality is denoted ℓ. The group Λ will usually be specialized to 2^A, the group of subsets of A under $\triangle$, the symmetric difference:

$$B \triangle C = (B \cup C) \setminus (B \cap C) = (B \setminus C) \cup (C \setminus B).$$

Since 2^A is isomorphic to $(\mathbf{Z}/2)^\ell$, a homomorphism $\sigma\colon G \longrightarrow 2^A$ carries the same information as a sequence of "ℓ" homomorphisms $G \longrightarrow \mathbf{Z}/2$. The theory may readily be generalized to homomorphisms $G \longrightarrow (\mathbf{Z}/p)^\ell$ where p need not be a prime, using the "partition groups" p^A in place of $(\mathbf{Z}/p)^\ell$.

Note that 2^A is a ring, using intersection of sets for multiplication.

Let $\Lambda = 2^A$, so that 2^E is a subgroup of 2^A for each $E \subset A$. We shall abbreviate $\mathcal{T}_{2^A}^{2^E}(G, y, \sigma)$ to $\mathcal{T}^E(G)$. Our aim here is to define and analyze a functor, for $F \subset A$,

$$\kappa_F\colon \mathcal{T}^E(G) \rightsquigarrow \mathcal{T}^{E \triangle F}(G).$$

In the simpler cases $F \subset E$ or $F \cap E = \emptyset$, this corresponds to restricting and inducing respectively between the appropriate subgroups of G when we pass to the "real world" interpretation from Section 2. The relations in 3.4 below then yield the appropriate interpretation of κ_F for general F. First we define ρ_C on $\mathcal{T}^E$.

DEFINITION. If (V, β) is an object in $\mathcal{T}^E(G)$, and $C \subset A$, let $\rho_C(V, \beta) = (\rho_C V, \rho_C \beta)$ where $(\rho_C \beta)(S) := (-1)^{|S \cap C|} \beta(S)$. On morphisms, let $\rho_C \varphi = \varphi$. It is easy to see that $\rho_C(V, \beta)$ and $\rho_C \varphi$ are well defined, are in $\mathcal{T}^E(G)$, and that $\rho_C\colon \mathcal{T}^E(G) \rightsquigarrow \mathcal{T}^E(G)$ is a functor satisfying 1.1. It follows that

$$\rho_C = \prod_{a \in C} \rho_a \text{ where } \rho_a := \rho_{\{a\}}.$$

Before launching into the definition of κ_F, the thoroughly modern reader may prefer to read the universal property of κ_F given after Theorem 3.3.

Fix an object $\mathcal{V} = (V, \beta)$ of $\mathcal{T}^E(G)$ for some $E \subset A$, and another set $F \subset A$. Let

$$\{z_{C,D} \mid C \subset E \cap F, D \subset F\}$$

be a set of complex numbers (necessarily non-zero) with

(a) $z_{\emptyset,D} = 1$ for all $D \subset F$;

(b) $z_{C_1 \triangle C_2, D} = z_{C_1, C_2 \triangle D} z_{C_2, D}$ for $C_1, C_2 \subset E \cap F$ and $D \subset F$;

(c) $z_{C,D \triangle T} = z_{C,D}$ for $C \subset E \cap F$, $D \subset F$, $T \subset F \setminus E$.

The definition of the operation κ_F will be based on such a set for later convenience. As we shall see, a simpler definition exists which shows that κ_F is independent (up to natural isomorphism) of the choices of $z_{C,D}$.

EXAMPLE (A). $z_{C,D} = 1$ for all C and D.

EXAMPLE (B). $z_{C,D} = (-1)^{|C \cap D|}(-i)^{|C|}$ where $i^2 = -1$.

EXAMPLE (C). Let α be an involution on A for which $\alpha(E) = E$ and $\alpha(F) = F$. Let

$$z_{C,D} = (-1)^{|C| + |C \cap \alpha D|} i^{|C \cap \alpha C|}.$$

Actually example (b) comes from (c) by taking α to be the identity map. Verification of these uses the identity

$$i^{|B \triangle C|} = (-1)^{|B \cap C|} i^{|B|} i^{|C|}.$$

DEFINITION. Define an object $\kappa_F \mathcal{V} = (\kappa_F V, \kappa_F \beta)$ of $\mathcal{T}^{E \triangle F}(G)$ as follows. Let

$$\kappa_F V = \{\zeta \in V^{2^F} \mid \zeta(C \triangle D) = z_{C,D}\beta(C)[\zeta(D)] \text{ for } C \subset E \cap F \text{ and } D \subset F\};$$

with action

$$(g \cdot \zeta)(B) = g \cdot \zeta(B);$$

and grading

$$(\kappa_F V)_B = \{\zeta \in \kappa_F V \mid \zeta(D) \in V_{B \triangle D} \text{ for all } D \subset F\}.$$

For each $S \subset E \triangle F$, define $(\kappa_F \beta)(S) \colon \kappa_F V \longrightarrow \kappa_F V$ by

$$\{[(\kappa_F \beta)(S)](\zeta)\}(B) = \beta(S \cap E)\{\zeta[B \triangle (S \cap F)]\}.$$

If $\varphi \colon V \longrightarrow W$ is a morphism in $\mathcal{T}^E(G)$ from $\mathcal{V}$ to $\mathcal{W}$, define $\kappa_F \varphi \colon \kappa_F V \longrightarrow \kappa_F W$ by $(\kappa_F \varphi)(\zeta) = \varphi \circ \zeta$. The following may then be verified in a straightforward manner:

(i) $\kappa_F V = \sum \oplus_{B \subset A} (\kappa_F V)_B$ as a vector space.

(ii) $(g, \zeta) \longmapsto g \cdot \zeta$ is a well-defined linear action of G on $\kappa_F V$ with $y \cdot \zeta = e^{i\pi/m}\zeta$.

(iii) If $\zeta \in (\kappa_F V)_B$, then $g \cdot \zeta \in (\kappa_F V)_{B \triangle \sigma g}$.

[So $\kappa_F V$ is a well-defined object in $\mathcal{T}(G)$.]

(iv) $(\kappa_F \beta)(S)$ is a linear map sending $(\kappa_F V)_B$ into $(\kappa_F V)_{B \triangle S}$.

(v) $(\kappa_F \beta)(S)$ commutes with the action of G.

(vi) $(\kappa_F \beta)(\emptyset) = \mathrm{id}$ and $(\kappa_F \beta)(S) \circ (\kappa_F \beta)(T) = (\kappa_F \beta)(S \triangle T)$.

[So $\kappa_F \mathcal{V}$ is a well-defined object in $\mathcal{T}^{E \triangle F}(G)$.]

(vii) $\kappa_F\varphi$ is a morphism in $\mathcal{T}^{E\triangle F}(G)$ from $\kappa_F\mathcal{V}$ to $\kappa_F\mathcal{W}$.

(viii) $\kappa_F(\mathrm{id}) = \mathrm{id}$ and $\kappa_F(\varphi_1 \circ \varphi_2) = (\kappa_F\varphi_1) \circ (\kappa_F\varphi_2)$

(ix) The canonical linear isomorphisms

$$(V \oplus W)^{2^F} \longleftrightarrow V^{2^F} \oplus W^{2^F}$$
$$\zeta \longmapsto (\pi_V \circ \zeta, \pi_W \circ \zeta)$$
$$\big[B \longmapsto \big(\zeta_1(B), \zeta_2(B)\big)\big] \longleftarrow (\zeta_1, \zeta_2),$$

where π_V, π_W are the projections for $V \oplus W$, have the following properties.

(a) They map $\kappa_F(V \oplus W)$ to and from $\kappa_F V \oplus \kappa_F W$.

(b) They preserve grading and commute with the action.

(c) They are natural with respect to pairs

$$\varphi_1 \colon (V, \beta) \longrightarrow (V', \beta') \text{ and } \varphi_2 \colon (W, \gamma) \longrightarrow (W', \gamma')$$

of maps in $\mathcal{T}^E(G)$.

These statements prove the following.

THEOREM 3.1. *The above definition yields a functor $\kappa_F \colon \mathcal{T}^E(G) \longrightarrow \mathcal{T}^{E\triangle F}(G)$ and a natural isomorphism*

$$\kappa_F(\mathcal{V} \oplus \mathcal{W}) \cong (\kappa_F\mathcal{V}) \oplus (\kappa_F\mathcal{W}).$$

COROLLARY 3.2. *The functor κ_F acts on isomorphism classes to produce a homomorphism*

$$\kappa_F \colon T^E(G) \longrightarrow T^{E\triangle F}(G)$$

of groups (for which we use the same name).

REMARK. For short term gain but long term pain, $\kappa_F V$ could simply have been defined to be $V^{2^{F\setminus E}}$ with the same formulae for everything else. Restriction of the functions ζ from 2^F to $2^{F\setminus E}$ would give an isomorphism in $\mathcal{T}^{E\triangle F}(G)$. This shows that our definition of κ_F is independent of the choices of $z_{C,D}$.

The functor κ_F may also be defined implicitly by a universal property, a specialization of the following theorem. Continue with a fixed $\mathcal{V} = (V, \beta)$ in $\mathcal{T}^E(G)$. Consider the class of all quadruples (D, Z, γ, ω) where $D \subset A$, $D \cap E \cap F = \emptyset$, $(Z, \gamma) \in \mathcal{T}^D(G)$, and $\omega \colon V \longrightarrow Z$ is a map in $\mathcal{T}(G)$ such that $\gamma(S) \circ \omega = \omega \circ \beta(S)$ for all $S \subset D \cap E$.

DEFINITION. Define $\delta_{\mathcal{V},F} = \delta \colon V \longrightarrow \kappa_F V$ by

$$[\delta(v)](B) = \begin{cases} z_{B,\emptyset}\beta(B)(v) & \text{if } B \subset E \cap F\,; \\ 0 & \text{if not.} \end{cases}$$

THEOREM 3.3. *The quadruple $(E\triangle F, \kappa_F V, \kappa_F\beta, \delta)$ is in the class just specified. It is universal in that class in the following sense. Given (D, Z, γ, ω) there is a map $\hat{\omega}\colon \kappa_F V \longrightarrow Z$ such that*

(i) $\omega = \hat{\omega}\circ\delta$;

(ii) $\hat{\omega}$ *is linear;*

(iii) $\gamma(S)\circ\hat{\omega} = \hat{\omega}\circ(\kappa_F\beta)(S)$ *for all* $S\subset D\cap(E\triangle F)$.

If $D\supset E\triangle F$, *then* $\hat{\omega}$ *is unique with respect to satisfying (i), (ii), and (iii). For any D, the map* $\hat{\omega}$ *may be chosen to be a* $\mathcal{T}(G)$*-map.*

REMARK. Specializing to quadruples whose first component D is $E\triangle F$ then gives a universal property implicitly defining κ_F. The previous construction(s) may be thought of as proof of existence from this point of view.

PROOF. It is straightforward to check that δ is in $\mathcal{T}(G)$ and that $(\kappa_F\beta)(S)\circ\delta = \delta\circ\beta(S)$ for all $S\subset E\setminus F$. Given (D, Z, γ, ω) define

$$\hat{\omega}(\zeta) = \sum_{B\subset D\cap F}\gamma(B)\left\{\omega[\zeta(B)]\right\}.$$

Then (ii) is clear. To check (i),

$$\begin{aligned}\hat{\omega}[\delta(v)] &= \sum_{B\subset D\cap F}\gamma(B)\{\omega[\delta(v)(B)]\}\\ &= \sum_{B\subset D\cap F\cap E\cap F}\gamma(B)\{\omega[z_{B,\emptyset}\beta(B)(v)]\}\ \text{by definition of}\ \delta\\ &= \gamma(\emptyset)\{\omega[z_{\emptyset,\emptyset}\beta(\emptyset)(v)]\}\ \text{since}\ D\cap E\cap F=\emptyset\\ &= \omega(v),\end{aligned}$$

as required. To verify (iii), let $S\subset D\cap(E\triangle F)$. Then

$$\begin{aligned}[\hat{\omega}\circ(\kappa_F\beta)(S)](\zeta) &= \sum_{B\subset D\cap F}\gamma(B)\{\omega[(\kappa_F\beta)(S)(\zeta)(B)]\}\\ &= \sum_{B\subset D\cap F}\gamma(B)\Big\{\omega\big[\beta(S\cap E)\{\zeta[B\triangle(S\cap F)]\}\big]\Big\}.\end{aligned}$$

But $S\cap E\subset D\cap E$ so $\omega\circ\beta(S\cap E) = \gamma(S\cap E)\circ\omega$. We obtain

$$\sum_{B\subset D\cap F}[\gamma(B)\circ\gamma(S\cap E)\circ\omega]\big(\zeta[B\triangle(S\cap F)]\big).$$

Now

$$S\cap F\subset D\cap(E\triangle F)\cap F = (D\cap E\cap F)\triangle(D\cap F) = D\cap F,$$

so we can make the variable change $B\longmapsto B\triangle(S\cap F)$, yielding

$$\begin{aligned}\sum_{B\subset D\cap F}\{\gamma[B\triangle(S\cap F)]\circ\gamma(S\cap E)\circ\omega\}\zeta(B) &= \sum_{B\subset D\cap F}\gamma(B\triangle S)\{\omega[\zeta(B)]\}\\ &\qquad\text{since}\ (S\cap F)\triangle(S\cap E)=S\\ &= \gamma(S)\big(\hat{\omega}(\zeta)\big)\end{aligned}$$

as required.

To prove uniqueness, note that $\kappa_F V$ is spanned by the set

$$\{(\kappa_F\beta)(C)[\delta(v)] \mid v \in V, C \subset E\triangle F\},$$

and $\hat{\omega}$ is determined on elements of this set by (i) and (iii), as long as $D \supset E\triangle F$. The map $\hat{\omega}$ as defined in this proof is easily seen to preserve grading and to commute with the action, proving the final statement in the theorem.

We shall use the functors ρ_C and κ_F to generate a ring K, which will act on the disjoint union $\mathcal{T}^*(G)$ of the $\mathcal{T}^E(G)$, and on the graded group $T^*(G)$ of all $T^E(G)$ making it a graded module. To do this we must find all the general relations which these functors satisfy under composition.

THEOREM 3.4. *There are natural isomorphisms*

(i) $\rho_C\kappa_F \cong \kappa_F\rho_C$;
(ii) $\rho_C\kappa_F \cong \rho_{C\setminus F}\kappa_F$;
(iii) $\kappa_F\kappa_J \cong \bigoplus_{C\subset F\cap J}\kappa_{F\triangle J}\rho_C \cong \kappa_J\kappa_F$.

COROLLARY 3.5. *Letting* $\kappa_\iota := \kappa_{\{\iota\}}$, *we have*

(iv) $\rho_C\kappa_F \cong \kappa_F$ *if* $C \subset F$;
(v) $\kappa_F\kappa_J \cong \kappa_{F\cup J}$ *if* $F \cap J = \emptyset$;
(vi) $\kappa_F^2 \cong \bigoplus_{C\subset F}\rho_C$.
(vii) $\rho_J\kappa_\iota \cong \kappa_\iota\rho_J$; $\rho_\iota\kappa_\iota \cong \kappa_\iota$; $\kappa_\iota^2 \cong \mathrm{id}\oplus\rho_\iota$;
(viii) $\kappa_\iota^3 \cong \kappa_\iota \oplus \kappa_\iota$; $\kappa_\iota\kappa_J \cong \kappa_J\kappa_\iota$.

PROOF. These are all immediate from 3.4. (The reader may have also noticed that (vi) follows by induction on $|F|$ from (v) and the last identity in (vii). Also (iv) follows by induction on $|C|$ from the initial case, $|C| = 1$.)

Before proving 3.4, let us digress to define the ring K.

COROLLARY 3.6. *Reinterpreting the functors* κ_F *and* ρ_C *as operators on the graded abelian group* $T^*(G) := \{T^E(G) \mid E \subset A\}$, *all the identities (i) to (viii) hold with* $\cong$ *replaced by* $=$, *and* $\oplus$ *replaced by* $+$, *and* id *replaced by* 1.

DEFINITION. Temporarily regarding κ_ι as a "variable", define a commutative ring

$$K := \mathbf{Z}[\kappa_\iota : i \in A]/\langle\kappa_\iota^3 - 2\kappa_\iota : i \in A\rangle,$$

with grading over 2^A determined by having κ_ι in grading $\{i\}$. Formally define

$$\rho_\iota = \kappa_\iota^2 - 1, \quad \rho_C = \prod_{\iota\in C}\rho_\iota, \quad \kappa_F = \prod_{\iota\in F}\kappa_\iota.$$

We used (viii) to define K, and now all the relations (i) to (vii) follow formally in K [with replacements as in 3.6]. As an abelian group, the F-th component K^F of K is free abelian with basis

$$\{\rho_C\kappa_F \mid C \cap F = \emptyset\}.$$

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Thus, $K \cong \mathbf{Z}^{3\ell}$ as an abelian group. In fact, K is the tensor power of "ℓ" copies of $\mathbf{Z}[x]/\langle x^3 - 2x\rangle$. By identifying the formal symbols in K with the operators in 3.6, each $T^*(G)$ becomes a 2^A-graded module over the 2^A-graded ring K. Note that we have used "external" gradings for K and $T^*(G)$ [inhomogeneous elements are never considered], whereas it was more convenient for objects V in $\mathcal{T}(G)$ to have "internal" gradings. The fact that 2^A occurs both times as the group of grading parameters is more or less accidental: for example, p^A-graded representations produce a collection of Grothendieck groups which form a 2^A-graded module.

PROOF OF 3.4. First we shall show that, to prove (ii) and (iii), it suffices to give direct proofs of (iv), (v), and (vi). Then we give the proofs of (i), (iv), (v), and (vi).

