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NEAR-RINGS WITH CHAIN CONDITIONS ON RIGHT ANNIHILATORS

by A. OSWALD
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Throughout this note, N will denote a (Left) near-ring with two-sided zero. Definitions of basic concepts can be found in (**9**).

We prove first that a right ideal I in a d.g. near-ring has a right identity if and only if $x \in xI$ for each $x \in I$. This enables us to study the structure of regular d.g. near-rings with chain conditions on right annihilators. Specifically we will prove that a regular d.g. near-ring with both the maximum and the minimum conditions on right annihilators is a finite direct sum of near-rings which are either rings of matrices over division rings or non-rings of the form $M_G(\Gamma)$ for a suitable type 2 N-module Γ. Finally we consider the case of maximum condition on N-subgroups.

These results generalise some results of Heatherly (**5**).

The idea for our first result is taken from (**4**).

Theorem 1. *Let N be a d.g. near-ring and I be a right ideal of N which, as a near-ring, has the minimum condition on right annihilators. Then the near-ring I has a right identity if and only if $x \in xI$ for each $x \in I$.*

Proof. The necessity is immediate. Let N be d.g. by S and for $a \in I$ define $L_a = \{s - as : s \in S\}$. Choose $e \in I$ such that $l_I(L_e) = \{x \in I : xL_e = (0)\}$ is maximal where $b \in l_I(L_e)$ if and only if $bL_e = (0)$ and $b \in I$. Suppose $l_I(L_e) \neq I$. Choose $y \in I$ with $yL_e \neq (0)$ and $s \in S$ with $y(s - es) \neq 0$. Then $(y - ye)s \neq 0$ so $y - ye \neq 0$. Also $y - ye \in (y - ye)I$ so $y - ye = (y - ye)e'$ for some $e' \in I$. Writing $e' = \Sigma \pm s_i$ where $s_i \in S$ we then have

$$\begin{aligned} y &= (y - ye)e' + ye \\ &= y\left(\sum \pm (s_i - es_i) + e\right) \\ &= y\left(\sum \pm s_i + u + e\right) \quad \text{for some} \quad u \in I \\ &= yf, \quad \text{where} \quad f = e' + u + e. \end{aligned}$$

Now $y \in l_I(L_f)$ but $y \notin l_I(L_e)$. Also $z \in l_I(L_e)$ implies $zf = z(\Sigma \pm (s_i - es_i) + e) = ze$ and so $z \in l_I(L_f)$. This contradicts the maximality of $l_I(L_e)$ and so $l_I(L_e) = I$. But then if $r \in I$ we have $rL_e = (0)$ and so $(r - re)S = (0)$ from which $(r - re)N = (0)$. In particular $(r - re)I = (0)$ and thus $r = re$ and e is a right identity.

Corollary 1. *A d.g. near-ring with minimum condition on right annihilators has a right identity if and only if $x \in xN$ for each $x \in N$.*

As a first application of this theorem we give an alternative proof of a theorem of Szeto (**11**). We say that a near-ring N is a *subdirect sum* of near-rings N_λ if and only if there exist ideals I_λ of N with $\cap_\lambda I_\lambda = (0)$ and $N_\lambda \simeq N/I$ as near-rings. Then as in Stewart (**10**) we have.

Lemma 1. *A near-ring N has no nilpotent elements if and only if it is isomorphic to a subdirect sum of near-rings without proper divisors of zero.*

Lemma 2. (Beidleman (**1**)). *If N is a regular near-ring and $0 \neq b \in N$ then $bN = fN$ for some idempotent $f \in N$.*

Theorem 2. (Szeto (**11**)). *A regular d.g. near-ring N has no nilpotent elements if and only if it is a subdirect sum of division near-rings.*

Proof. The sufficiency is immediate. Suppose N has no nilpotent elements. Then by Lemma 1 it is a subdirect sum of near-rings without proper divisors of zero each of which is d.g., regular and trivially has the minimum condition on right annihilators. Let N_1 be one such subdirect summand. From Corollary 1 it has a right identity, e say. Then $0 \neq x \in N_1$ implies $xN_1 = fN_1$ for some idempotent $f \in N_1$. Since $r(f) = \{x \in N_1 : fx = 0\} = (0)$ we get $N_1 = fN_1 = xN_1$ and so $e = xy$ for some $y \in N_1$. Thus $N_1 \backslash \{0\}$ is a group and N_1 is a division near-ring as required.

Observe that since N_1 is a division near-ring then by Ligh (**7**) it has abelian addition. Consequently from Fröhlich (**3**; 4.4.1) it is distributive and hence is a ring. We thus get

Corollary 2. *A regular d.g. near-ring has no (non-zero) nilpotent elements if and only if it is a ring and a subdirect sum of division rings.*

If I is an ideal of the near-ring N and $x \in I$ we denote by $Sg_I(x)(Sg_N(x))$ the I-subgroup (N-subgroup) of I (N) generated by x. Clearly $Sg_N(x) \subseteq I$ and $Sg_I(x) \subseteq Sg_N(x)$.

