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A MAXIMALITY CRITERION FOR NILPOTENT COMMUTATIVE MATRIX ALGEBRAS

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Let A be a commutative algebra contained in $M_n(F)$, F a field. Then A is nilpotent if there exists v such that $A^v=(0)$, and is said to have nilpotency class k (denoted $Cl(A)=k$) if $A^k=(0)$, but $A^{k-1}\neq(0)$. A well known result asserts that matrix algebras are nilpotent if and only if every element is nilpotent. Let $\mathbf{N}=\{A \mid A$ is a nilpotent commutative subalgebra of $M_n(F)\}$.

If $A\in\mathbf{N}$, and A is not properly contained in any other algebra in $\mathbf{N}$, then A is maximal in $\mathbf{N}$. Let $\mathbf{M}=\{A \mid A$ is maximal in $\mathbf{N}\}$. For $a, b\in M_n(F)$, if $a^v=b^v=0$ and $ab=ba$, then $(ab)^v=(a+b)^{2v}=0$. Therefore $A\in\mathbf{M}$ if and only if $a'a=aa'$, for all $a\in A$ and $(a')^k=0$ for some k imply $a'\in A$.

Let $\mathbf{A}_k=\{A \mid A$ is maximal among those algebras of class k in $\mathbf{N}\}$. Clearly $\mathbf{M}\subset\bigcup_k \mathbf{A}_k$. We prove the converse.

THEOREM. *If $M\in\mathbf{A}_k$ for some k, then $M\in\mathbf{M}$, or $M=(0)$.*

Proof. Let M be a nontrivial algebra in $\mathbf{A}_k$. Then M is contained in some $N\in\mathbf{M}$. (See [1, p. 35].) For a nontrivial algebra $C\in\mathbf{N}$, with $Cl(C)=s>1$, define

$$H_C=\{x\in C \mid xC=(0)\}.$$

$H_C\neq(0)$ since $C^{s-1}\subseteq H_C$.

LEMMA. $H_M=H_N$

Proof. (i) $H_N\subseteq H_M$

Since $x\in H_N$ implies $xM=(0)$, it suffices to show $H_N\subseteq M$. Let S be the algebra generated by M and H_N. Then $M\subseteq S$, and $Cl(M)=Cl(S)$; thus, since $M\in\mathbf{A}_k$, we have $M=S$. Therefore $H_N\subseteq M$.

(ii) $H_M\subseteq H_N$

Since N is nilpotent, there exists $x\in F^n-\{0\}$, such that $ax=0$ for all $a\in N$ (See [1]); and there exists $y\in F^n-\{0\}$, linearly independent of x such that $y\notin NF^n$. Complete x, y to a basis $y, e_2, \ldots, e_{n-1}, x$ for F^n and rewrite the matrices of N in terms of this basis.

Then all the matrices of N are of the form:

$$(1)\qquad \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \cdot \\ & & & * & & & \cdot \\ & & & & & & \cdot \\ & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & 0 \end{bmatrix}_{n\times n}$$

Define z to be the matrix with a 1 in the $(n, 1)$ position, and 0 elsewhere. It is clear from (1) that $zN=Nz=(0)$, $z^2=0$, and so $z \in N$, since $N \in \mathbf{M}$. Hence $z \in H_N$. Suppose there exists $b' \in H_M - H_N$; i.e. there exists $a' \in N-M$, such that $a'b'=b'a'\neq 0$ but $b'M=Mb'=(0)$. Choose $W \in F^n-\{0\}$ such that $a'b'W\neq 0$. Define c to be the $n\times n$ matrix whose first column is $b'W$, and whose remaining columns are 0. Since $zb'=0$, $zb'W=0$; hence $zc=0$. Thus the (1, 1) entry of c is 0, and therefore $c^2=0$. Since $bb'=0$, $bc=0$ for all $b \in M$. From (1), $cN=(0)$; so $cM=(0)=Mc$. But the class of the algebra generated by c and M is clearly equal to $Cl(M)$; hence $c \in M$, since $M \in \mathbf{A}_k$. However $ca'=0\neq a'c$ by construction, so $c \notin N$. This is a contradiction since $c \in M$ and $M \subseteq N$. Therefore, no such b' exists, i.e. $H_M \subseteq H_N$. This completes the proof of the lemma.

Continuing the proof of the theorem:

If $M^{k-p}N^p=(0)$, then $M^{k-(p+1)}N^p \subseteq H_M=H_N$ and so $M^{k-(p+1)}N^{p+1}=(0)$. Applying this k times to $M^k=(0)$, $(p=0)$ we reach the conclusion $N^k=(0)$. That is, $Cl(M)=Cl(N)$. But since $M \subseteq N$, and $M \in \mathbf{A}_k$, this implies $M=N$. So M is maximal, as desired.

Remark. If F is an integral domain (instead of a field), the result is also true. We may reduce the problem to that of a field by considering the quotient field; the induced commutative nilpotent algebra is maximal of its class.

Reference

1. D. A. Suprenenko, and R. I. Tyshkevich, *Commutative matrices*, Academic Press, New York, 1968.

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