To deduce (iii) from (i), (iv), (v), and (vi):

$$\begin{aligned}
\kappa_F\kappa_J &\cong (\kappa_{F\setminus J}\kappa_{F\cap J})(\kappa_{J\cap F}\kappa_{J\setminus F}) \text{ by (v)}\\
&\cong (\kappa_{F\setminus J})\Big(\bigoplus_{C\subset F\cap J}\rho_C\Big)(\kappa_{J\setminus F}) \text{ by (vi)}\\
&\cong (\kappa_{F\setminus J}\kappa_{J\setminus F})\Big(\bigoplus_{C\subset F\cap J}\rho_C\Big) \text{ by (i)}\\
&\cong (\kappa_{F\triangle J})\Big(\bigoplus_{C\subset F\cap J}\rho_C\Big) \text{ by (v).}
\end{aligned}$$

To deduce (ii),

$$\rho_C\kappa_F = \rho_{C\setminus F}\rho_{C\cap F}\kappa_F \cong \rho_{C\setminus F}\kappa_F \text{ by (iv).}$$

To prove (i) and (iv), fix C, F and $\mathcal{V} = (V,\beta) \in \mathcal{T}^E(G)$. Consider pairs $(\mathcal{Y},\omega)$ where $\mathcal{Y} = (Y,\gamma) \in \mathcal{T}^{E\triangle F}(G)$ and $\omega\colon \rho_C V \longrightarrow Y$ is a map in $\mathcal{T}(G)$ such that, for all $S \subset E \setminus F$,

$$\gamma(S)\circ\omega = (-1)^{|C\cap S|}\omega\circ\beta(S).$$

Except for the sign $(-1)^{|C\cap S|}$, this gives the universal property defining $\kappa_F\mathcal{V}$. Using that property, it is easily verified that both $(\rho_C\kappa_F\mathcal{V}, \rho_C\delta_{\mathcal{V}})$ and $(\kappa_F\rho_C\mathcal{V}, \delta_{\rho_C\mathcal{V}})$ are universal among such pairs. This proves (i). But when $C \subset F$, the set $C\cap S$ is empty, the sign disappears, and so (iv) follows as well. [Following through the details, the reader can verify that an isomorphism $\varphi\colon \rho_C\kappa_F\mathcal{V} \longrightarrow \kappa_F\rho_C V$ for (i) is given by

$$[\varphi(\zeta)](B) = (-1)^{|B\cap C|}\zeta(B).$$

In particular, although $\rho_C\kappa_F V = \kappa_F\rho_C V$ in $\mathcal{T}(G)$, the identity function is not a map in $\mathcal{T}^{E\triangle F}(G)$, from $\rho_C\kappa_F\mathcal{V}$ to $\kappa_F\rho_C\mathcal{V}$, except when C and F are disjoint.]

To prove (v), we shall show that $\kappa_J\kappa_F\mathcal{V}$ has the universal property characterizing $\kappa_{F\cup J}\mathcal{V}$. For this, the more general form given in 3.3 will be useful. Suppose given (D, Z, γ, ω) such that

$$D \supset E\triangle(F\cup J); \quad D\cap E\cap(F\cup J) = \emptyset; \quad (Z,\gamma) \in \mathcal{T}^D(G);$$

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and $\omega\colon V \longrightarrow Z$ satisfies $\gamma(S) \circ \omega = \omega \circ \beta(S)$ for $S \subset D \cap E$. Consider the diagram

$$\begin{tikzcd}
V \arrow[r, "\delta_{\mathcal{V},F}"] \arrow[dr, "\omega"'] & \kappa_F V \arrow[r, "\delta_{\kappa_F\mathcal{V},J}"] \arrow[d, "\omega_1"] \arrow[d, phantom, shift right=5ex, "(*)"] \arrow[d, phantom, shift left=5ex, "(**)"] & \kappa_J\kappa_F V \arrow[dl, "\omega_2"] \\
& Z &
\end{tikzcd}$$

By considering the object $\big(Z, \{\gamma(S) \mid S \subset D \setminus J\}\big)$ of $\mathcal{T}^{D\setminus J}(G)$, and noting that $(D \setminus J) \cap E \cap F = \emptyset$, $D \setminus J \supset E\triangle F$, and $(D \setminus J) \cap E \subset D \cap E$, it follows from 3.3 [with D in 3.3 changed to $D \setminus J$] that there exists a unique $\mathcal{T}(G)$-map ω_1 such that $(*)$ commutes and $\gamma(S) \circ \omega_1 = \omega_1 \circ (\kappa_F\beta)(S)$ for all $S \subset (D \setminus J) \cap (E\triangle F) = E\triangle F = D \cap (E\triangle F)$. Since $D \cap (E\triangle F) \cap J = \emptyset$, and $D \supset (E\triangle F)\triangle J$ we may apply 3.3 again [this time with E in 3.3 changed to $E\triangle F$, and F in 3.3 changed to J, and $\mathcal{V}$ changed to $\kappa_F\mathcal{V}$] to deduce the existence of a unique $\mathcal{T}(G)$-map ω_2 making $(**)$ commute, and such that $\gamma(S) \circ \omega_2 = \omega_2 \circ (\kappa_J\kappa_F\beta)(S)$ for all $S \subset D \cap (E\triangle F\triangle J)$ [which equals $E\triangle F\triangle J$] as required.

As for (vi), the object $\bigoplus_{C\subset F} \rho_C\mathcal{V}$ is clearly isomorphic to $\lambda_F\mathcal{V} = (\lambda_F V, \lambda_F\beta)$, where

$$\begin{gathered}
\lambda_F V = V^{2^F} \text{ as a vector space,} \\
\text{with grading} (\lambda_F V)_B := \{\eta \in V^{2^F} \mid \eta(C) \in V_{B\triangle C} \text{ for all } C \subset F\}, \\
\text{with action} (g \cdot \eta)(C) := g \cdot \big(\eta(C)\big); \\
\text{and where} (\lambda_F\beta)(S)(\eta)(C) = (-1)^{|S\cap C|}\beta(S)\big(\eta(C)\big) \text{ for } S \subset E.
\end{gathered}$$

We shall realize $\kappa_F V$ as $V^{2^{F\setminus E}}$ as indicated in the remark after 3.2. Thus, as a space,

$$\kappa_F\kappa_F V = (V^{2^{F\setminus E}})^{2^{F\cap E}},$$

since $F \setminus (E\triangle F) = F \cap E$. Define

$$\varphi\colon \lambda_F V \longrightarrow \kappa_F\kappa_F V$$

by

$$\varphi(\eta)(C)(D) = \sum_{\substack{B\subset F \\ B\setminus F=D}} (-1)^{|B\cap C|}\beta[C\triangle(B\cap E)]\big(\eta(B)\big)$$

for all $C \subset F \cap E$ and $D \subset F \setminus E$. Linearity of φ and naturality are easy to see. We must prove that φ is bijective, preserves grading, commutes with the action of G, and that

$$\varphi \circ (\lambda_F\beta)(S) = (\kappa_F\kappa_F\beta)(S) \circ \varphi$$

for all $S \subset E$.

Since $\lambda_F V$ and $\kappa_F\kappa_F V$ both have dimension equal to $(\dim V)^{2^{|F|}}$, bijectivity follows from injectivity. To prove injectivity suppose that $\varphi(\eta)$ is zero. Writing $B = J \cup D$ for $J \subset E \cap F$, the equation $\varphi(\eta)(C)(D) = 0$ becomes

$$\sum_{J\subset F\cap E} (-1)^{|C\cap J|}\beta(J)\,[\eta(J\cup D)] = 0.$$

for all $C \subset F \cap E$ and $D \subset F \setminus E$. But, given a finite set W, a subset $\{v_X : X \subset W\}$ of a vector space, and linear relations $\sum_{X\subset W}(-1)^{|Y\cap X|}v_X = 0$ for all $Y \subset W$, we get $v_Z = 0$ for all $Z \subset W$ by applying $\sum_{Y\subset W}(-1)^{|Y\cap Z|}$ and interchanging summation variables. Thus $\eta(J \cup D) = 0$ for all $J \subset F \cap E$ and $D \subset F \setminus E$, so $\eta = 0$.

If η is in $(\lambda_F V)_R$, then $\beta\big(C\triangle(B\cap E)\big)\big(\eta(B)\big)$ is in $V_{R\triangle B\triangle C\triangle(B\cap E)} = V_{R\triangle C\triangle(B\setminus E)} = V_{R\triangle C\triangle D}$ for all $C \subset F\cap E$, $D \subset F\setminus E$, $B \subset F$ with $B\setminus E = D$. Thus $\varphi(\eta)(C)$ is in $(\kappa_F V)_{R\triangle C}$ for all $C \subset F \cap E$, and so $\varphi(\eta)$ is in $(\kappa_F\kappa_F V)_R$, as required.

The equation

$$\varphi(g\cdot\eta)(C)(D) = \big(g\cdot\varphi(\eta)\big)(C)(D)$$

is an elementary calculation with the definitions.

Finally, iterating the definition of $(\kappa_F\beta)(S)$ yields the formula

$$(\kappa_F\kappa_F\beta)(S)(\zeta)(C)(D) = \beta(S\setminus F)\,\{\zeta[C\triangle(S\cap F)](D)\}\,.$$

Thus

$$\begin{aligned}
[(\kappa_F\kappa_F\beta)(S)\,\circ\,\varphi](\eta)(C)(D)&\\
&= \beta(S\setminus F)\{\varphi(\eta)[C\triangle(S\cap F)](D)\}\\
&= \beta(S\setminus F)\sum_{\substack{B\subset F\\ B\setminus E=D}}(-1)^{|B\cap[C\triangle(S\cap F)]|}\beta[C\triangle(S\cap F)\triangle(B\cap E)]\big(\eta(B)\big)\\
&= \sum_{\substack{B\subset F\\ B\setminus E=D}}(-1)^{|B\cap S|+|B\cap C|}\beta[C\triangle(B\cap E)\triangle S]\big(\eta(B)\big)\\
&= \sum_{\substack{B\subset F\\ B\setminus E=D}}(-1)^{|B\cap C|}\beta[C\triangle(B\cap E)]\{(-1)^{|B\cap S|}\beta(S)\big(\eta(B)\big)\}\\
&= \varphi[(\lambda_F\beta)(S)(\eta)](C)(D),
\end{aligned}$$

as required.

4. **The operation $\overset{\alpha}{\curlyvee}$ on $\mathcal{G}(2^A, m)$.** Now fix $\alpha\colon A \longrightarrow A$ such that $\alpha^2 = \mathrm{id}$. Let (G, y, σ) and (G', y', σ') be objects in $\mathcal{G}(2^A, m)$.

PROPOSITION 4.1. *(i) The following operation on the set $G \times G'$ defines a group $G \times^\alpha_\ell G'$: letting $z = y^m$, define*

$$(g, g')(h, h') = (z^{|\sigma g'\cap\alpha\sigma h|}gh, g'h').$$

(ii) Defining $\overline{\sigma}(g, g') := \sigma(g)\triangle\sigma'(g')$ gives a homomorphism $\overline{\sigma}$ from $G \times^\alpha_\ell G'$ to 2^A.

(iii) The kernel of $\overline{\sigma}$ contains $\{(y^i, y'^j) \mid 1 \le i, j \le 2m\}$, which is a central subgroup of $G \times^\alpha_\ell G'$.

PROOF. (We'll omit the prime on σ' when that on its argument eliminates ambiguity.) To check associativity

$$[(g,g')(h,h')](k,k') = (z^{|\sigma g'\cap\alpha\sigma h|+|\sigma(g'h')\cap\alpha\sigma k|}ghk, g'h'k').$$

Bracketing the other way yields the same answer except that the exponent of z is

$$\begin{aligned} |\sigma h'\cap\alpha\sigma k| + |\sigma g'\cap\alpha\sigma(hk)| &= |\sigma h'\cap\alpha\sigma k| + |(\sigma g'\cap\alpha(\sigma h\triangle\sigma k)| \\ &= |\sigma h'\cap\alpha\sigma k| + |(\sigma g'\cap\alpha\sigma h)\triangle(\sigma g'\cap\alpha\sigma k)| \\ &\equiv |\sigma h'\cap\alpha\sigma k| + |\sigma g'\cap\alpha\sigma h| + |\sigma g'\cap\alpha\sigma k| \pmod 2. \end{aligned}$$

This agrees (mod 2) with the previous exponent of z, as suffices, since $z^2 = 1$. It is easy to see that $(1_G, 1_{G'})$ is the identity element, and that (g,g') has inverse $(z^{|\sigma g'\cap\alpha\sigma g|}g^{-1}, g'^{-1})$. The proof of (ii) and (iii) are equally obvious calculations.

DEFINITION. Let $G \overset{\alpha}{\mathsf{Y}} G'$ be the object in $\mathcal{G}(2^A, m)$ whose underlying group is the quotient

$$(G\times_\ell^\alpha G')/\{(y^{-1}, y')^i \mid 1\le i\le 2m\},$$

with central element $y_{G\overset{\alpha}{\mathsf{Y}}G'}$ equal to the image of $(y,1)$, and with homomorphism $\sigma_{G\overset{\alpha}{\mathsf{Y}}G'}$ obtained by passing to the quotient with $\overline{\sigma}$.

Elements of $G\overset{\alpha}{\mathsf{Y}}G'$ will be denoted by ordered pairs which represent them. The same symbols y and σ will normally be used for the "add-on" components of several different objects in a discussion, as long as no ambiguity threatens. Thus we have

$$\sigma(g,g') = \sigma g\triangle\sigma g'$$

with three meanings for σ, and

$$y = (y,1) = (1,y)$$

with three meanings for y.

REMARK. The formula

$$(g,g')(h,h') = (gh, z'^{|\sigma g'\cap\alpha\sigma h|}g'h')$$

will define a group $G\times_r^\alpha G'$ which in general is not isomorphic to $G\times_\ell^\alpha G'$. It does however produce a quotient group as in the above definition which is naturally isomorphic to $G\overset{\alpha}{\mathsf{Y}}G'$.

DEFINITION. If $\theta\colon G\longrightarrow H$ and $\theta'\colon G'\longrightarrow H'$ are morphisms in $\mathcal{G}(2^A, m)$ define

$$\theta\overset{\alpha}{\mathsf{Y}}\theta'\colon G\overset{\alpha}{\mathsf{Y}}G'\longrightarrow H\overset{\alpha}{\mathsf{Y}}H' \text{ by } (g,g')\longmapsto(\theta g, \theta' g').$$

It is easy to check that this is well defined and behaves with respect to composition and identity maps, yielding the following.

PROPOSITION 4.2. *The above definitions give a functor* $\overset{\alpha}{\curlyvee}\colon \mathcal{G}(2^A, m) \times \mathcal{G}(2^A, m) \rightsquigarrow \mathcal{G}(2^A, m)$.

Let "Tr" = (Tr, y, σ) be the trivial object in which Tr is the cyclic group of order $2m$ generated by y, and σ is the trivial homomorphism. The following basic properties of $\overset{\alpha}{\curlyvee}$ are proved by straightforward calculation.

PROPOSITION 4.3. *There are natural isomorphisms:*

(i) $(G \overset{\alpha}{\curlyvee} H) \overset{\alpha}{\curlyvee} K \longrightarrow G \overset{\alpha}{\curlyvee} (H \overset{\alpha}{\curlyvee} K)$
$\big((g, h), k\big) \longmapsto \big(g, (h, k)\big)$;

(ii) $t\colon G \overset{\alpha}{\curlyvee} H \longrightarrow H \overset{\alpha}{\curlyvee} G$
$(g, h) \longmapsto z^{|\sigma h \cap \alpha\sigma g|}(h, g)$;

(iii) $G \overset{\alpha}{\curlyvee} \mathrm{Tr} \longleftarrow G \longrightarrow \mathrm{Tr} \overset{\alpha}{\curlyvee} G$
$(g, 1) \longleftarrow g \longmapsto (1, g)$.

PROOF OF (ii). We shall do the calculation for this one in order to make a point below. The product of the images of (g, h) and (g', h') is $z^N(hh', gg')$, where

$$N = |\sigma h \cap \alpha\sigma g| + |\sigma h' \cap \alpha\sigma g'| + |\sigma g \cap \alpha\sigma h'|.$$

The image of the product of (g, h) and (g', h') is similar, except that the exponent of z is

$$\begin{aligned}
&|\sigma h \cap \alpha\sigma g'| + |\sigma(hh') \cap \alpha\sigma(gg')| \\
&\quad = |\sigma h \cap \alpha\sigma g'| + |(\sigma h \triangle \sigma h') \cap (\alpha\sigma g \triangle \alpha\sigma g')| \\
&\quad \equiv |\sigma h \cap \alpha\sigma g'| + |\sigma h \cap \alpha\sigma g| + |\sigma h \cap \alpha\sigma g'| + |\sigma h' \cap \alpha\sigma g| + |\sigma h' \cap \alpha\sigma g'| \pmod 2.
\end{aligned}$$

This agrees with N, since $|\sigma g \cap \alpha\sigma h'| = |\sigma h' \cap \alpha\sigma g|$ using the fact that α is an involution.

REMARK. The last little argument shows that it makes no difference if we alter the appearance of the exponent in the definition of the multiplication to $|\sigma h \cap \alpha\sigma g'|$, or of that in 4.3 (ii) to $|\sigma g \cap \alpha\sigma h|$. Note that the power of z is definitely needed in 4.3 (ii).

5. The operation $\overset{\alpha}{\bowtie}$. Continue with $\Lambda = 2^A$ and the involution α on A. First we shall define a binary operation on the category $\mathcal{T}$.