Theorem 3. *Let N be d.g. and I be an ideal of N which, as a near-ring, has the minimum condition on right annihilators and $A^2 = A$ for each I-subgroup A of I. Then I has an identity which is a central idempotent of N.*

Proof. If L is an I-subgroup of I and $x \in L$ then xN is an I-subgroup of I. Also $xN = xNxN \subseteq xI \subseteq L$ so each I-subgroup of I is an N-subgroup of N contained in I. Hence if $x \in I$ then $Sg_N(x) \subseteq Sg_I(x)$ and thus $Sg_I(x) = Sg_N(x)$. Now $x \in Sg_N(x) = Sg_N(x)Sg_N(x)$ and so $x = uv$ for some $u, v \in Sg_N(x)$. If N is distributively generated by S then $v = \Sigma \pm s_i$ where $s_i \in S$ and $u = \Sigma xr_j$ where $r_j \in N$ or r_j is formal identity. Hence $uv = \Sigma xr_j \Sigma \pm s_i = x\Sigma \pm r_j s_i \subseteq xN = xNxN \subseteq xI$. Applying Theorem 1 we see that I has a right identity e. Now suppose that $y \in N$ with $z = ey - ye$. Then $z \in I$ and

$ez = ey - e(ye) = 0$ from which $IezI = (0)$ and $IzI = (0)$. Then $(zI)^2 = (0)$ so that $zI = (0)$ and $z = 0$. Thus e is a central idempotent of N and a two-sided identity for I.

Corollary 3. *If N is a regular, d.g. near-ring with the minimum condition on right annihilators then N has a two-sided identity.*

Proof. N regular implies $A^2 = A$ for each N-subgroup A of N.

Corollary 4. *A d.g. near-ring with the minimum condition on N-subgroups and no nilpotent N-subgroups has a two-sided identity.*

Proof. Such a near-ring is completely reducible (**8**) and hence regular.

Theorem 4. *A d.g. near-ring N which is regular with the minimum condition on right annihilators is a direct sum of ideals which are simple d.g. near-ring with identity.*

Proof. If I is a non-zero ideal of N then I has an identity e and so $I \cap r(e) = (0)$. Now $N = I \oplus r(e)$ and if $x \in r(e)$ and $n \in N$ then $enx = nex = 0$ so that $r(e)$ is an ideal of N and hence has a two-sided identity f. Now $I \subseteq r(f)$ and $r(f) \cap r(e) = (0)$ from which $I = r(f)$. It follows that every ideal of N is of the form $r(f)$ for some central idempotent f of N. Let $r(e_1)$ be a minimal non-zero ideal of N. There is an ideal of N, I_1, with $I_1 \cap r(e_1) = (0)$ and $I_1 \oplus r(e_1) = N$. Choose $e_2 \in I_1$ with $r(e_2)$ a minimal non-zero ideal of N in I_1. For some ideal I_2 of N we have $N = I_2 \oplus r(e_1) \oplus r(e_2)$. In this way we construct a descending chain $I_1 \supseteq I_2 \supseteq \ldots$ of right annihilators. It follows that for some k, $N = r(e_1) \oplus \ldots \oplus r(e_k)$. If I is an ideal of N and if B is an ideal of I then $BN \subseteq B$ when I is a direct summand of N. Furthermore, if e is the identity of I then $eN = Ne$ and $NB = NeB = eNB \subseteq IB \subseteq B$. Hence the ideals $r(e_i)$ are simple and we have the result.

The following result is proven in the same way as in Koh (**6**).

Lemma 3. *If N is regular and I is a maximal annihilator right ideal of N then there is a minimal N-subgroup $B = eN$, where e is an idempotent, with $I \cap B = (0)$ and $I + B = N$.*

An N-module Γ is *type*-2 if $\Gamma N \neq (0)$ and Γ has no proper N-subgroups. If N has an identity then Γ has no proper N-subgroups if and only if $\gamma N = \Gamma$ or $\gamma N = (0)$ for each $\gamma \in \Gamma$. Hence if N is a regular near-ring with the maximum condition on right annihilators then N has an N-subgroup which is a type-2 N-module. If, in addition, N has no non-trivial two-sided ideals this N-subgroup Γ will be such that $\Gamma a = (0)$ implies that $a = 0$ and Γ will be a type-2 faithful N-module. In such a case we say that N is 2-primitive on Γ.