DEFINITION. Given objects V and V' in $\mathcal{T}(G)$ and $\mathcal{T}(G')$ respectively, define an object $V \overset{\alpha}{\otimes} V'$ in $\mathcal{T}(G \overset{\alpha}{\curlyvee} G')$ as follows. The underlying vector space is $V \otimes V'$. Grading is defined by

$$(V \otimes V')_B := \sum_{C \subset A} \oplus (V_C \otimes V'_{B \triangle C}),$$

identifying $V_C \otimes V'_D$ with a subspace of $V \otimes V'$. The action of $G \overset{\alpha}{\curlyvee} G'$ is defined by

$$(g, g') \cdot (v \otimes v') = (-1)^{|\sigma g' \cap \alpha C|} gv \otimes g'v' \text{ for } v \in V_C.$$

PROPOSITION 5.1. $V \overset{\alpha}{\otimes} V'$ *is well defined.*

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PROOF. Evidently we have a well defined graded vector space. The right hand side of the action formula is $v \otimes v'$ when $(g, g') = (y^{-1}, y')^j$ for any j, and is bilinear as a function of (v, v') for each (g, g'). Thus each element of $G \overset{\alpha}{\curlyvee} G'$ gives a well defined linear endomorphism of $V \otimes V'$, which is multiplication by $e^{\imath\pi/m}$ or by 1 when the element is $(y, 1)$ or $(1, 1)$ respectively. If $v \in V_C$ and $v' \in V'_D$, the right side is in $V_{C\triangle\sigma g} \otimes V'_{D\triangle\sigma g'} \subset (V \otimes V')_{C\triangle D\triangle\sigma(g,g')}$ as required. It remains to check associativity of the action. We have, for $v \in V_C$,

$$(h, h') \cdot [(g, g') \cdot (v \otimes v')] = (-1)^N hg \cdot v \otimes h'g' \cdot v'$$
$$\big[(h, h')(g, g')\big] \cdot v \otimes v' = (-1)^M hg \cdot v \otimes h'g' \cdot v'$$

where

$$N = |\sigma g' \cap \alpha C| + |\sigma h' \cap \alpha(C\triangle\sigma g)|$$
$$M = |\sigma h' \cap \alpha\sigma g| + |\sigma(h'g') \cap \alpha C|.$$

These agree mod 2, which suffices by linearity.

DEFINITION. Given morphisms $\varphi: V \longrightarrow W$ in $\mathcal{T}(G)$ and $\varphi': V' \longrightarrow W'$ in $\mathcal{T}(G')$, define $\varphi \overset{\alpha}{\otimes} \varphi'$ to be $\varphi \otimes \varphi': V \otimes V' \longrightarrow W \otimes W'$. (Recall that the underlying vector space of $\overset{\alpha}{\otimes}$ applied to two objects is simply the usual tensor product of the underlying vector spaces of the objects.)

It is easily checked that $\varphi \overset{\alpha}{\otimes} \varphi'$ is a morphism in $\mathcal{T}(G \overset{\alpha}{\curlyvee} G')$ and that $\overset{\alpha}{\otimes}$ behaves properly on compositions and identity morphisms, yielding the first part of the following.

PROPOSITION 5.2. *The above definitions yield a functor* $\overset{\alpha}{\otimes}: \mathcal{T}(G) \times \mathcal{T}(G') \rightsquigarrow \mathcal{T}(G \overset{\alpha}{\curlyvee} G')$. *The following are natural isomorphisms.*

(0) $\rho_B(V \overset{\alpha}{\otimes} W) = V \overset{\alpha}{\otimes} \rho_B W \longrightarrow (\rho_B V) \overset{\alpha}{\otimes} W$
For $w \in W_D$, $v \otimes w \longmapsto (-1)^{|B\cap D|} v \otimes w$,

(i) $(V \oplus W) \overset{\alpha}{\otimes} V' \longleftrightarrow (V \overset{\alpha}{\otimes} V') \oplus (W \overset{\alpha}{\otimes} V')$
$(v, w) \otimes v' \longleftrightarrow (v \otimes v', w \otimes v')$

(ii) $(V \overset{\alpha}{\otimes} V') \overset{\alpha}{\otimes} V'' \longleftrightarrow V \overset{\alpha}{\otimes} (V' \overset{\alpha}{\otimes} V'')$
$(v \otimes v') \otimes v'' \longleftrightarrow v \otimes (v' \otimes v'')$

where V'' *is in* $\mathcal{T}(G'')$ *and we identify* $(G \overset{\alpha}{\curlyvee} G') \overset{\alpha}{\curlyvee} G''$ *with* $G \overset{\alpha}{\curlyvee} (G' \overset{\alpha}{\curlyvee} G'')$ *using the isomorphism given in 4.3 (i).*

(iii) $\tau: V \overset{\alpha}{\otimes} V' \longleftrightarrow V' \overset{\alpha}{\otimes} V$
$v \otimes v' \longleftrightarrow (-1)^{|C\cap\alpha D|} v' \otimes v$

for $v \in V_C$, $v' \in V'_D$, *where we identify* $G \overset{\alpha}{\curlyvee} G'$ *with* $G' \overset{\alpha}{\curlyvee} G$ *using the isomorphism* t *in 4.3 (ii). [It is safer to write this* $V \overset{\alpha}{\otimes} V' \cong t^*(V' \overset{\alpha}{\otimes} V)$*; see the definition of restriction at the start of Section 6.]*

PROOF OF (iii) The checks required are all mechanical, with (iii) being the most interesting in showing that the map commutes with the action The image of (g, g') $(v \otimes v')$ on the right side is

$$(-1)^{|\sigma g \cap \alpha C|}(-1)^{|(C \triangle \sigma g) \cap \alpha(D \triangle \sigma g)|} g'v' \otimes gv,$$

whereas the action of $z^{|\sigma g \cap \alpha \sigma g|}(g', g)$ on the given right side is

$$(-1)^{|C \cap \alpha D|}(-1)^{|\sigma g \cap \alpha \sigma g|}(-1)^{|\sigma g \cap \alpha D|} g'v' \otimes gv$$

The exponents agree (mod 2), as required, using twice that α is an involution

Now restrict the sets E and E' to be in

$$2^{\alpha} = \{B \in 2^A \mid \alpha B = B\}$$

Our objective is to produce a more general operation

$$\overset{\alpha}{\bowtie} \ \mathcal{T}^E(G) \times \mathcal{T}^{E}(G') \rightsquigarrow \mathcal{T}^{E \triangle E}(G \overset{\alpha}{\curlyvee} G')$$

which is, among other things, functorial, associative, and bilinear with respect to the action of the ring

$$K_{\alpha} = \{\rho_C \kappa_F \quad F \in 2^{\alpha}, C \in 2^A\}$$

There will be some choices involved, since, given such an operation $\overset{\alpha}{\bowtie}$, taking $(\mathcal{V}, \mathcal{V}')$ to $\rho_{E \cap E} \mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ can be seen to also have these properties

To begin, let E and E' be any subsets of A, not necessarily invariant under α

DEFINITION OF $\psi(R, R')$ Given objects $\mathcal{V} = (V, \beta)$ in $\mathcal{T}^E(G)$ and $\mathcal{V}' = (V', \beta')$ in $\mathcal{T}^E(G')$, for all $R \subset E$ and $R' \subset E'$, define

$$\psi_{\mathcal{V}\mathcal{V}}(R, R') = \psi(R, R') \quad V \otimes V' \longrightarrow V \otimes V'$$

by, for each subset D of E' ,

$$\psi(R, R')|_{V \otimes V_D} = (-1)^{|R \cap \alpha D|} \beta(R) \otimes \beta'(R')$$

PROPOSITION 5 3 *The function $\psi(R, R')$ is an isomorphism in $\mathcal{T}(G \overset{\alpha}{\curlyvee} G')$ between $V \overset{\alpha}{\otimes} V'$ and $\rho_{R \triangle R}$ $V \overset{\alpha}{\otimes} V'$ If also $T \subset E$ and $T' \subset E'$, then*

$$\begin{aligned} \psi(R \triangle T, R' \triangle T') &= (-1)^{|T \cap \alpha R|} \psi(T, T') \circ \psi(R, R') \\ &= (-1)^{|R \cap \alpha T|} \psi(R, R') \circ \psi(T, T') \end{aligned}$$

COROLLARY 5 4 *For $R \subset E \cap E'$, the automorphism $\psi(R, R)$ of $V \overset{\alpha}{\otimes} V'$ has the properties*

$$\psi(R, R) \circ \psi(T, T) = (-1)^{|R \cap \alpha T|} \psi(R \triangle T, R \triangle T) \ [\textit{for } T \subset E \cap E']$$
$$\psi(R, R)^2 = (-1)^{|R \cap \alpha R|} \mathrm{id}$$
$$\psi(\emptyset, \emptyset) = \mathrm{id}$$

PROOF OF 5.4. The last equation is immediate from the definition, the first is a special case of 5.3, and the middle one is immediate from the other two.

PROOF OF 5.3. Analogous properties of the maps $\beta(R)$ and $\beta'(R')$ give the last equation, the bijectivity of $\psi(R,R')$, and that it alters grading by $R\triangle R'$. To prove that it commutes with the action let $v \in V_C$ and $v' \in V'_D$. Then

$$\begin{aligned} \psi(R,R')[(g,g')\cdot(v\otimes v')] &= (-1)^{|\sigma g' \cap \alpha C|}(-1)^{|R\cap\alpha(D\triangle\sigma g')|}\beta(R)(gv)\otimes\beta'(R')(g'v') \\ (g,g')[\psi(R,R')(v\otimes v')] &= (-1)^{|R\cap\alpha D|}(-1)^{|\sigma g'\cap\alpha(C\triangle R)|}g\cdot\beta(R)(v)\otimes g'\cdot\beta(R')(v'). \end{aligned}$$

But $\beta(R)$ and $\beta'(R')$ commute with the action, and the exponents agree (mod 2), as required.

Now we shall be able to write $V\otimes V'$ as a direct sum of "$2^{|E\cap E'|}$" subspaces, each of which is an intersection of eigenspaces for the operators $\psi(R,R)$. These operators have eigenvalues either ± 1 or $\pm i$, depending on R, by 5.4. To split $V\otimes V'$ we just write down formulae for projectors $\pi^{(C)}\colon V\otimes V' \longrightarrow V\otimes V'$, one for each $C\subset E\cap E'$.

DEFINITION OF $\pi^{(C)}$. With the above notation, let

$$\pi^{(C)} = 2^{-|E\cap E'|}\sum_{D\subset E\cap E'}(-1)^{|D\cap C|}i^{|D\cap\alpha D|}\psi(D,D).$$

PROPOSITION 5.6. *The maps $\pi^{(C)}$ are endomorphisms in $\mathcal{T}(G\overset{\alpha}{\curlyvee}G')$ of $V\overset{\alpha}{\otimes}V'$ satisfying*

(i) $\psi(R,R)\circ\pi^{(C)} = (-1)^{|R\cap C|}(-i)^{|R\cap\alpha R|}\pi^{(C)}$ *for* $R\subset E\cap E'$;

(ii) $\pi^{(C)}\circ\pi^{(B)} = \delta_{B,C}\pi^{(B)}$ *(using the Kronecker delta);*

(iii) $\sum_{C\subset E\cap E'}\pi^{(C)} = \mathrm{id}$.

COROLLARY 5.7. *In $\mathcal{T}(G\overset{\alpha}{\curlyvee}G')$, the object $V\overset{\alpha}{\otimes}V'$ splits as $\bigoplus_{C\subset E\cap E'}\operatorname{Im}\pi^{(C)}$. Furthermore,*

$$\operatorname{Im}\pi^{(C)} = \bigcap_{R\subset E\cap E'}[(-1)^{|R\cap C|}(-i)^{|R\cap\alpha R|}]\text{-}\textit{eigenspace of }\psi(R,R).$$

PROOF OF 5.7. Being linear combinations of the $\mathcal{T}(G\overset{\alpha}{\curlyvee}G')$-morphisms $\psi(D,D)$, the maps $\pi^{(C)}$ are also morphisms, and so have images which are sub-objects of $V\overset{\alpha}{\otimes}V'$. The splitting of $V\overset{\alpha}{\otimes}V'$ is then immediate from 5.6 (ii) and (iii). By 5.6 (i), $\operatorname{Im}\pi^{(C)}$ is a subspace of the given intersection of eigenspaces. But these intersections are a family of linearly independent subspaces, whereas the family $\{\operatorname{Im}\pi^{(C)} \mid C\subset E\cap E'\}$ spans $V\otimes V'$, so we must have equality.

In the Proof of 5.6 there are two identities which are needed later as well. If B and C are sets, then

$$|B\triangle C| = |B| + |C| - 2|B\cap C|.$$

Thus

$$i^{|B\triangle C|} = i^{|B|}i^{|C|}(-1)^{|B\cap C|} \tag{5.8}$$

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If $C \subset B$, then, using the Kronecker delta,

$$\sum_{D \subset B} (-1)^{|C \cap D|} = \delta_{\emptyset, C} 2^{|B|} \tag{5.9}$$

PROOF OF 5.6. (i) By 5.4 and the definition of $\pi^{(C)}$,

$$\begin{aligned}\psi(R,R) \circ \pi^{(C)} &= 2^{-|E \cap E'|} \sum_{D \subset E \cap E'} (-1)^{|D \cap C|} i^{|D \cap \alpha D|} (-1)^{|R \cap \alpha D|} \psi(R \triangle D, R \triangle D)\\ &= 2^{-|E \cap E'|} \sum_{B \subset E \cap E'} (-1)^{|(R \triangle B) \cap C|} i^{|(R \triangle B) \cap \alpha(R \triangle B)|} (-1)^{|R \cap \alpha(R \triangle B)|} \psi(B,B).\end{aligned}$$

Iterating (5.8), the coefficient of $\psi(B,B)$ inside the summation is

$$(-1)^{|R \cap C| + |B \cap C|} i^{|R \cap \alpha R|} i^{|R \cap \alpha B|} i^{|B \cap \alpha R|} i^{|B \cap \alpha B|} (-1)^N (-1)^{|R \cap \alpha R|} (-1)^{|R \cap \alpha B|}$$

where

$$\begin{aligned} N = {} & |R \cap \alpha R \cap R \cap \alpha B| + |R \cap \alpha R \cap B \cap \alpha R| + |R \cap \alpha R \cap B \cap \alpha B|\\ & + |R \cap \alpha B \cap B \cap \alpha R| + |R \cap \alpha B \cap B \cap \alpha B| + |B \cap \alpha R \cap B \cap \alpha B|\\ \equiv {} & |R \cap \alpha R \cap \alpha B| + |R \cap \alpha R \cap B| + |R \cap \alpha B \cap B| + |B \cap \alpha R \cap \alpha B| \pmod 2. \end{aligned}$$

Since α is an involution, we obtain

$$|R \cap \alpha R \cap \alpha B| = |R \cap \alpha R \cap B| \text{ and } |R \cap B \cap \alpha B| = |B \cap \alpha B \cap \alpha R|$$

giving $N \equiv 0 \pmod 2$, and we also obtain $|R \cap \alpha B| = |B \cap \alpha R|$, so the coefficient becomes

$$\begin{aligned}&(-1)^{|R \cap C|} (-1)^{|B \cap C|} i^{|R \cap \alpha R|} (-1)^{|R \cap \alpha B|} i^{|B \cap \alpha B|} (-1)^{|R \cap \alpha R|} (-1)^{|R \cap \alpha B|}\\ &\qquad = (-1)^{|R \cap C|} (-i)^{|R \cap \alpha R|} (-1)^{|B \cap C|} i^{|B \cap \alpha B|}.\end{aligned}$$

Thus

$$\begin{aligned}\psi(R,R) \circ \pi^{C} &= (-1)^{|R \cap C|} (-i)^{|R \cap \alpha R|} 2^{-|E \cap E'|} \sum_{B \subset E \cap E'} (-1)^{|B \cap C|} i^{|B \cap \alpha B|} \psi(B,B)\\ &= (-1)^{|R \cap C|} (-i)^{|R \cap \alpha R|} \pi^{(C)},\end{aligned}$$

as required.

(ii):

$$\begin{aligned}\pi^{(C)} \circ \pi^{(B)} &= 2^{-|E \cap E'|} \sum_{D \subset E \cap E'} (-1)^{|D \cap C|} i^{|D \cap \alpha D|} \psi(D,D) \circ \pi^{(B)}\\ &= 2^{-|E \cap E'|} \sum_{D \subset E \cap E'} (-1)^{|D \cap C|} i^{|D \cap \alpha D|} (-1)^{|D \cap B|} (-i)^{|D \cap \alpha D|} \pi^{(B)}\\ &= 2^{-|E \cap E'|} \pi^{(B)} \sum_{D \subset E \cap E'} (-1)^{|(D \cap B) \triangle (D \cap C)|}\\ &= 2^{-|E \cap E'|} \pi^{(B)} \sum_{D \subset E \cap E'} (-1)^{|D \cap (B \triangle C)|}\\ &= 2^{-|E \cap E'|} \pi^{(B)} \delta_{\emptyset, B \triangle C} 2^{|E \cap E'|} \text{ by (5.9)}\\ &= \delta_{B,C} \pi^{(B)}\end{aligned}$$

(iii):

$$\begin{aligned}\sum_{C\subset E\cap E'} \pi^{(C)} &= 2^{-|E\cap E'|} \sum_{B\subset E\cap E'} i^{|B\cap\alpha B|}\psi(B,B) \sum_{C\subset E\cap E'} (-1)^{|B\cap C|} \\ &= 2^{-|E\cap E'|} \sum_{B\subset E\cap E'} i^{|B\cap\alpha B|}\psi(B,B)\delta_{B,\emptyset}2^{|E\cap E'|} \text{ by (5.9)} \\ &= i^{|\emptyset|}\psi(\emptyset,\emptyset) = \mathrm{id}\,.\end{aligned}$$

DEFINITION. Let C_0 be the set of those elements in $E\cap E'$ which are fixed by α. Abbreviate $\pi^{(C_0)}$ to π, and let $U_{\mathcal{V}\mathcal{V}'} := \mathrm{Im}\,\pi$.

The object $U_{\mathcal{V}\mathcal{V}'}$ of $\mathcal{T}(G \overset{\alpha}{\curlyvee} G')$ will be the first coordinate of $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$. The choice referred to earlier was just made when we chose the image of $\pi^{(C_0)}$ rather than some other $\pi^{(C)}$.

At this point we must take E and E' in 2^α; that is, $\alpha E = E$ and $\alpha E' = E'$. The subgroup 2^α is isomorphic to $(\mathbf{Z}/2)^{\ell_1+\ell_2}$, where A has "ℓ_1" fixed points under α, and "ℓ_2" doubletons interchanged by α (so that $\ell = \ell_1 + 2\ell_2$).

PROPOSITION 5.10. *(i) For $S \subset E\triangle E'$, let $(\beta \overset{\alpha}{\otimes} \beta')(S) = \psi(S\cap E, S\cap E')$. Then the pair $(V \overset{\alpha}{\otimes} V', \beta \overset{\alpha}{\otimes} \beta')$ is an object in $\mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G')$.*

(ii) If $S \subset E\triangle E'$, then π commutes with $\psi(S\cap E, S\cap E')$. In particular, $\psi(S\cap E, S\cap E')$ maps $U_{\mathcal{V}\mathcal{V}'}$ into itself.

PROOF. The first part is immediate from 5.3. As for (ii), it suffices to prove, for each $B \subset E\cap E'$, that $\psi(S\cap E, S\cap E')$ commutes with $\psi(B,B)$. Using 5.4, we need only note that

$$\begin{aligned}|B\cap\alpha(S\cap E')| + |S\cap E\cap\alpha B| &= |\alpha B\cap S\cap E'| + |\alpha B\cap S\cap E| \\ &\equiv |\alpha B\cap S\cap(E\triangle E')| = |\alpha B\cap S| = |\emptyset| = 0 \pmod 2,\end{aligned}$$

since $\alpha B \subset \alpha(E\cap E') = E\cap E'$ and $S \subset E\cup E' \setminus E\cap E'$.

DEFINITION OF $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$. Let $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ be the pair $(U_{\mathcal{V}\mathcal{V}'}, \gamma)$ where $\gamma(S)$ is the restriction to $U_{\mathcal{V}\mathcal{V}'}$ of $(\beta \overset{\alpha}{\otimes} \beta')(S) = \psi(S\cap E, S\cap E')$.

THEOREM 5.11. *(i) The object $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ of $\mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G')$ is well defined.*

(ii) We have isomorphisms in $\mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G')$

$$(V \overset{\alpha}{\otimes} V', \beta \overset{\alpha}{\otimes} \beta') \cong \bigoplus_{C\subset E\cap E'} \rho_C(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}') \cong \kappa^2_{E\cap E'}(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}').$$

PROOF. (i) The map $\gamma(S)$ is a well defined linear map by 5.10. Since $\psi(S\cap E, S\cap E')$ is an isomorphism from $V \overset{\alpha}{\otimes} V'$ to $\rho_{(S\cap E)\triangle(S\cap E')}V \overset{\alpha}{\otimes} V'$ by 5.3, and since $(S\cap E)\triangle(S\cap E') = S$, it follows that $\gamma(S)$ is an isomorphism from $U_{\mathcal{V}\mathcal{V}'}$ to $\rho_S U_{\mathcal{V}\mathcal{V}'}$. It remains to prove that $\gamma(S)\gamma(T) = \gamma(S\triangle T)$ for subsets S and T of $E\triangle E'$. But

$$\psi(S\cap E, S\cap E')\psi(T\cap E, T\cap E') = \psi\big((S\triangle T)\cap E, (S\triangle T)\cap E'\big)$$

by 5.3, as suffices.

(ii) By using 5.7, this will follow if we can find an isomorphism between $\rho_C(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}')$ and $(\operatorname{Im} \pi^{(C_0\triangle\alpha C)}$, restriction of $\beta \overset{\alpha}{\otimes} \beta')$. The map $\psi(C,\emptyset)$ yields such an isomorphism.

PROPOSITION 5.12. *If $\varphi\colon \mathcal{V} \longrightarrow \mathcal{W}$ and $\varphi'\colon \mathcal{V}' \longrightarrow \mathcal{W}'$ are maps in $\mathcal{T}^E(G)$ and $\mathcal{T}^{E'}(G')$, respectively, then $\varphi\otimes\varphi'$ maps $U_{\mathcal{V}\mathcal{V}'}$ into $U_{\mathcal{W}\mathcal{W}'}$. The resulting restriction is a morphism from $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ to $\mathcal{W} \overset{\alpha}{\bowtie} \mathcal{W}'$ in $\mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G')$, which we shall denote $\varphi \overset{\alpha}{\bowtie} \varphi'$. Then we have a functor*

$$\overset{\alpha}{\bowtie}\colon \mathcal{T}^E(G)\times\mathcal{T}^{E'}(G') \longrightarrow \mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G').$$

The proof of each part is a short calculation.

Now we shall give a universal property which characterizes $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ up to a unique isomorphism. Consider the class of all pairs $(\mathcal{W},\nu)$, where $\mathcal{W} = (W,\delta)$ is an object in $\mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G')$ and $\nu\colon V\otimes V' \longrightarrow W$ is a morphism in $\mathcal{T}(G \overset{\alpha}{\curlyvee} G')$ such that the following two diagrams commute:

$$\begin{tikzcd}
V\otimes V' \arrow[dd, "\pi"'] \arrow[dr, "\nu"] & \\
& W \\
V\otimes V' \arrow[ur, "\nu"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V\otimes V' \arrow[r, "\nu"] \arrow[d, "\psi(S\cap E,S\cap E')"'] & W \arrow[d, "\delta(S)"] \\
V\otimes V' \arrow[r, "\nu"'] & W
\end{tikzcd}$$

for all $S\subset E\triangle E'$. [The first is equivalent to the commuting of

$$\begin{tikzcd}
V\otimes V' \arrow[dd, "\psi(B,B)"'] \arrow[dr, "i^{|B\cap\alpha B|}\nu"] & \\
& W \\
V\otimes V' \arrow[ur, "\nu"'] &
\end{tikzcd}$$

for all $B\subset E\cap E'$.]

DEFINITION. Let $\mu\colon V \otimes V' \longrightarrow U_{\mathcal{V}\mathcal{V}'}$ be given by π essentially; *i.e.* $\mu(x) = \pi(x)$; *i.e.* π factors as $V \otimes V' \xrightarrow{\mu} U_{\mathcal{V}\mathcal{V}'} \hookrightarrow V \otimes V'$.

PROPOSITION 5.13. *The pair* $(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}', \mu)$ *is a pair as above and is universal among such pairs in that, given* $(\mathcal{W}, \nu)$ *there is a unique linear map* $\hat{\nu}\colon U_{\mathcal{V}\mathcal{V}'} \longrightarrow W$ *such that* $\nu = \hat{\nu} \circ \mu$. *Furthermore,* $\hat{\nu}$ *is a morphism* $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}' \longrightarrow \mathcal{W}$ *in* $T^{E\triangle E'}(G \overset{\alpha}{\mathrm{Y}} G')$.

PROOF. It is immediate from the construction that $(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}', \mu)$ has the required properties. The existence of a linear map $\hat{\nu}$ with $\nu = \hat{\nu} \circ \mu$ is equivalent to having $\operatorname{Ker} \pi \subset \operatorname{Ker} \nu$, which is immediate from the first diagram which ν satisfies. Since μ is surjective, $\hat{\nu}$ is unique, and is a morphism in $\mathcal{T}(G \overset{\alpha}{\mathrm{Y}} G')$. Now consider the diagram

$$\begin{tikzcd}
V \otimes V' \arrow[dr, "\mu"'] \arrow[drr, "\nu"] \arrow[ddd, "\psi(S\cap E.S\cap E')"'] & & \\
& U \arrow[r, "\hat{\nu}"] \arrow[d, "\gamma(S)"'] & W \arrow[d, "\delta(S)"] \\
& U \arrow[r, "\hat{\nu}"'] & W \\
V \otimes V' \arrow[ur, "\mu"] \arrow[urr, "\nu"'] & &
\end{tikzcd}$$

The triangles, the trapezoid and the entire diagram commute. Hence, so does the square, as required, since μ is surjective.

REMARK. One could alternatively formulate a universal property for pairs $(\mathcal{W}, \underline{\nu})$, for bilinear maps $\underline{\nu}\colon V \times V' \longrightarrow W$ satisfying certain properties. Take $\underline{\nu}$ to be $V \times V' \longrightarrow V \otimes V' \xrightarrow{\nu} W$ to work out what these properties are.

Next are the distributive, associative and "commutative" properties of the operation $\overset{\alpha}{\bowtie}$. Properties (o), (i) and (iii) follow using the universal property, whereas a direct argument (surprisingly intricate, it seems) appears to be easier for proving associativity.

THEOREM 5.14. *There exist natural isomorphisms:*

(o) $\rho_D(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}') \cong \mathcal{V} \overset{\alpha}{\bowtie} \rho_D\mathcal{V}' \cong (\rho_D\mathcal{V}) \overset{\alpha}{\bowtie} \mathcal{V}'$;

(i) $(\mathcal{V}_1 \oplus \mathcal{V}_2) \overset{\alpha}{\bowtie} \mathcal{V}' \cong (\mathcal{V}_1 \overset{\alpha}{\bowtie} \mathcal{V}') \oplus (\mathcal{V}_2 \overset{\alpha}{\bowtie} \mathcal{V}')$ *for* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *in* $\mathcal{T}^E(G)$, *and* $\mathcal{V}'$ *in* $\mathcal{T}^{E'}(G')$;

(ii) $(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}') \overset{\alpha}{\bowtie} \mathcal{V}'' \cong \mathcal{V} \overset{\alpha}{\bowtie} (\mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}'')$ *for* $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{V}''$ *in* $\mathcal{T}^E(G)$, $\mathcal{T}^{E'}(G')$, $\mathcal{T}^{E''}(G'')$, *respectively, [with* E'', E' *and* E *all invariant under* α, *of course];*

(iii) $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}' \cong \rho_{C_0}\mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}$ *where* $G \overset{\alpha}{\curlyvee} G'$ *is identified with* $G' \overset{\alpha}{\curlyvee} G$ *by means of the isomorphism* $(g, g') \overset{t}{\longmapsto} z^{|\sigma g' \cap \alpha\sigma g|}(g', g)$ *of 4.3 (ii).*

[Recall that C_0 consists of those a in $E \cap E'$ with $\alpha a = a$. Referring forward to Section 6, it is better to write this $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}' \cong t^*\rho_{C_0}\mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}$.]

PROOF. (o) This may be proved using the universal property, by considering maps which shift grading by D. Alternatively, the isomorphisms of 5.2 (o) preserve the direct sum decompositions of 5.7.

(i) By abstract nonsense, taking $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ in the universal property, it suffices to find a map

$$\zeta \colon (V_1 \oplus V_2) \otimes V' \longrightarrow U_{\mathcal{V}_1\mathcal{V}'} \oplus U_{\mathcal{V}_2\mathcal{V}'}$$

such that the pair $[(\mathcal{V}_1 \overset{\alpha}{\bowtie} \mathcal{V}') \oplus (\mathcal{V}_2 \overset{\alpha}{\bowtie} \mathcal{V}'), \zeta]$ is universal. Such a map is determined by

$$\zeta \colon (v_1, v_2) \otimes v' \longmapsto (\pi_{\mathcal{V}_1\mathcal{V}'}(v_1 \otimes v'), \pi_{\mathcal{V}_2\mathcal{V}'}(v_2 \otimes v')),$$

as may be readily checked. [We have subscripted the maps π in the obvious way, to distinguish them.]

(iii) The map [using τ from 5.2 (iii)]

$$\lambda \colon V \otimes V' \xrightarrow{\ \tau\ } V' \otimes V \xrightarrow{\psi(C_0,\emptyset)} V' \otimes V \xrightarrow{\mu_{V'V}} U_{\mathcal{V}'\mathcal{V}}$$

gives a pair $(t^*\rho_{C_0}\mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}, \lambda)$ which is universal in the same context that $(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}', \mu_{VV'})$ is universal, as suffices. To check this, first note that λ is a morphism in $\mathcal{T}^{E\triangle E'}(G \overset{\alpha}{\curlyvee} G')$ since it is a composite

$$V \overset{\alpha}{\otimes} V' \longrightarrow t^*(V' \overset{\alpha}{\otimes} V) \longrightarrow \rho_{C_0}t^*(V' \overset{\alpha}{\otimes} V) \longrightarrow t^*\rho_{C_0}U_{V'V}$$

of such morphisms. The commutativity of the first required diagram for λ is, for $B \subset E \cap E'$,

$$\begin{tikzcd}
V \otimes V' \arrow[r, "\tau"] \arrow[d, "\psi_{VV'}(B.B)"'] & V' \otimes V \arrow[r, "\psi(C_0.\emptyset)"] \arrow[d, "(-1)^{|B\cap\alpha B|}\psi_{V'V}(B{,}\,B)"'] & V' \otimes V \arrow[dr, "\mu_{V'V}"] \arrow[d, "\psi_{V'V}(B.B)"'] & \\
& & & U_{\mathcal{V}'\mathcal{V}} \\
V \otimes V' \arrow[r, "\tau"'] & V' \otimes V \arrow[r, "\psi(C_0.\emptyset)"'] & V' \otimes V \arrow[ur, "\mu_{V'V}"'] &
\end{tikzcd}$$

The triangle commutes, being one of the diagrams in the universal property for $\mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}$. The middle square commutes by 5.3. The left square commutes since both paths send $v \otimes v' \in V_C \otimes V'_D$ to $(-1)^N \beta'(B)(v') \otimes \beta(B)(v)$ with $N = |B \cap \alpha B| + |C \cap \alpha B| + |C \cap \alpha D|$. The commutativity of the second diagram is, for $S \subset E \triangle E'$,

$$\begin{tikzcd}
V \otimes V' \arrow[r, "\tau"] \arrow[d] & V' \otimes V \arrow[r, "\psi(C_0, \emptyset)"] \arrow[d] & V' \otimes V \arrow[r, "\mu_{\nu'\nu}"] \arrow[d] & U_{\nu'\nu} \arrow[d] \\
V \otimes V' \arrow[r, "\tau"'] & V \otimes V' \arrow[r, "\psi(C_0, \emptyset)"'] & V' \otimes V \arrow[r, "\mu_{\nu'\nu}"'] & U_{\nu'\nu}
\end{tikzcd}$$

where, from left to right, the vertical arrows are $\psi_{\mathcal{V},\mathcal{V}'}(S \cap E, S \cap E')$, $\psi_{\mathcal{V}'\mathcal{V}}(S \cap E', S \cap E)$ twice, and $\gamma_{\mathcal{V}'\mathcal{V}}(S)$. Verification from right-to-left is similar to the previous one: by a diagram for the universality of $\mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}$, by 5.3, and by a direct calculation, this time the answer being

$$(-1)^{|C \cap \alpha D| + |S \cap E' \cap \alpha C|} \beta'(S \cap E')(v') \otimes \beta(S \cap E)(v),$$

and depending on E and E' being invariant under α. Finally, we check that λ is universal. Given $(\mathcal{W}, \nu)$ consider the diagram

$$\begin{tikzcd}
V \otimes V' \arrow[rr, "\tau"] \arrow[drrr, "\nu"'] & & V' \otimes V \arrow[rr, "\psi(C_0, \emptyset)"] \arrow[dr, "\varphi_1"] & & V' \otimes V \arrow[rr, "\mu_{\nu\nu'}"] \arrow[dl, "\varphi_2"'] & & U_{\nu\nu'} \arrow[dlll, "\varphi"] \\
& & & W & & &
\end{tikzcd}$$

Linear maps φ_1, then φ_2, then φ, exist uniquely to make the triangles commute because the maps at the top are isomorphisms in the first two cases and universal in the case of φ.

(ii) Rather than attempting to formulate a universal property which both ("tri") functors satisfy, we shall give some explicit formulae. First note that, by 5.4, for a, b in $E \cap E'$ and $R \subset E \cap E'$,

$$\psi(a, a) \circ \psi(b, b) = \psi(b, b) \circ \psi(a, a)$$

and

$$\psi(R, R) = (-1)^{\mathrm{forb}(R)} \prod_{a \in R} \psi(a, a)$$

where forb(R) is the number of two element sets $\{a, \alpha a\}$ contained in R. It follows that

$$U_{\mathcal{V}\mathcal{V}'} = \bigcap_{a \in E \cap E'} I_a\big(\psi(a, a)\big)$$

where $I_a(T)$ is the $i^{|a\cap\alpha a|}$-eigenspace of T. Given a third object $\mathcal{V}'' = (V'', \beta'') \in \mathcal{T}^{E''}(G'')$, define endomorphisms $\psi_{12}(a)$, $\psi_{23}(a)$ and $\psi_{13}(a)$ of $V \otimes V' \otimes V''$ by

$$\psi_{12}(a) = \psi_{vv'}(a, a) \otimes \mathrm{id} \ \text{ for } a \in E \cap E'$$
$$\psi_{23}(a) = \mathrm{id} \otimes \psi_{v'v''}(a, a) \text{ for } a \in E' \cap E''$$

and

$$[\psi_{13}(a)](v \otimes v' \otimes v'') = (-1)^{|a\cap(C\triangle D)|}\beta(a)(v) \otimes v' \otimes \beta''(a)(v'')$$

for $v \in V$, $v' \in V'_C$, $v'' \in V''_D$ and $a \in E \cap E''$. Now

$$\begin{aligned}
&U_{\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}', \mathcal{V}''} \\
&= \bigcap_{a\in(E\triangle E')\cap E''} I_a\Big(\psi_{\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}', \mathcal{V}''}(a, a)\Big) \\
&= \Big[\bigcap_{a\in E'\cap E''\setminus E\cap E''} I_a\Big(\psi_{23}(a, a)\Big)\Big] \cap \Big[\bigcap_{a\in E\cap E''\setminus E'\cap E''} I_a\Big(\psi_{13}(a, a)\Big)\Big] \cap [U_{\mathcal{V},\mathcal{V}'} \otimes V''] \\
&= \Big[\bigcap_{\substack{a\in E'\cap E'' \\ a\notin E\cap E'\cap E''}} I_a\Big(\psi_{23}(a, a)\Big)\Big] \cap \Big[\bigcap_{\substack{a\in E\cap E'' \\ a\notin E\cap E'\cap E''}} I_a\Big(\psi_{13}(a, a)\Big)\Big] \cap \Big[\bigcap_{a\in E\cap E'} I_a\Big(\psi_{12}(a, a)\Big)\Big].
\end{aligned}$$

Similarly, removing the condition $a \notin E \cap E' \cap E''$ from the left term and placing it on the right term gives a formula for $U_{\mathcal{V}, \mathcal{V}' \overset{\alpha}{\bowtie} \mathcal{V}''}$. Now let

$$\varphi = \prod_{b\in E\cap E'\cap E''} \Big(\psi_{12}(b) + \psi_{23}(b)\Big).$$

Using the following identities (readily verified using 5.4):

$$\psi_{12}(a)^2 = \psi_{23}(a)^2 = (-1)^{|a\cap\alpha a|};$$
$$\psi_{12}(a) \circ \psi_{23}(a) = (-1)^{|a\cap\alpha a|}\psi_{23}(a) \circ \psi_{12}(a);$$

and for $a \neq b$, letting $[S, T] = S \circ T - T \circ S$,

$$\begin{aligned}
[\psi_{12}(a), \psi_{12}(b)] &= [\psi_{23}(a), \psi_{23}(b)] = [\psi_{12}(a), \psi_{23}(b)] \\
&= [\psi_{13}(a), \psi_{12}(b)] = [\psi_{13}(a), \psi_{23}(b)] = 0;
\end{aligned}$$

we see that φ is independent of the order in the product, that it maps $U_{\mathcal{V}\bowtie\mathcal{V}', \mathcal{V}''}$ into $U_{\mathcal{V}, \mathcal{V}'\bowtie\mathcal{V}''}$, and that

$$\varphi^2 = (-2)^{|E\cap E'\cap E''|}\,\mathrm{id}\,.$$

It is clear that φ preserves grading and commutes with the action since each of $\psi_{12}(b)$ and $\psi_{23}(b)$ do. Thus it remains only to check that φ commutes with the maps $\beta_{\mathcal{V}\bowtie\mathcal{V}', \mathcal{V}''}(S)$ and $\beta_{\mathcal{V}, \mathcal{V}'\bowtie\mathcal{V}''}(S)$ for all $S \subset E\triangle E'\triangle E''$. This follows using 5.3.