In the case where N is a ring Γ will be a faithful ring module so that N will be a regular simple primitive ring and if N also has the minimum condition on right annihilators then N will have an identity. The set of minimal right ideals will be non-empty and the sum of all of them will be an ideal containing Γ and hence will be

N. It follows that N will be the sum of finitely many minimal right ideals and hence N will have the minimum condition on right ideals so that N will be a ring of matrices over a suitable division ring.

Turning to the case where N is a non-ring we have

Theorem 5. *If N is a non-ring with identity and the minimum condition on right annihilators which is 2-primitive on an N-module Γ which is an N-subgroup of N then* $N = M_G(\Gamma)$ where $G = \mathrm{Aut}_N\ (\Gamma)$ and $M_G(\Gamma) = \{f : \Gamma \to \Gamma : f\alpha = \alpha f$ for each $\alpha \in G\}$.

Proof. This is the same as in Betsch (**2**; Thm 2.5) making use of the fact that since $\Gamma \subseteq N$, $r(\gamma_1) \cap r(\gamma_2) = r(\{\gamma_1, \gamma_2\})$ is an annihilator right ideal of N.

Observe that in view of (**2**; Thm 5.9) we have

Corollary 5. *A non-ring N with identity and the minimum condition on right annihilators which is 2-primitive on an N-module Γ which is an N-subgroup of N has both the minimum and the maximum conditions on right ideals.*

Theorems 4 and 5 and the intervening discussion now yield the result mentioned in the introduction.

Theorem 6. *A d.g. near-ring which is regular and has the minimum and maximum conditions on right annihilators is a finite direct sum of ideals which are d.g. near-rings each of which is either a ring of matrices over a suitable division ring or a non-ring of the form $M_G(\Gamma)$ for a suitable type-2 near-ring module Γ.*

An N-subgroup A of N is *module-essential* if whenever $A \cap B = (0)$ with B a right ideal of N then $B = (0)$. The N-subgroup A is *essential* when this is true for B an N-subgroup of N.

Theorem 7. *Let N be a near-ring in which module essential N-subgroups are essential. If N is regular with maximum condition on right annihilators and no infinite direct sums of right ideals then N has minimum condition on right annihilators.*

Proof. Let $A_1 \supset A_2 \supset \dots$ be a properly descending chain of right annihilators and U be a left annihilator minimal subject to $l(A_k) \subsetneq U \subseteq l(A_{k+1})$. Then $U \neq (0)$ and $UA_k = (0)$. Choose $u \in U$ with $uA_k \neq (0)$. Since N is regular, N has no nilpotent N-subgroups so $uA_k uA_k \neq (0)$ and so for some $a \in A_k$, $uauA_k \neq (0)$. If $y \in A_k$ with $auy \in A_{k+1} \cap auA_k$ then $Uauy = (0)$. Now $l(A_k) \subseteq l(y)$ so $l(A_k) \subseteq l(y) \cap U \subseteq U$. Since U is minimal either $U = l(y) \cap U$ or $l(A_k) = l(y) \cap U$. As $uauy = 0$ we have $uau \in l(y) \cap U$ whereas $uauA_k \neq (0)$. It follows that $U = l(y) \cap U$ and so $U \subseteq l(y)$ and $Uy = (0)$. Then $auy = 0$ and so $A_{k+1} \cap auA_k = (0)$. For each $k \geq 1$ choose a right ideal X_k maximal subject to $A_{k+1} \cap X_k = (0)$. Then $A_{k+1} + X_k$ is module-essential and hence essential in N. Writing $C_k = A_k \cap X_k$ we can find, since $l(A_k) \neq l(A_{k+1})$, an N-subgroup $B_k \subset A_k$ with $B_k \neq (0)$ and $A_{k+1} \cap B_k = (0)$. Let $b \in B_k \cap (A_{k+1} + X_k)$ with $b \neq 0$. Then $b = t + x$, $t \in A_{k+1}$, $x \in X_k$ so $x = -t + b \in A_k \cap X_k$ and so $A_k \cap X_k \neq (0)$. It follows that

C_k is a non-zero right ideal of N and $C_k \cap A_{k+1} = (0)$. We then have a chain $C_1 \subsetneq C_1 \oplus C_2 \subsetneq \ldots$ which must terminate and so N has the minimum condition on right annihilators.

Corollary 6. *If N is a regular d.g. near-ring in which module-essential N-subgroups are essential and if N has the maximum condition on right annihilators and no infinite direct sums of right ideals then N is a finite direct sum of ideals which are either rings of matrices over division rings or non-rings of the form $M_G(\Gamma)$ for a suitable type-2 N-module Γ.*

Obviously Theorem 7 holds when N is a regular near-ring in which module essential N-subgroups are essential and N has the maximum condition on N-subgroups. In this case the conclusion of Corollary 6 again follows.

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DEPARTMENT OF MATHEMATICS AND STATISTICS
TEESSIDE POLYTECHNIC
MIDDLESBROUGH
CLEVELAND