DEFINITION. The collection $\{T^E(G) : E \in 2^\alpha\}$, which we shall denote $T^\alpha(G)$, is a group graded over 2^α, and in fact is a graded module over the 2^α-graded ring K_α defined after 5.2.

By 5.12, the isomorphism class of $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ depends only on the isomorphism classes of $\mathcal{V}$ and of $\mathcal{V}'$. By 5.14 (i), the operation $\overset{\alpha}{\bowtie}$, after passing to isomorphism classes, extends biadditively to all of $T^E(G) \times T^{E'}(G')$, and so yields

$$\overset{\alpha}{\bowtie}: T^\alpha(G) \times T^\alpha(G') \longrightarrow T^\alpha(G \overset{\alpha}{\curlyvee} G')$$

[which is K_α-bilinear as we see in Section 9]. It is immediate that the associative and "commutative" laws of 5.14 (ii) and (iii) hold also for this version of $\overset{\alpha}{\bowtie}$. Either the "commutativity", or a direct argument, proves the other distributive law (linearity in the second variable), which is needed for the above extension of $\overset{\alpha}{\bowtie}$ from $\mathcal{T}$ to T.

6. **Restricting and inducing.** In this section, Λ will again be a general abelian group. Let $\theta: G \longrightarrow H$ be a morphism in $\mathcal{G}(\Lambda, m)$. Let W be an object in $\mathcal{T}(H)$. Define $\theta^* W$, the *restriction of W along θ*, to be the following object in $\mathcal{T}(G)$: as a graded vector space, $\theta^* W := W$, and the action of G is $g \cdot w := \theta(g) \cdot w$.

It is immediate that $\theta^* W$ is in $\mathcal{T}(H)$. Now suppose further that $\mathcal{W} = (W, \beta)$ is an object in $\mathcal{T}^\Gamma(H)$ for some subgroup Γ of Λ. Define $\theta^*\mathcal{W}$ to be $(\theta^* W, \beta)$. Finally, if $\varphi: W_1 \longrightarrow W_2$ is a morphism in $\mathcal{T}^\Gamma(H)$ from $\mathcal{W}_1$ to $\mathcal{W}_2$, let $\theta^*(\varphi) := \varphi$. The following is then immediate.

PROPOSITION 6.1. *(i) Each θ yields a functor*

$$\theta^*: \mathcal{T}^\Gamma(H) \longrightarrow \mathcal{T}^\Gamma(G).$$

(ii) Sending G to $\mathcal{T}^\Gamma(G)$, and θ to θ^*, gives a contravariant functor from $\mathcal{G}(\Lambda, m)$ to the category whose objects are categories and whose morphisms are covariant functors.

The next proposition is proved directly from the definitions without difficulty.

PROPOSITION 6.2. *(i) If $\mathcal{W}'$ is also in $\mathcal{T}^\Gamma(H)$, then*

$$\theta^*(\mathcal{W} \oplus \mathcal{W}') = (\theta^*\mathcal{W}) \oplus (\theta^*\mathcal{W}').$$

(ii) When $\Lambda = 2^A$ and $\Gamma = 2^E$, the functors θ^* commute with both ρ_C and κ_C for all $C \subset A$.

PROPOSITION 6.3. *Given morphisms $\theta: G \longrightarrow H$ and $\theta_1: G_1 \longrightarrow H_1$ in $\mathcal{G}(2^A, m)$, an involution α on A, objects $\mathcal{W}$ and $\mathcal{W}_1$ in $\mathcal{T}^E(H)$ and $\mathcal{T}^{E_1}(H_1)$ respectively for subsets E and E_1 invariant under α, we have*

$$(\theta \overset{\alpha}{\curlyvee} \theta_1)^*(\mathcal{W} \overset{\alpha}{\bowtie} \mathcal{W}_1) = (\theta^*\mathcal{W}) \overset{\alpha}{\bowtie} (\theta_1^*\mathcal{W}_1).$$

The proof is simply to observe that none of the ingredients in constructing $\overset{\alpha}{\bowtie}$ differ when comparing $\mathcal{W} \overset{\alpha}{\bowtie} \mathcal{W}_1$ to $(\theta^*\mathcal{W}) \overset{\alpha}{\bowtie} (\theta_1^*\mathcal{W}_1)$; the group action itself is not used directly in the construction.

To define induced graded representations, assume $\theta: G \longrightarrow H$ is an *injective* map in $\mathcal{G}(\Lambda, m)$. Let $\mathcal{V} = (V, \beta)$ be an object in $\mathcal{T}^\Gamma(G)$.

DEFINITION. An *inducing* of $\mathcal{V}$ to H is a pair $(\theta_*\mathcal{V}, \iota)$, where $\theta_*\mathcal{V} = (W, \gamma)$ is an object in $\mathcal{T}^\Gamma(H)$ and $\iota: V \longrightarrow W$ is a $\mathcal{T}^\Gamma(G)$-map from $\mathcal{V}$ to $\theta^*\mathcal{W}$ which is universal among pairs $(\mathcal{Y}, \zeta)$ of objects $\mathcal{Y} = (Y, \delta)$ in $\mathcal{T}^\Gamma(H)$ and $\mathcal{T}^\Gamma(G)$-maps $\zeta: V \longrightarrow \theta^* Y$. That is, there is a unique $\mathcal{T}^\Gamma H$-map $\hat{\zeta}: W \longrightarrow Y$ with $\zeta = \hat{\zeta} \circ \iota$. The object $\theta_*\mathcal{V}$ is said to be *induced from* $\mathcal{V}$.

The usual abstract nonsense shows that, modulo the question of existence of $\theta_*\mathcal{V}$, we have

$$\theta_*(\mathcal{V} \oplus \mathcal{V}') \cong (\theta_*\mathcal{V}) \oplus (\theta_*\mathcal{V}'),$$
$$\mathrm{id}_* = \mathrm{id},$$

and

$$(\theta_1 \circ \theta_2)_* = \theta_{1_*} \circ \theta_{2_*}.$$

The definition may be summarized by

$$\mathrm{Map}_{\mathcal{T}^\Gamma(H)}(\theta_*\mathcal{V}, \mathcal{W}) \cong \mathrm{Map}_{\mathcal{T}^\Gamma(G)}(\mathcal{V}, \theta^*\mathcal{W}).$$

This reciprocity relation is generalized in Proposition 7.8.

THEOREM 6.4. *(i) Each injective θ defines a functor*

$$\theta_*: \mathcal{T}^\Gamma(G) \longrightarrow \mathcal{T}^\Gamma(H).$$

(ii) Sending G to $\mathcal{T}^\Gamma(G)$ and θ to θ_* gives a covariant functor to the category of categories from $\mathcal{G}!(\Lambda, m)$, the category obtained by removing the non-injective maps from $\mathcal{G}(\Lambda, m)$.

PROOF. It remains to construct an inducing $(\theta_*\mathcal{V}, \iota)$ of each $\mathcal{V}$, and to define θ_* on morphisms in $\mathcal{T}^\Gamma(G)$. The latter is done by the diagram

$$\begin{tikzcd} V_1 \arrow[r, "\iota_1"] \arrow[d, "\varphi"'] & W_1 \arrow[d, "\theta_*\varphi"] \\ V_2 \arrow[r, "\iota_2"'] & W_2 \end{tikzcd} \qquad \begin{array}{l} \text{where } \theta_*\mathcal{V}_1 = (W_1, \gamma_1); \\ \\ \text{where } \theta_*\mathcal{V}_2 = (W_2, \gamma_2). \end{array}$$

A morphism $\theta_*\varphi$ in $\mathcal{T}^\Gamma(H)$ exists and is unique by the universal property for $\theta_*\mathcal{V}_1$, which also easily yields functoriality of θ_*.

As for existence, given $\mathcal{V} = (V, \beta)$ define $\theta_*\mathcal{V} = (\theta_*V, \theta_*\beta)$[$= (W, \gamma)$ say] as follows. Let $W := V^{H/\theta G}$ as a vector space. To define the action of H and the grading, pick a cross section

$$s: H/\theta G \longrightarrow H,$$

that is, $s[k]\theta(G) = [k] := k\theta(G)$ for all $k \in H$ Define the grading by having ξ in W_B if and only if $\xi[k]$ is in $V_{B\Delta\sigma(s[k])}$ for all $[k]$. Define the action by

$$(h \cdot \xi)[k] = \theta^{-1}(s[k]^{-1}hs[h^{-1}k]) \cdot \xi[h^{-1}k].$$

It is straightforward to verify that we have a well-defined object in $\mathcal{T}(H)$. Define $\gamma = \theta_* \beta$ by

$$\big(\gamma(S)(\xi)\big)[k] = \alpha(S)(\xi[k]) \text{ for } S \in \Gamma.$$

Then (W, γ) is an object in $\mathcal{T}^{\Gamma}(H)$. Finally, define $\iota\colon V \longrightarrow W$ by

$$\big(\iota(v)\big)[k] = \begin{cases} v & \text{if } k \in \theta(G); \\ 0 & \text{otherwise.} \end{cases}$$

To prove universality, given $\mathcal{Y} = (Y, \delta)$ as in the definition, and given $\zeta\colon V \longrightarrow \theta^* Y$, define $\hat{\zeta}$ by

$$\hat{\zeta}(\xi) = \sum_{[k] \in H/\theta(G)} s[k] \cdot h(\xi[k]).$$

Checking that $\zeta = \hat{\zeta} \cdot \iota$ and that $\hat{\zeta}$ is in $\mathcal{T}^{\Gamma}(H)$ are mechanical. Uniqueness of $\hat{\zeta}$ is also straightforward.

PROPOSITION 6.5. *Taking $\Lambda = 2^A$ and $\Gamma = 2^E$ for $E \subset A$, we have natural isomorphisms for all C and F:*

(i) $\theta_* \rho_C \cong \rho_C \theta_*$

(ii) $\theta_* \kappa_F \cong \kappa_F \theta_*$.

PROOF. The proof of (i) is easy. As for (ii), let $\mathcal{V} = (V, \beta) \in \mathcal{T}^E(G)$. Using the universal properties in 3.2 for κ_F and in the definition for θ_*, let

$$\theta_* \mathcal{V} = (W_1, \gamma_1) \in \mathcal{T}^E(H) \text{ with } \iota_1\colon V \longrightarrow W_1;$$
$$\kappa_F \theta_* \mathcal{V} = (U_1, \alpha_1) \in \mathcal{T}^{E \triangle F}(H) \text{ with } \delta_1\colon W_1 \longrightarrow U_1;$$
$$\kappa_F \mathcal{V} = (W_2, \gamma_2) \in \mathcal{T}^{T \triangle F}(G) \text{ with } \delta_2\colon V \longrightarrow W_2;$$
$$\theta_* \kappa_F \mathcal{V} = (U_2, \alpha_2) \in \mathcal{T}^{E \triangle F}(H) \text{ with } \iota_2\colon W_2 \longrightarrow U_2.$$

Then it is easy to check that both the pairs $(\kappa_F \theta_* \mathcal{V}, V \xrightarrow{\delta_1 \iota_1} U_1)$ and $(\theta_* \kappa_F \mathcal{V}, V \xrightarrow{\iota_2 \delta_2} U_2)$ are universal in the class of pairs $(\mathcal{Z}, V \xrightarrow{\omega} Z)$ for $\mathcal{Z} = (Z, \pi) \in \mathcal{T}^{E \triangle F}(H)$ and $\mathcal{T}^{E \setminus F}(G)$-maps ω, where $\mathcal{V}$ and $\mathcal{Z}$ are made into $\mathcal{T}^{E \setminus F}(G)$-objects by forgetting $\beta(S)$ and $\pi(S)$ for any S not in $E \setminus F$ (that is, we are considering the objects $\kappa_{E \cap F} \mathcal{V}$ and $\kappa_{F \setminus E} \theta^* \mathcal{Z}$). Universality means the existence of a unique $\mathcal{T}^{E \triangle F}(H)$-map $\hat{\omega}$ from U_1 or U_2 into Z which factors ω. In each case, one applies the two universal properties, of κ_F and of inducing, to verify this. It follows that (ii) holds.

Reverting to general Λ and Γ, the functoriality of θ^* and θ_* implies that they can operate on isomorphism classes. Since they commute with direct sums, we obtain group homomorphisms

$$\theta^*\colon T^{\Gamma}(H) \longrightarrow T^{\Gamma}(G) \text{ for all } \theta\colon G \longrightarrow H;$$

and

$$\theta_*\colon T^{\Gamma}(G) \longrightarrow T^{\Gamma}(H) \text{ for injective } \theta\colon G \longrightarrow H.$$

When $\Lambda = 2^A$ and $\Gamma = 2^E$, these maps commute with the module action of the ring K. For each involution α on A, the maps θ^* commute with the biadditive maps $\overset{\alpha}{\bowtie}$.

7. Infernal Hom. Again specialize Λ to 2^A, where A has a given involution α, and specialize Γ to $2^E, 2^F, \ldots$ for α-invariant subsets $E, F, \ldots$ of A. Let $\mathcal{T}^\alpha(G)$ denote the disjoint union of the categories $\mathcal{T}^E(G)$ for sets E invariant under α. The construction in this section will later be used to define a K_α-valued inner product on $\mathcal{T}^\alpha(G)$, where K_α is the subring of K defined after 5.2. Sooner than that, it will be used to prove the central result in Part I, which describes all objects of $\mathcal{T}^\alpha(G \overset{\alpha}{\curlyvee} G')$ by using the operation $\overset{\alpha}{\bowtie}$.

Suppose given $\mathcal{V} = (V, \beta) \in \mathcal{T}^E(G)$ and $\mathcal{W} = (W, \gamma) \in \mathcal{T}^F(G \overset{\alpha}{\curlyvee} L)$, where G and L are both objects in $\mathcal{G}(2^A, m)$. We aim to define an object

$$\text{“}\mathcal{H}\text{”} = \mathcal{H}^\alpha(\mathcal{V}, \mathcal{W}) \in \mathcal{T}^{E \triangle F}(L)$$

in such a way that the relation of $\mathcal{H}$ to $\overset{\alpha}{\bowtie}$ will be analogous to that of Hom to $\otimes$.

DEFINITION. For $B \subset A$, let H_B be the space of all linear maps $\varphi \colon V \longrightarrow W$ such that:
(i) $\varphi(V_C) \subset W_{C \triangle B}$ for all $C \subset A$;
(ii) $\varphi(g \cdot v) = (g, 1) \cdot \varphi(v)$ for all $g \in G$ and $v \in V$;
(iii) For all $S \subset E \cap F$, the following commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ (-1)^{|B \cap \alpha S|}\beta(S) \downarrow & & \downarrow \gamma(S)\,. \\ V & \xrightarrow{\varphi} & W \end{array}$$

Thus, when $\alpha = \mathrm{id}$, the space H_B consists of all $\mathcal{T}^{E \cap F}(G)$-maps from $\kappa_{E \setminus F}\mathcal{V}$ to $\kappa_{F \setminus E}\rho_B \iota^* \mathcal{W}$, where $\iota \colon G \longrightarrow G \overset{\alpha}{\curlyvee} L$ is the embedding, $g \longmapsto (g, 1)$, of 4.3. When $E \cap F = \emptyset$, the space H_B consists of all $\mathcal{T}(G)$-maps from V to $\rho_B W$.

Let H be the subspace of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ spanned by $\cup_B H_B$. It becomes a graded vector space, since the H_B are clearly linearly independent subspaces by (i). Define an action of L on H as follows. For $\ell \in L$ and $v \in V_C$, let

$$(\ell \cdot \varphi)(v) := (-1)^{|\sigma\ell \cap \alpha C|}(1, \ell) \cdot \varphi(v).$$

PROPOSITION 7.1. *This action is well-defined, making H into an object in $\mathcal{T}(L)$.*

PROOF. The map $\ell \cdot \varphi$ is certainly linear. Let $\varphi \in H_B$ and $v \in V_C$. Then

$$(\ell \cdot \varphi)(v) \in W_{C \triangle B \triangle \sigma\ell} = (\rho_{B \triangle \sigma\ell} W)_C,$$

so that $\ell \cdot \varphi$ behaves with respect to the grading. Also

$$\begin{aligned} (\ell \cdot \varphi)(g \cdot v) &= (-1)^{|\sigma\ell \cap \alpha(C \triangle \sigma g)|}(1, \ell) \cdot \varphi(g \cdot v) \\ &= (-1)^{|\sigma\ell \cap \alpha(C \triangle \sigma g)|}[z^{|\sigma\ell \cap \alpha\sigma g|}(g, \ell)] \cdot \varphi(v) \\ &= (-1)^{|\sigma\ell \cap \alpha C|}(g, \ell) \cdot \varphi(v) \\ &= (g, 1) \cdot [(-1)^{|\sigma\ell \cap \alpha C|}(1, \ell) \cdot \varphi(g \cdot v)] \\ &= (g, 1) \cdot [(\ell \cdot \varphi)(v)], \end{aligned}$$

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verifying (ii) in the definition. As for (iii),

$$\begin{aligned}[\gamma(S)\circ(\ell\cdot\varphi)](v) &= (-1)^{|\sigma\ell\cap\alpha C|}\gamma(S)[(1,\ell)\cdot\varphi(v)]\\ &= (-1)^{|\sigma\ell\cap\alpha C|}(1,\ell)\cdot\gamma(S)\big(\varphi(v)\big),\\ &\qquad \text{since } \gamma(S) \text{ commutes with the action of } G\overset{\alpha}{\curlyvee}L\\ &= (-1)^{|\sigma\ell\cap\alpha C|}(-1)^{|B\cap\alpha S|}(1,\ell)\varphi[\beta(S)(v)], \text{ by (iii) for } \varphi\,,\end{aligned}$$

whereas

$$[(\ell\cdot\varphi)\circ\beta(S)](v) = (-1)^{|\sigma\ell\cap\alpha(C\triangle S)|}(1,\ell)\cdot\varphi[\beta(S)(v)].$$

The last two exponents differ (mod 2) by $|(B\triangle\sigma\ell)\cap\alpha S|$, as required. To complete the proof, $(\ell'\ell)\cdot\varphi = \ell'\cdot(\ell\cdot\varphi)$ is easy to verify, and $1\cdot\varphi=\varphi$ is obvious.

To obtain an object $\mathcal{H}=(H,\delta)$ in $\mathcal{T}^{E\triangle F}(L)$, let $\varphi\in H_B$ and $R\subset E\triangle F$, and define

$$\delta(R)(\varphi) = (-1)^{|E\cap B\cap\alpha R|}i^{|E\cap R\cap\alpha R|}\gamma(R\cap F)\circ\varphi\circ\beta(R\cap E).$$

PROPOSITION 7.2. *This defines an object in $\mathcal{T}^{E\triangle F}(L)$, whose full name will be $\mathcal{H}^{\alpha}(\mathcal{V},\mathcal{W})$.*

PROOF. First we shall show that $\delta(R)(\varphi)$ above is in $H_{B\triangle R}$. For this the scalar factor $\pm i^N$ in the definition is irrelevant. We certainly have a linear map from V to W, taking V_C into $W_{C\triangle B\triangle R}$, and commuting with the action of G (since each of the three factors do). To check (iii), use the diagram

$$\begin{tikzcd}
V \arrow[r, "\beta(R\cap E)"] \arrow[d, "(-1)^{|(B\triangle R)\cap\alpha S|}\beta(S)"'] & V \arrow[r, "\varphi"] \arrow[d, "(-1)^{|B\cap\alpha S|}\beta(S)"] & W \arrow[r, "\gamma(R\cap F)"] \arrow[d, "\gamma(S)"] & W \arrow[d, "\gamma(S)"] \\
V \arrow[r, "\beta(R\cap E)"'] & V \arrow[r, "\varphi"'] & W \arrow[r, "\gamma(R\cap F)"'] & W
\end{tikzcd}$$

The right and left squares commute by defining properties of γ and β, using also that

$$R\cap\alpha S\subset(E\triangle F)\cap\alpha(E\cap F) = (E\triangle F)\cap E\cap F=\emptyset.$$

The middle square is (iii) for φ.

Next we check that $\delta(R)$ commutes with the action of L. Let $v\in V_C$. Then

$$\begin{aligned}&[\delta(R)(\ell\cdot\varphi)](v)\\ &\quad= i^{|E\cap R\cap\alpha R|}(-1)^{|E\cap(B\triangle\sigma\ell)\cap\alpha R|}[\gamma(R\cap F)\circ(\ell\cdot\varphi)\circ\beta(R\cap E)](v)\\ &\quad= i^{|E\cap R\cap\alpha R|}(-1)^{|E\cap B\cap\alpha R|+|E\cap\sigma\ell\cap\alpha R|+|\sigma\ell\cap\alpha[C\triangle(R\cap E)]|}\gamma(R\cap F)\{(1,\ell)\cdot[\varphi\circ\beta(R\cap E)](v)\}\\ &\quad= i^{|E\cap R\cap\alpha R|}(-1)^{|E\cap B\cap\alpha R|+|\sigma\ell\cap\alpha C|}(1,\ell)\cdot[\gamma(R\cap F)\circ\varphi\circ\beta(R\cap E)](v)\\ &\quad= (-1)^{|\sigma\ell\cap\alpha C|}(1,\ell)\cdot[\delta(R)(\varphi)](v)\\ &\quad= (\ell\cdot[\delta(R)(\varphi)])(v)\end{aligned}$$

as required.

Since $\delta(\emptyset)$ is the identity map, it remains only to check that $\delta(R)\circ\delta(T)=\delta(R\triangle T)$ for all subsets R and T of $E\triangle F$. Let $\varphi\in H_B$. Then

$$\begin{aligned}[\delta(R)\circ\delta(T)](\varphi) &= (-1)^{|E\cap B\cap\alpha T|}i^{|E\cap T\cap\alpha T|}\delta(R)[\gamma(T\cap F)\circ\varphi\circ\beta(T\cap E)]\\ &= (-1)^{|E\cap B\cap\alpha T|+|E\cap(B\triangle T)\cap\alpha R|}i^{|E\cap T\cap\alpha T|}i^{|E\cap R\cap\alpha R|}\eta,\end{aligned}$$

where

$$\eta=\gamma(R\cap F)\circ\gamma(T\cap F)\circ\varphi\circ\beta(T\cap E)\circ\beta(R\cap E)=\gamma[(R\triangle T)\cap F]\circ\varphi\circ\beta[(R\triangle T)\cap E].$$

On the other hand,

$$\delta(R\triangle T)(\varphi)=(-1)^{|E\cap B\cap\alpha(R\triangle T)|}i^{|E\cap(R\triangle T)\cap\alpha(R\triangle T)|}\eta.$$

These agree, as required, by the following calculation:

$$i^{|E\cap(R\triangle T)\cap\alpha(R\triangle T)|}=(-1)^N i^{|E\cap R\cap\alpha R|}i^{|E\cap R\cap\alpha T|}i^{|E\cap T\cap\alpha R|}i^{|E\cap T\cap\alpha T|},$$

where N is a sum of six terms which are the cardinalities of the intersections of the four exponents of i in pairs [as in the proof of 5.6 (i)]. This yields zero (mod 2). Also, the middle two powers of i are equal, so their product is $(-1)^{|E\cap T\cap\alpha R|}$ as required.

DEFINITION. Continue with the previous notation. Since $E\triangle(E\triangle F)=F$, we have, in $\mathcal{T}^F(G\overset{\alpha}{\curlyvee}L)$, an object $\mathcal{V}\overset{\alpha}{\otimes}\mathcal{H}=(V\overset{\alpha}{\otimes}H,\beta\overset{\alpha}{\otimes}\delta)$ from section 5. It is a direct sum of objects $\rho_C(\mathcal{V}\overset{\alpha}{\bowtie}\mathcal{H})$ as C ranges over subsets of $E\cap(E\triangle F)=E\setminus F$. Let the linear map

$$\nu\colon V\otimes H\longrightarrow W$$

be the evaluation map, determined by sending $v\otimes\varphi$ to $\varphi(v)$.

PROPOSITION 7.3. *The map ν is a morphism in $\mathcal{T}^F(G\overset{\alpha}{\curlyvee}L)$ from $\mathcal{V}\overset{\alpha}{\otimes}\mathcal{H}$ to $\mathcal{W}$.*

COROLLARY 7.4. *Let $\mathcal{W}$ be a non-zero object in $\mathcal{T}^F(G\overset{\alpha}{\curlyvee}L)$. Then there are objects $\mathcal{V}$ in $\mathcal{T}^F(G)$ and $\mathcal{H}$ in $\mathcal{T}^{\emptyset}(L)$, and a non-zero $\mathcal{T}^F(G\overset{\alpha}{\curlyvee}L)$-morphism from $\mathcal{V}\overset{\alpha}{\bowtie}\mathcal{H}[=\mathcal{V}\overset{\alpha}{\otimes}\mathcal{H}]$ to $\mathcal{W}$.*

PROOF OF 7.4. In the previous discussion, let $E=F$. Let $\mathcal{V}$ be any object (for example, $i^*\mathcal{W}$ itself) in $\mathcal{T}^F(G)$ for which a non-zero $\mathcal{T}^E(G)$ morphism $\varphi\colon\mathcal{V}\longrightarrow i^*\mathcal{W}$ exists. Such a φ is in $H_\emptyset$, and ν is non-zero, and is in $\mathcal{T}^F(G\overset{\alpha}{\curlyvee}L)$ by 7.3.

PROOF OF 7.3. Let $v\in V_C$ and $\varphi\in H_B$. Then $\nu(v\otimes\varphi)=\varphi(v)$ is in $W_{B\triangle C}$, so ν preserves the grading. It also commutes with the action of $G\overset{\alpha}{\curlyvee}L$, since

$$\begin{aligned}\nu[(g,\ell)\cdot(v\otimes\varphi)] &= \nu[(-1)^{|\sigma\ell\cap\alpha C|}(g\cdot v)\otimes(\ell\cdot\varphi)]\\ &= (-1)^{|\sigma\ell\cap\alpha C|}(\ell\cdot\varphi)(g\cdot v)\\ &= (-1)^{|\sigma\ell\cap\alpha C|}(g,1)\cdot[(\ell\cdot\varphi)(v)] \text{ since } \ell\cdot\varphi \text{ is in } H_B\\ &= (g,1)\cdot[(1,\ell)\cdot\varphi(v)] \text{ by definition of } \ell\cdot\varphi\\ &= (g,\ell)\cdot\varphi(v)\\ &= (g,\ell)\cdot\nu(v\otimes\varphi)\end{aligned}$$

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as required. It remains to prove that for $S \subset F$, the following diagram commutes

$$\begin{tikzcd}
V \otimes H \arrow[r, "\nu"] \arrow[d, "(\beta \overset{\alpha}{\otimes} \delta)(S) = \psi[S\cap E, S\cap(E\triangle F)]"'] & W \arrow[d, "\gamma(S)"] \\
V \otimes H \arrow[r, "\nu"'] & W
\end{tikzcd}$$

Letting $\varphi \in H_B$, we have, since $S \cap (E\triangle F) = S \setminus E$,

$$\begin{aligned}
[\nu \circ (\beta \overset{\alpha}{\otimes} \delta)(S)](v \otimes \varphi) &= (-1)^{|S\cap E\cap\alpha B|}\nu[\beta(S\cap E)(v) \otimes \delta(S \setminus E)(\varphi)] \\
&= (-1)^{|S\cap E\cap\alpha B|}\delta(S \setminus E)(\varphi)[\beta(S\cap E)(v)] \\
&= (-1)^{|S\cap E\cap\alpha B|}(-1)^{|E\cap B\cap\alpha(S\setminus E)|}i^{|E\cap(S\setminus E)\cap\alpha(S\setminus E)|}\gamma[(S \setminus E)\cap F] \\
&\qquad \circ \varphi \circ \beta[(S\setminus E)\cap E] \circ \beta(S\cap E)(v) \\
&= (-1)^{|S\cap E\cap\alpha B|}\gamma(S\setminus E) \circ \varphi \circ \beta(S\cap E)(v) \\
&= (-1)^{|S\cap E\cap\alpha B|}(-1)^{|B\cap\alpha(S\cap E)|}\gamma(S\setminus E) \\
&\qquad \circ \gamma(S\cap E) \circ \varphi(v) \text{ by (iii) for } \varphi \\
&= \gamma(S) \circ \varphi(v) = \gamma(S)[\nu(v\otimes\varphi)], \text{ as required.}
\end{aligned}$$

PROPOSITION 7.5. *The map ν factors as*

$$\mathcal{V} \overset{\alpha}{\otimes} \mathcal{H} \xrightarrow{\mu} \mathcal{V} \overset{\alpha}{\bowtie} \mathcal{H} \xrightarrow{\mathcal{E}} \mathcal{W},$$

where $\mathcal{E}$ is a map in $\mathcal{T}^F(L)$.

PROOF. The last part of the previous proof checked the second diagram in the universal property after 5.12 for $\overset{\alpha}{\bowtie}$, so it remains only to check the first. That is, for all $S \subset E \cap (E\triangle F) = E \setminus F$,

$$\begin{tikzcd}
V \otimes H \arrow[dr, "i^{|S\cap\alpha S|}\nu"] \arrow[dd, "\psi_{\mathcal{V}\mathcal{H}}(S,S)"'] & \\
& V \otimes H \arrow[dl, "\nu"] \\
W &
\end{tikzcd}$$

commutes. To see this, for $v \in V$ and $\varphi \in H_B$ we have, with δ as defined before 7.2,

$$\begin{aligned}[\nu \circ \psi(S,S)](v \otimes \varphi) &= \nu[(-1)^{|S \cap \alpha B|}\beta(S)(v) \otimes \delta(S)(\varphi)] \\ &= (-1)^{|S\cap\alpha B|}\delta(S)(\varphi)[\beta(S)(v)] \\ &= (-1)^{|S\cap\alpha B|}(-1)^{|E\cap B\cap\alpha S|}i^{|E\cap S\cap\alpha S|}\gamma(S\cap F)\circ\varphi\circ\beta(S\cap E)[\beta(S)(v)] \\ &= i^{|S\cap\alpha S|}\varphi(v) \text{ since } S\cap F=\emptyset \text{ and } S\cap E = S \\ &= i^{|S\cap\alpha S|}\nu(v\otimes\varphi),\end{aligned}$$

as required.

To describe an adjointness relating $\overset{\alpha}{\bowtie}$ to $\mathcal{H}^\alpha$, it is convenient to first relate $\overset{\alpha}{\otimes}$ to a larger object Hom^α, which decomposes as a direct sum over $C \subset E\cap F$ of $\rho_C \circ \mathcal{H}^\alpha$ [just as $\overset{\alpha}{\otimes}$ decomposed in terms of $\rho_C \circ \overset{\alpha}{\bowtie}$]. This discussion below also makes condition (iii) in the definition of $\mathcal{H}^\alpha$ appear to arise more naturally.

With $\mathcal{V}$ and $\mathcal{W}$ as before, let $\mathrm{Hom}^\alpha(\mathcal{V},\mathcal{W})$ be defined [as a graded L-representation] to have B-th component equal to the subspace of $\mathrm{Hom}_C(V,W)$ defined by (i) and (ii) in the definition of $\mathcal{H}^\alpha(\mathcal{V},\mathcal{W})$, *i.e.*

(i) $\varphi(V_C) \subset W_{C\triangle B}$ for all $C \subset A$;

(ii) $\varphi(g\cdot v) = (g,1)\cdot\varphi(v)$ for all $g \in G$ and $v \in V$;

and with the same formula for the action of L, *i.e.*

$$(\ell\cdot\varphi)(v) := (-1)^{|\sigma\ell\cap\alpha C|}(1,\ell)\cdot\varphi(v).$$

If $R \subset E$ and $T \subset F$, let

$$\xi(R,T)\colon \mathrm{Hom}^\alpha(\mathcal{V},\mathcal{W})_B \longrightarrow \mathrm{Hom}^\alpha(\mathcal{V},\mathcal{W})_{B\triangle R\triangle T}$$

be given by

$$\xi(R,T)(\varphi) = (-1)^{|B\cap\alpha R|}\gamma(T)\circ\varphi\circ\beta(R).$$

It is easy to check that $\xi(R,T)$ is well-defined, and, because of the sign $(-1)^{|B\cap\alpha R|}$, it commutes with the action of L. Furthermore

$$\xi(R_1,T_1)\circ\xi(R_2,T_2) = (-1)^{|(R_2\triangle T_2)\cap\alpha R_1|}\xi(R_1\triangle R_2, T_1\triangle T_2).$$

It follows that $\mathrm{Hom}^\alpha(\mathcal{V},\mathcal{W})$ can be made into an object in $\mathcal{T}^{E\triangle F}(L)$ by taking the structure maps, for $R \subset E\triangle F$, to be $i^{|E\cap R\cap\alpha R|}\xi(R\cap E, R\cap F)$. [The power of i is needed to make the maps compose correctly.] These maps commute with $\xi(S,S)$ for each $S \subset E\cap F$, and $\xi(S,S)^2 = 1$. Then $\mathcal{H}^\alpha(\mathcal{V},\mathcal{W})$ is given from this point of view as the intersection over all $S \subset E\cap F$ of the $+1$-eigenspace of $\xi(S,S)$. [This is (iii) in the definition of $\mathcal{H}^\alpha$.] The appropriate projectors are

$$2^{-|E\cap F|}\sum_{S\subset E\cap F}(-1)^{|C\cap S|}\xi(S,S),$$

one for each $C \subset E \cap F$, the case $C = \emptyset$ projecting onto $\mathcal{H}^\alpha$. For general C, the above projector maps onto the intersection over all $B \subset E \cap F$ of the $(-1)^{|B\cap C|}$-eigenspace of $\psi(B, B)$, giving an object isomorphic to $\rho_C \mathcal{H}^\alpha(\mathcal{V}, \mathcal{W})$. Thus

$$\mathrm{Hom}^\alpha(\mathcal{V}, \mathcal{W}) \cong \bigoplus_{C\subset E\cap F} \rho_C \mathcal{H}^\alpha(\mathcal{V}, \mathcal{W}).$$

Now for adjointness, suppose given objects $\mathcal{V}$ in $\mathcal{T}^E(G)$, $\mathcal{V}'$ in $\mathcal{T}^{E'}(G')$ and $\mathcal{W}$ in $\mathcal{T}^F(G \overset{\alpha}{\curlyvee} G' \overset{\alpha}{\curlyvee} L)$, where, of course, each of E, E' and F is invariant under α. The following proposition will have its proof described rather than presented in detail, since we won't use it in Part I [and a less tedious proof may emerge].

PROPOSITION 7.6. *There is a commutative diagram in which the horizontal maps are isomorphisms in $\mathcal{T}^{E\triangle E'\triangle F}(L)$, and the vertical maps are given by using the above projectors, and on the left, the inclusion of $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ into $\mathcal{V} \overset{\alpha}{\otimes} \mathcal{V}'$:*

$$\begin{array}{ccc} \mathrm{Hom}^\alpha(\mathcal{V} \overset{\alpha}{\otimes} \mathcal{V}', \mathcal{W}) & \cong & \mathrm{Hom}^\alpha[\mathcal{V}', \mathrm{Hom}^\alpha(\mathcal{V}, \mathcal{W})] \\ \downarrow & & \downarrow \\ \mathcal{H}^\alpha(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}', \mathcal{W}) & \cong & \mathcal{H}^\alpha[\mathcal{V}', \mathcal{H}^\alpha(\mathcal{V}, \mathcal{W})] \end{array}$$

One can see that the existence of an isomorphism on the bottom is not unreasonable, given one on the top, as follows. Using the decompositions of $\overset{\alpha}{\otimes}$ and Hom^α in terms of $\overset{\alpha}{\bowtie}$ and $\mathcal{H}^\alpha$, both upper objects can be seen to be direct sums of ρ_C applied to the corresponding lower object over all $C \subset (E \cap E') \cup (E \cap F) \cup (E' \cap F)$. This part of the proof is made precise by giving the upper isomorphism explicitly, and then proving that there are linear maps both ways at the bottom which make the diagram commute. Because the vertical maps are surjective, these lower maps are unique, are mutually inverse and are morphisms in $\mathcal{T}^{E\triangle E'\triangle F}(L)$.

Now it is easy to give isomorphisms at the top which are in $\mathcal{T}(L)$, using the usual adjointness, namely

$$\mathcal{B}(\varphi)(v')(v) = \varphi(v \otimes v') \text{ for } \varphi \text{ on the left;}$$
$$\mathcal{B}^{-1}(\psi)(v \otimes v') = \psi(v')(v) \text{ for } \psi \text{ on the right.}$$

Then $\mathcal{B}$ and $\mathcal{B}^{-1}$ are certainly mutually inverse if well defined, and checking well-definition, and behaviour re grading and action of L, are straightforward. When one comes to check that $\mathcal{B}$ commutes with the structure maps corresponding to $S \subset E\triangle E'\triangle F$, there is a scalar factor problem if $S \cap E \cap E' \cap F \neq \emptyset$, and more seriously, initially a mystery if $E \cap E' \cap F \neq \emptyset$. The following is the key point in the proof [and led to the alteration of the section title from "Internal Hom"]. On either upper object there is a map $\eta(S, S', T)$, for each $S \subset E$, $S' \subset E'$, and $T \subset F$, defined using $\beta(S)$, $\beta'(S')$ and $\gamma(T)$, where $\mathcal{V} = (V, \beta)$, $\mathcal{V}' = (V', \beta')$ and $\mathcal{W} = (W, \gamma)$. For example, on the left

$$\eta(S, S', T)(\varphi)(v \otimes v') = (-1)^{|C'\cap\alpha S|+|B\cap\alpha(S\triangle S')|}\gamma(T)\,\{\varphi[\beta(S)(v) \otimes \beta'(S')(v')]\}$$

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for $\varphi \in H_B$, $v' \in V'_{C'}$. This map commutes with the action and changes grading by $S \triangle S' \triangle T$. In particular, the intersections of eigenspaces of $\eta(S, S', S\triangle S')$, for $S \subset E$ and $S' \subset E'$ such that $S\triangle S' \subset F$, provide decompositions. Much of this information has already been used up in decomposing Hom^α and $\overset{\alpha}{\otimes}$ in terms of $\mathcal{H}^\alpha$ and $\overset{\alpha}{\bowtie}$. Essentially the extra information is contained in the operators $\eta(S, S', S\triangle S')$ where both S and S' are subsets of $E \cap E' \cap F$. One can then define an adjointness isomorphism for the top line of the diagram in the proposition by using a different multiple of $\mathcal{B}$ for each summand arising from these last operators. Once the appropriate multiples are chosen, the remainder of the proof is mechanical, but tedious. It is hoped that a more palatable proof will be found.

A second result, not needed till Part II, with a similar proof, is "*coadjointness*". Suppose given objects: $\mathcal{V} = (V, \beta)$ in $\mathcal{T}^E(G)$; $\mathcal{W} = (W, \gamma)$ in $\mathcal{T}^F(G \overset{\alpha}{\curlyvee} L)$; and $\mathcal{W}' = (W', \gamma')$ in $\mathcal{T}^{F'}(G')$. Then there is a $\mathcal{T}(L \overset{\alpha}{\curlyvee} G')$ isomorphism

$$\mathcal{C}\colon \mathrm{Hom}^\alpha(\mathcal{V}, \mathcal{W}) \overset{\alpha}{\otimes} \mathcal{W}' \longrightarrow \mathrm{Hom}^\alpha(\mathcal{V}, \mathcal{W} \overset{\alpha}{\otimes} \mathcal{W}')$$

given by $\mathcal{C}(\varphi \otimes w')(v) = \varphi(v) \otimes w'$. As with the adjointness, a problem arises with the structure maps for $S \subset E \cap F \cap F'$, as well as a scalar factor problem for all S if $E \cap F \cap F' \neq \emptyset$. The solution is exactly analogous to the case of adjointness. Both the domain and codomain of $\mathcal{C}$ admit self maps, indexed by $(S \subset E, T \subset F, T' \subset F')$, which preserve degree when $S = T \triangle T'$. The extra information, beyond that used to decompose $\overset{\alpha}{\otimes}$ in terms of $\overset{\alpha}{\bowtie}$ and Hom^α in terms of $\mathcal{H}^\alpha$, is essentially contained in the self maps when T, T' and $S = T\triangle T'$ are subsets of $E \cap F \cap F'$. These define a decomposition of the domain and codomain of $\mathcal{C}$. Multiplying $\mathcal{C}$ by a suitable scalar factor on each summand, the following result may be proved.

PROPOSITION 7.7. *There are $\mathcal{T}^{E\triangle F \triangle F'}(L \overset{\alpha}{\curlyvee} G)$ isomorphisms giving a commutative diagram*

$$\begin{array}{ccc}
\mathrm{Hom}^\alpha(\mathcal{V}, \mathcal{W}) \overset{\alpha}{\otimes} \mathcal{W}' & \cong & \mathrm{Hom}^\alpha(\mathcal{V}, \mathcal{W} \overset{\alpha}{\otimes} \mathcal{W}') \\
\downarrow & & \downarrow \\
\mathcal{H}^\alpha(\mathcal{V}, \mathcal{W}) \overset{\alpha}{\bowtie} \mathcal{W}' & \cong & \mathcal{H}^\alpha(\mathcal{V}, \mathcal{W} \overset{\alpha}{\bowtie} \mathcal{W}')
\end{array}$$

Generalizing the formula before 6.4, essentially the definition of inducing, is the following reciprocity formula.

PROPOSITION 7.8. *Suppose that $\theta\colon G \longrightarrow G'$ is an injective map in $\mathcal{G}(2^A, m)$, and that we have objects $\mathcal{V}$ in $\mathcal{T}^E(G)$ and $\mathcal{W}$ in $\mathcal{T}^F(G' \overset{\alpha}{\curlyvee} L)$. Then $\mathcal{H}(\theta_* \mathcal{V}, \mathcal{W})$ and $\mathcal{H}(\mathcal{V}, (\theta \overset{\alpha}{\curlyvee} \mathrm{id})^* \mathcal{W})$ are isomorphic in $\mathcal{T}^{E\triangle F}(L)$.*

PROOF. Let

$$\mathcal{V} = (V, \beta);\ \ \mathcal{W} = (W, \gamma);\ \ (H, \delta) = \mathcal{H} = \mathcal{H}(\theta_* \mathcal{V}, \mathcal{W});\ \text{and}$$
$$(H', \delta') = \mathcal{H}' = \mathcal{H}\big(\mathcal{V}, (\theta \overset{\alpha}{\curlyvee} \mathrm{id})^* \mathcal{W}\big).$$

Given φ in H'_B, we use the construction in 6.4 of $\theta_* V$ as $V^{G'/\theta G}$ to define $\varphi_+ : \theta_* V \longrightarrow W$ by

$$\varphi_+(\xi) = \sum_{[y] \in G'/\theta G} s([y]) \cdot \varphi(\xi[y])$$

where $s: G'/\theta G \longrightarrow G'$ is a fixed section with $s[1] = 1$. Straightforward calculations show that

(i) if ξ is in $(\theta_* V)_C$, then $\varphi_+(\xi)$ is in $W_{C \triangle B}$;

(ii) $\varphi_+(g' \cdot \xi) = g' \cdot \varphi_+(\xi)$;

(iii) for all $S \subset E \cap F$,

$$\gamma(S) \circ \varphi_+ = (-1)^{|B \cap \alpha S|} \varphi_+ \circ (\theta_* \beta)(S).$$

Thus φ_+ is in H_B, as required.

Conversely, given ψ in H_D, define $\psi_- : V \longrightarrow W$ by $\psi_-(v) = \psi(\xi_v)$, where

$$\xi_v[y] = \begin{cases} v & \text{if } [y] = [1]; \\ 0 & \text{if not.} \end{cases}$$

It is easily seen that ψ_- is in H'_D, and that $(\psi_-)_+ = \psi$ and $(\varphi_+)_- = \varphi$ for all ψ and φ. Thus we have a pair of mutually inverse, linear, gradation preserving maps. To check that $\ell \cdot (\psi_-) = (\ell \cdot \psi)_-$, evaluate both sides at $v \in V_C$, yielding $(-1)^{|\sigma \ell \cap \alpha C|}(1, \ell) \cdot \psi_-(v)$. It remains only to check that $[\delta(R)(\psi)]_- = \delta'(R)(\psi_-)$ for all $R \subset E \triangle F$. This reduces to

$$\xi_{\beta(R \cap E)(v)}[y] = \beta(R \cap E)(\xi_v[y])$$

[which is immediate from the definition of ξ_v], since the desired equality, when evaluated at v, then at $[y]$, becomes $\gamma(R \cap F) \circ \psi$ applied to the equation above.

8. **Irreducibles for $G \overset{\alpha}{\curlyvee} G'$.** The elementary aspects of classical representation theory carry over to $\mathcal{T}^\Gamma$ almost verbatim. Here Γ and Λ are general once again. The following could be deduced from the "real world" interpretation of Section 2, but deductions directly analogous to the methods for ordinary representations are probably simplest. This has been delayed till now to emphasize the independence of all the previous machinery from decompositions into irreducibles, *etc.*

A *sub-object* (W, γ) of $(V, \beta) \in \mathcal{T}^\Gamma(G)$ is a subspace W of V which is invariant under G [and with that restricted action]; such that $W = \sum \oplus_B (V_B \cap W)$ [and with grading $W_B = V_B \cap W$]; such that, for $S \in \Gamma$, $\beta(S)(W) \subset W$ [and with $\gamma(S)$ equal to a restriction of $\beta(S)$]. It follows that (W, γ) is also in $\mathcal{T}^\Gamma(G)$. An equivalent definition is essentially that the inclusion of W into V is a morphism in $\mathcal{T}^\Gamma(G)$.

DEFINITION. An object (V, β) is *irreducible* if and only if it has exactly two sub-objects. It follows that they are the zero object and (V, β) itself, and that $V \neq \{0\}$.

If (W, γ) is a sub-object of (V, β), then there is a quotient object $(V/W, \delta)$, with the following structure:

$$(V/W)_B = (V_B + W)/W = \text{ image of } V_B/W_B;$$
$$g \cdot (v + W) = (g \cdot v) + W;$$

and

$$\delta(S)(v + W) = \beta(S)(v) + W.$$

Then $(V/W, \delta) \in \mathcal{T}^\Gamma(G)$, and the inclusion and projection

$$W \hookrightarrow V \longrightarrow V/W$$

are morphisms in $\mathcal{T}^\Gamma(G)$.

The usual averaging trick shows that, when G is finite, any $\mathcal{T}^\Gamma(G)$-morphism $V \xrightarrow{\varphi} U$ which is surjective has a right inverse $U \xrightarrow{\psi} V$ in $\mathcal{T}^\Gamma(G)$. To see this, one chooses any linear gradation-preserving right inverse $\eta\colon U \longrightarrow V$, and then one sets

$$\psi(u) = |G|^{-1} \sum_{g \in G} g \cdot \eta(g^{-1} \cdot u).$$

Since kernels and images of $\mathcal{T}^\Gamma(G)$-morphisms are evidently sub-objects of the domains and codomains respectively, it follows that any sub-object (W, γ) of (V, β) has a complementary sub-object (W_1, γ_1): Let $U = V/W$ with φ being the projection, and then let W_1 be the image of ψ. Since all spaces are finite dimensional, it follows that every object is a direct sum of (finitely many) irreducible objects.

By the remark above about kernels and images, the usual proof of Schur's lemma is valid in this context. Thus any non-zero $\mathcal{T}^\Gamma(G)$-morphism between two irreducible objects is an isomorphism, and any $\mathcal{T}^\Gamma(G)$- endomorphism of an irreducible is a scalar multiple of an identity map.

DEFINITION. Given two objects (V_i, β_i) in $\mathcal{T}^\Gamma(G)$, let $\langle (V_1, \beta_1), (V_2, \beta_2) \rangle_{\mathbf{Z}}$ be the dimension of the vector space of all $\mathcal{T}^\Gamma(G)$-morphisms between them; say, from V_1 to V_2.

The notation $\langle\, , \rangle$ will be reserved for a K-bilinear inner product with values in K (to be defined later), which specializes to $\langle\, , \rangle_{\mathbf{Z}}$, at least when $\Gamma = 2^E$, $\Lambda = 2^A$, by composing with a map $K \longrightarrow \mathbf{Z}$.

It is elementary to check that $\langle\, , \rangle_{\mathbf{Z}}$ is bi-additive with respect to $\oplus$. If $\mathcal{V}$ and $\mathcal{W}$ are irreducible, the above analogue of Schur's lemma yields

$$\langle \mathcal{V}, \mathcal{W} \rangle_{\mathbf{Z}} = \begin{cases} 1 \text{ if } \mathcal{V} \cong \mathcal{W}; \\ 0 \text{ if not.} \end{cases}$$

In particular, $\langle\, , \rangle_{\mathbf{Z}}$ is symmetric, and it follows that the irreducible summands $\mathcal{V}_i$ in a decomposition

$$\mathcal{V} \cong \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots$$

are unique up to order and isomorphism. Thus

PROPOSITION 8.1. *The group $T^\Gamma(G)$ is free abelian, a basis consisting of the isomorphism classes of irreducibles in $\mathcal{T}^\Gamma(G)$ [of which there are finitely many], when G is finite.*

Now we can prove the initial version of the main result of this part.

THEOREM 8.2. *Let $\Lambda = 2^A$, where A has a given involution α, let G and L be finite objects in $\mathcal{G}(2^A, m)$, and let $\mathcal{W}$ be any irreducible in $\mathcal{T}^E(G \overset{\alpha}{\curlyvee} L)$ where E is α-invariant. Then there exist irreducibles $\mathcal{V}$ in $\mathcal{T}^E(G)$ and $\mathcal{V}'$ in $\mathcal{T}^\emptyset(L)$ such that $\mathcal{W}$ occurs as a summand in $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$.*

PROOF. Let $\mathcal{V}$ be any irreducible occurring in $\iota^*\mathcal{W}$, where $\iota: G \longrightarrow G \overset{\alpha}{\curlyvee} L$ is the embedding $g \longmapsto (g, 1)$. Thus there is a non-zero $\mathcal{T}^E(G)$-morphism from $\mathcal{V}$ to $\iota^*\mathcal{W}$. By 7.4, there is an object $\mathcal{H}$ in $\mathcal{T}^\emptyset(L)$ and a non-zero $\mathcal{T}^E(G \overset{\alpha}{\curlyvee} L)$-morphism from $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{H}$ to $\mathcal{W}$. When restricted to the sub-object $\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ for at least one irreducible summand $\mathcal{V}'$ of $\mathcal{H}$, it remains non-zero [indeed surjective] by Schur's lemma. By using a right inverse $\mathcal{W} \longrightarrow \mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'$ in $\mathcal{T}^E(G \overset{\alpha}{\curlyvee} L)$, the result follows.

REMARK. Since

$$(\rho_F\mathcal{V}) \overset{\alpha}{\bowtie} (\rho_F\mathcal{V}') \cong \rho_F^2\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}' = \mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}',$$

there are trivial reasons why $\mathcal{V}$ and $\mathcal{V}'$ will not necessarily be unique when $\ell > 0$. We shall see later that uniqueness does hold modulo this K-bilinearity of $\overset{\alpha}{\bowtie}$. Furthermore

$$(\kappa_F\mathcal{V}) \overset{\alpha}{\bowtie} (\kappa_F\mathcal{V}') \cong {\kappa_F}^2\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}' \cong \bigoplus_{C \subset F} (\rho_C\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}'),$$

as we see in the next section, so a $\overset{\alpha}{\bowtie}$-product of irreducibles may not be irreducible. When $\ell = 1$, this is essentially the only such annoyance, but, as we shall see in Part II, for $\ell > 1$ there are also less obvious reasons for this to happen.

9. **The K-bilinearity of $\overset{\alpha}{\bowtie}$.** Recall Tr, the trivial object in $\mathcal{G}(2^A, m)$. It is cyclic of order $2m$, generated by y, and σ_{Tr} is the trivial homomorphism. There is a unique object One $\in \mathcal{T}(\mathrm{Tr}) = \mathcal{T}^\emptyset(\mathrm{Tr})$ for which

$$\mathrm{One}_B = \begin{cases} \mathbf{C} & \text{if } B = \emptyset; \\ 0 & \text{if not.} \end{cases}$$

One may directly verify that the irreducible objects in $\mathcal{T}^F(\mathrm{Tr})$ are $\rho_C\kappa_F\,\mathrm{One}$ for $C \cap F = \emptyset$. We have

$$(\kappa_F\,\mathrm{One})_B \cong \begin{cases} \mathbf{C} & \text{if } B \subset F; \\ 0 & \text{if not.} \end{cases}$$

If 1_B is a chosen generator for $(\kappa_F\,\mathrm{One})_B$, the map $\beta(S)$ for the object $\kappa_F\,\mathrm{One}$ may be chosen to send 1_B to $1_{B\triangle S}$ for all $S \subset F$.

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PROPOSITION 9.1. *The map $K_\alpha \longrightarrow T^\alpha(\mathrm{Tr})$ sending λ to $\lambda\,\mathrm{One}$ is an isomorphism of graded K_α-modules.*

This is immediate from the statement above about the irreducibles. This holds for any α. The involution α plays little role in this section, since with σ_{Tr} trivial, $\mathrm{Tr}\overset{\alpha}{\curlyvee}G$ is independent of α.

Recall now, for any G in $\mathcal{G}(2^A, m)$, the isomorphisms of 4.3:

$$\begin{array}{ccccc} \mathrm{Tr}\overset{\alpha}{\curlyvee}G & \xleftarrow{\theta_G} & G & \xrightarrow{\tilde{\theta}_G} & G\overset{\alpha}{\curlyvee}\mathrm{Tr} \\ (1,g) & \longleftarrow & g & \longmapsto & (g,1) \end{array}$$

PROPOSITION 9.2. *For all F and all $\mathcal{V}$ in $\mathcal{T}^\alpha(G)$,*

$$\theta_G^*(\kappa_F\,\mathrm{One}\overset{\alpha}{\bowtie}\mathcal{V}) \cong \kappa_F\mathcal{V} \cong \tilde{\theta}_G^*(\mathcal{V}\overset{\alpha}{\bowtie}\kappa_F\,\mathrm{One}).$$

REMARK. The case $F=\emptyset$ of this makes the name "One" seem reasonable. This proposition shows how the action of the ring K_α on $\mathcal{T}^\alpha G$ is essentially a special case of the $\overset{\alpha}{\bowtie}$ product.

PROOF. If $\mathcal{V}=(V,\beta)$, recall that $\kappa_F\mathcal{V}=(\kappa_F V,\kappa_F\beta)$, with $\kappa_F V$ a subspace of V^{2^F}. A map for the right-hand isomorphism is given by sending $\zeta\in V^{2^F}$ to $\sum_{C\subset F}\zeta(C)\otimes 1_C \in V\otimes\kappa_F\,\mathrm{One}$. This is evidently a linear isomorphism. It is readily checked that this map preserves grading and commutes with the action of G. An elementary calculation shows that the subspace $\kappa_F V$ of V^{2^F} maps onto the subspace $U_{\mathcal{V},\kappa_F\mathrm{One}}$ of $V\otimes\kappa_F\,\mathrm{One}$. This uses example (c) of $z_{C,D}$ before the definition of κ_F. The calculation of the righthand vertical arrow in the required commutative diagram, for $S\subset E\triangle F$,

$$\begin{array}{rcccl} & \kappa_F V & \cong & U_{\mathcal{V},\kappa_F\mathrm{One}} & \\ (\kappa_F\beta)(S) & \downarrow & & \downarrow & \gamma(S) \\ & \kappa_F V & \cong & U_{\mathcal{V},\kappa_F\mathrm{One}} & \end{array}$$

where $\gamma(S)$ is the structure map for $\mathcal{V}\overset{\alpha}{\bowtie}\kappa_F\,\mathrm{One}$, may be taken as the motivation for the formula in the definition for $(\kappa_F\beta)(S)(\zeta)(D)$.

The other isomorphism may be proved similarly; alternatively it is deduced from the "commutative" law in 5.14 (iii) as follows:

$$\begin{aligned} \theta_G^*(\kappa_F\,\mathrm{One}\overset{\alpha}{\bowtie}\mathcal{V}) &\cong \theta_G^* t^*\rho_{C_0}(\mathcal{V}\overset{\alpha}{\bowtie}\kappa_F\,\mathrm{One}) \\ &= \tilde{\theta}_G^*\rho_{C_0}(\mathcal{V}\overset{\alpha}{\bowtie}\kappa_F\,\mathrm{One}), \text{ since } \tilde{\theta}_G = t\circ\theta_G \\ &\cong \rho_{C_0}\kappa_F\mathcal{V} \\ &\cong \kappa_F\mathcal{V}, \text{ by 3.4 (ii) or (iv), since } C_0\subset E\cap F\subset F. \end{aligned}$$

REMARK. We have

$$\begin{aligned}\rho_C\mathcal{V} &\cong \rho_C\theta_G^*(\mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V}) \text{ taking } F = \emptyset \text{ in } 9.2\\ &\cong \theta_G^*\rho_C(\mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V})\\ &\cong \theta_G^*(\rho_C \mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V}) \text{ by } 5.4 \text{ (o)}.\end{aligned}$$

Similarly, $\cdots \cong \tilde{\theta}_G^*(\mathcal{V} \overset{\alpha}{\bowtie} \rho_C \mathrm{One})$. A second proof of this may be given which is analogous to, and easier than, the proof of 9.2. A third proof proceeds by induction on $|C|$, using 9.2, the relation $\kappa_F{}^2 = \sum_{C\subset F} \rho_C$, and decomposition into irreducibles.

PROPOSITION 9.3. *For all G, G', $\mathcal{V} \in \mathcal{T}^E(G)$, $\mathcal{V}' \in \mathcal{T}^{E'}(G')$, $F \subset A$ where E, E' and F are invariant under the involution α of A, we have*

$$\mathcal{V} \overset{\alpha}{\bowtie} \kappa_F\mathcal{V}' \cong \kappa_F(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}') \cong (\kappa_F\mathcal{V}) \overset{\alpha}{\bowtie} \mathcal{V}'.$$

PROOF. We shall use the associativity of $\overset{\alpha}{\bowtie}$, that is, 5.14 (ii), to give an easy formal proof of this. Note that

$$\mathrm{id}_G \overset{\alpha}{\mathrm{Y}} \theta_{G'} = \tilde{\theta}_G \overset{\alpha}{\mathrm{Y}} \mathrm{id}_{G'} \colon G \overset{\alpha}{\mathrm{Y}} G' \longrightarrow G \overset{\alpha}{\mathrm{Y}} \mathrm{Tr} \overset{\alpha}{\mathrm{Y}} G'$$

and

$$\theta_G \overset{\alpha}{\mathrm{Y}} \mathrm{id}_{G'} = \theta_{G \overset{\alpha}{\mathrm{Y}} G'} \colon G \overset{\alpha}{\mathrm{Y}} G' \longrightarrow \mathrm{Tr} \overset{\alpha}{\mathrm{Y}} G \overset{\alpha}{\mathrm{Y}} G'.$$

Thus we have

$$\begin{aligned}\mathcal{V} \overset{\alpha}{\bowtie} \kappa_F\mathcal{V}' &\cong \mathcal{V} \overset{\alpha}{\bowtie} \theta_{G'}^*(\kappa_F \mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V}')\\ &= (\mathrm{id}_G \overset{\alpha}{\mathrm{Y}} \theta_{G'})^*[\mathcal{V} \overset{\alpha}{\bowtie} (\kappa_F \mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V}')]\\ &\cong (\tilde{\theta}_G \overset{\alpha}{\mathrm{Y}} \mathrm{id}_{G'})^*[(\mathcal{V} \overset{\alpha}{\bowtie} \kappa_F \mathrm{One}) \overset{\alpha}{\bowtie} \mathcal{V}']\\ &= \tilde{\theta}_G^*(\mathcal{V} \overset{\alpha}{\bowtie} \kappa_F \mathrm{One}) \overset{\alpha}{\bowtie} \mathcal{V}'\\ &\cong (\kappa_F\mathcal{V}) \overset{\alpha}{\bowtie} \mathcal{V}'\\ &\cong \theta_G^*(\kappa_F \mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V}) \overset{\alpha}{\bowtie} \mathcal{V}'\\ &= (\theta_G \overset{\alpha}{\mathrm{Y}} \mathrm{id}_{G'})^*[(\kappa_F \mathrm{One} \overset{\alpha}{\bowtie} \mathcal{V}) \overset{\alpha}{\bowtie} \mathcal{V}']\\ &\cong (\theta_{G \overset{\alpha}{\mathrm{Y}} G'})^*[\kappa_F \mathrm{One} \overset{\alpha}{\bowtie} (\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}')]\\ &\cong \kappa_F(\mathcal{V} \overset{\alpha}{\bowtie} \mathcal{V}').\end{aligned}$$

REMARK. This result is the analogue of 5.14 (o) with κ_F replacing ρ_C. It can similarly be obtained by a direct proof, much more tedious than that for 5.14 (o). On the other hand 5.14 (o) has a proof as above, based on the associativity of $\overset{\alpha}{\bowtie}$. It has a third proof by induction on $|C|$ from the relation $\kappa_F{}^2 = \sum_{C\subset F} \rho_C$, using 10.3 and decomposition into irreducibles.

COROLLARY 9.4. *The map*

$$\overset{\alpha}{\bowtie}: T^{\alpha}G \times T^{\alpha}G' \longrightarrow T^{\alpha}(G \overset{\alpha}{\curlyvee} G')$$

is K_α-bilinear, and so it determines a morphism of 2^α-graded K_α-modules

$$T^{\alpha}G \otimes_{K_\alpha} T^{\alpha}G' \longrightarrow T^{\alpha}(G \overset{\alpha}{\curlyvee} G').$$

REMARK. When $\ell = 1 = m$ (so $\alpha = \mathrm{id}$) one main theorem of Hoffman-Humphreys [H-H1; Theorem 2.24] is that this last morphism is an isomorphism. This will follow from a later result which shows that in general the map is injective with finite cokernel. In part II we investigate its deviation from being an isomorphism. It will be an isomorphism whenever either $T^{\alpha}G$ or $T^{\alpha}G'$ is a free K_α-module. Freeness is automatic when $\ell = 1$ for any m, confirming the first sentence above.

10. **Summary of Parts II and III.** Taking $L = \mathrm{Tr}$ in Section 7 yields a map

$$\langle\, , \rangle : T^{E}(G) \times T^{F}(G) \longrightarrow K^{E \triangle F},$$

by passing to isomorphism classes with the biadditive map

$$\mathcal{T}^{E}(G) \times \mathcal{T}^{F}(G) \cong \mathcal{T}^{E}(G) \times \mathcal{T}^{F}(G \overset{\alpha}{\curlyvee} \mathrm{Tr}) \xrightarrow{\mathcal{H}} \mathcal{T}^{E \triangle F}(\mathrm{Tr}) \cong K^{E \triangle F} = K_{\alpha}^{E \triangle F}.$$

By 9.4, $\langle\, , \rangle$ is bilinear over K_α. Combining 7.7 and 7.6 gives the identity

$$\langle x \overset{\alpha}{\bowtie} x', y \overset{\alpha}{\bowtie} y' \rangle_{G \overset{\alpha}{\curlyvee} G'} = \langle x, y \rangle_G \langle x', y' \rangle_{G'}.$$

[By taking $G' = \mathrm{Tr}$, part of the bilinearity is essentially a special case of this.] Reciprocity,

$$\langle \theta_* x, z \rangle_H = \langle x, \theta^* z \rangle_G,$$

for injective maps $\theta: G \longrightarrow H$ in $\mathcal{G}$, follows from 7.7. In part II, these laws and others satisfied by the modules T^*G are studied abstractly. The main point will be that such a module with inner product and positivity is uniquely decomposable as a direct sum of indecomposable sub-objects. These are the submodules, one for each equivalence class of special irreducibles, generated by such special irreducibles. Here an irreducible x is *special* if it does not have the form $\kappa_F y$ for any $F \neq \emptyset$. Two such elements x and x' are equivalent if and only if $\langle x, x' \rangle \neq 0$. Attempted classification of such indecomposables is likely to lead to a combinatorial morass, if attempted for general ℓ and α. But for small ℓ it is quite easy. For example, when $\ell = 1$ [and so $\alpha = \mathrm{id}$], all such indecomposables are free on one generator x with $\langle x, x\rangle = 1$ [and so T^*G is always free over K, where a K-basis $\{x_\alpha\}$ of special irreducibles leads to the canonical $\mathbf{Z}$-basis $\{x_\alpha, \rho x_\alpha, \kappa x_\alpha\}$ of triple the size]. There are two indecomposables up to isomorphism, depending on the two choices for the grading of the generator. When $\ell = 2$ and $\alpha = \mathrm{id}$, there are six indecomposables, four of which are free on one generator, one for each of the four elements in the group 2^A of grading parameters. Letting $A = \{b, c\}$, the other two indecomposables are both generated by two special irreducibles, say x and y, in "antipodal" gradings [*i.e.* $\mathrm{gr}\, x \triangle \mathrm{gr}\, y = A$], with two relations, $\kappa_b x = \kappa_c y$ and $\kappa_c x = \kappa_b y$. Inner products are determined by

$$\langle x, x\rangle = 1 + \rho_A = \langle y, y\rangle \text{ and } \langle x, y\rangle = \kappa_A.$$

[See the remark below.] The $\mathbf{Z}$-basis of irreducibles, in the four different gradings, is

$$\{x, \rho_b x\} \cup \{y, \rho_b y\} \cup \{\kappa_b x\} \cup \{\kappa_c x\}.$$

Other relations are, for example,

$$\rho_A x = x \text{ and } \rho_b x = \rho_c x;$$

and similarly for y. To deduce one of these,

$$x + \rho_b x = \kappa_b^2 x = \kappa_b \kappa_c y = \kappa_c \kappa_b y = \kappa_c^2 x = x + \rho_c x.$$

REMARK. In general, for an irreducible x, there is a subgroup Γ of 2^A such that

$$\langle x, x\rangle = \sum_{C \subset \Gamma} \rho_C.$$

For x to be special, it is necessary and sufficient that $2^B \subset \Gamma$ only for $B = \emptyset$. In general, $x = \kappa_B x'$ for a special irreducible x' and the maximal B with $2^B \subset \Gamma$. When x is special, it generates one of the indecomposable submodules [necessarily freely] if and only if $\Gamma = \{\emptyset\}$. The other indecomposables are neither cyclic over K_α, nor are they free over K_α.

In part III, the application to projective representations of monomial groups will be given. For this we need an isomorphism, for $n > 1$,

$$M(H \wr S_n) \cong M(S_n) \oplus M(H) \oplus \mathrm{Hom}(H; \mathbf{Z}/2) \oplus \Lambda^2 \,\mathrm{Hom}(H; \mathbf{Z}/2) \oplus \mathrm{Hom}(H; \mathbf{Z}/2)^{1-\delta_{2n}}.$$

The existence of such an isomorphism is due to Read [R], and also follows from the Lyndon spectral sequence [M]. We shall give a very explicit isomorphism: each element on the right-hand side above [where the first two components are given in terms of cyclic covers of S_n and H] will produce an explicit cyclic cover of $H \wr S_n$. Varying n, [including $n = 0$ and 1] will produce sequences of cyclic covers as discussed in the introduction. An abstraction of these sequences (which generalize the Young systems of [H-H2]) will consist, among other things, of a sequence of objects Y_n in $\mathcal{G}(2^A, m)$ together with embeddings $Y_a \overset{\alpha}{\curlyvee} Y_b \longrightarrow Y_{a+b}$ Applying the functor T^α, using the operation $\overset{\alpha}{\bowtie}$, and inducing along the above embedding gives a map

$$(T^\alpha Y_a) \otimes_{K_\alpha} (T^\alpha Y_b) \longrightarrow T^\alpha Y_{a+b}$$

and thereby makes $\oplus_{n \geq 0} T^\alpha Y_n$ into an algebra over K_α, graded over $\mathbb{N}$, in which each homogeneous component is a module over K_α graded over 2^α Such algebras with inner product and positivity are likely to decompose uniquely into a tensor product of atoms, where an atom is an algebra whose component in lowest positive $\mathbb{N}$-grading is one of the indecomposable modules discussed above This should follow formally, imitating Zelevinski's argument [Z], which can be regarded as the case $\ell = 0$ The case $\ell = 1$ was done by the author and Michael Bean [B-H]

The other side of the coin is the classification of atoms Here it seems likely that each indecomposable (occurring in lowest grading 1) will give rise to at most one (perhaps exactly one) such atomic algebra Progress on this at the time of writing has concentrated on the example of $H \wr S_n$ where H is cyclic of order 2 [*i e* the hyperoctahedral group] Four of the eight sequences of covers are taken care of by the case $\ell = 1$ done in [H-H2] [In fact, one of them is really the case $\ell = 0$] The other four can be done with $\ell = 2$ and $\alpha \neq \mathrm{id}$, with $\ell = 4$ if we prefer $\alpha = \mathrm{id}$ The main complication is the failure of

$$\overset{\alpha}{\bowtie} \ (T^\alpha X) \otimes_{K_\alpha} (T^\alpha Y) \longrightarrow T^\alpha(X \overset{\alpha}{\curlyvee} Y)$$

to be an isomorphism, except when $\ell = 1$ or, more generally, when $T^\alpha X$ or $T^\alpha Y$ is free over K_α This leads to the failure of the above algebra to be a Hopf algebra However, by simultaneously and inductively working out all the groups $T^E(Y_{a_1} \overset{\alpha}{\curlyvee} Y_{a_2} \overset{\alpha}{\curlyvee} \quad)$, the core of the Hopfian methods can be extracted and used to make progress on the structure of these algebras

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