THE TWISTED GROUP ALGEBRA OF A FINITE NILPOTENT GROUP OVER A NUMBER FIELD

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Introduction. Let G be a finite group with neutral element e which operates trivially on the multiplicative group R^* of a commutative ring with identity 1. Let $H^2(G, R^*) = Z^2(G, R^*)/B^2(G, R^*)$ denote the second cohomology group of G with respect to the trivial G-module R^*. With every factor system (2-cocycle) $f \in Z^2(G, R^*)$ we associate the so called (central) twisted group algebra (R, G, f) of G over R (see [**4**, Chapter V, 23.7] or [**13**, §4] for a definition). If f is cohomologous to f', then the R-algebras (R, G, f) and (R, G, f') are isomorphic. Hence, up to R-algebra isomorphism, (R, G, f) is determined by the cohomology class $\bar{f} \in H^2(G, R^*)$ determined by f. If $R = k$ is a field of characteristic not dividing the order $|G|$ of G, then a computation of the discriminant of (k, G, f) shows that (k, G, f) is semisimple (see [**13**, 4.2]).

In this paper we develop a method which shows how to construct splitting fields of (k, G, f) in the case where k is a number field and G is nilpotent.

If $f = 1$ the algebra (k, G, f) is the ordinary group algebra. In this case the problem is completely solved by a well-known theorem of Roquette [**12**]. So we let $f \neq 1$ throughout. Partial results are contained in [**1**] and [**9**, §9].

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1. Reduction to groups with nondegenerate center. We say that the finite nilpotent group G has a nondegenerate center $Z(G)$ with respect to the factor system $f \in Z^2(G, k^*)$ if the symplectic pairing $\omega : Z(G) \times Z(G) \to k^*$ determined by the restriction $f|_{Z(G)}$ is nondegenerate [**13**, 2.2]. We reduce our problem to such a group G.

So let us start with an arbitrary finite nilpotent group G, a number field k and a factor system $f \in Z^2(G, k^*)$. To compute the index of a simple component of (k, G, f) we may assume without loss of generality that the central subgroup

$$Z(f) = \{z \in Z(G) \mid f(g, z) = f(z, g) \text{ for all } g \in G\}$$

of G is trivial. Otherwise we consider the factor group $G/Z(f)$. We get a factor system $t \in Z^2(G/Z(f), (k, Z(f), f)^*)$ and an isomorphism of k-algebras $(k, G, f) \cong ((k, Z(f), f), G/Z(f), t)$ [**11**, Theorem 2.1].

Now we decompose $(k, Z(f), f)$ into a direct sum of fields $(k, Z(f), f) \cong \bigtimes_i K_i$. We get factor systems $t_i \in Z^2(G/Z(f), K_i^*)$ and an isomorphism of k-algebras $(k, G, f) \cong \bigtimes_i (K_i, G/Z(f), t_i)$. This isomorphism allows us to view a simple component of (k, G, f) as a simple component of one of the algebras $(K_i, G/Z(f), t_i)$, and the index of the first algebra over k equals the index of the second over K_i. So we may assume inductively that $Z(f)$ is trivial. (The above technique was first applied in a somewhat different context in [**11**] and was used in connection with splitting field questions in [**9**, §9].)

Now we consider the kernel

$$Z_f = \{z \in Z(G) \mid f(z, z') = f(z', z) \text{ for all } z' \in Z(G)\}$$

of the symplectic pairing $\omega : Z(G) \times Z(G) \to k^*$, and the subgroup

$$Z^f = \{g \in G \mid f(g, z) = f(z, g) \text{ for all } z \in Z_f\}$$

of G. The pairing $\omega_f : G \times Z_f \to k^*$ determined by f (see [**13**, §2]) yields a homomorphism $\omega_f(\ , z) : g \mapsto \omega_f(g, z)$ of G into k^* for every $z \in Z_f$, and Z^f is the intersection of the kernels of these homomorphisms; in particular, Z^f is a normal subgroup of G. Furthermore, the pairing $\omega_f : G \times Z_f \to k^*$ yields a nondegenerate pairing of $G/Z^f \times Z_f$ into k^* which is also denoted by ω_f. Since $Z(f)$ is trivial, we get a canonical isomorphism between G/Z^f and $\mathrm{Hom}(Z_f, k^*)$ by mapping each $x \in G/Z^f$ to the linear character $\omega_f(x,\ )$ of Z_f. We note the following lemma.

Lemma 1.1. *The group G/Z^f is canonically isomorphic to* $\mathrm{Hom}(Z_f, k^*)$. *In particular, k contains a primitive root of unity of order* $\exp Z_f$.

Now let C be an algebraically closed field containing k. A simple character α of $C \otimes (k, Z_f, f)$, $\alpha \notin k$, yields an element of $\mathrm{Hom}(Z_f, C^*/k^*)$, which we can take to be faithful without loss of generality. So all values of $\alpha^{\exp Z_f}$ belong to k. Hence by the second part of Lemma 1.1 the field $k(\alpha)$ is a Kummer extension of k. Hence by the second part of Lemma 1.1 the field $k(\alpha)$ is a Kummer extension of k. Multiplication of α by all the linear characters $\omega(x,\ ) \in \mathrm{Hom}(Z_f, k^*)$, where $x \in G/Z^f$, yields exactly all k-automorphisms of $k(\alpha)$. Hence we have the next lemma.

Lemma 1.2. *If α is a simple character of $C \otimes (k, Z_f, f)$ then the field $k(\alpha)$ is a Kummer extension of k with Galois group $\mathcal{G}(\alpha)$ canonically isomorphic to G/Z^f.*

Now let χ be a simple character of $C \otimes (k, G, f)$ with values in k and let α be a simple component of $\mathrm{Res}^G_{Z_f}(\chi)$. A short computation shows that the fixed group $I(\alpha)$ of α is equal to Z^f. Clifford's theory, which is also valid for representations of twisted group algebras, therefore yields a simple character ξ of $C \otimes (k, Z^f, f)$ such that $\chi = \mathrm{Ind}^G_{Z^f}(\xi)$ and $\mathrm{Res}^{Z^f}_{Z_f}(\xi) = l\alpha$ for some natural number l. The last condition shows that the center $k(\xi)$ of the simple component $\mathcal{A}(Z^f, \xi)$ of (k, Z^f, f) corresponding to ξ is equal to $k(\alpha)$. A simple application of Frobenius' law of reciprocity (see e.g. [**9**, 4.13]) establishes the next lemma.

Lemma 1.3. *For every $x \in G/Z^f$ there is a σ in the Galois group $\mathcal{G}(\xi)$ of $k(\xi)$ over k such that $\xi^x = \xi^\sigma$.*

The following reduction theorem is contained in [**3**, 3.2] in a similar form for the case $f = 1$. The proof of it is easily carried over to the twisted case. For the convenience of the reader we recall the main steps.

Lemma 1.4. *Let χ be a simple character of $C \otimes (k, G, f)$ with values in k. If the index of $\mathcal{A}(Z^f, \xi)$ is equal to* 1 *then $\mathcal{A}(G, \chi)$ is similar to a crossed product $\Gamma(k(\xi)/k, c)$ for a factor system $c : \mathcal{G}(\xi) \times \mathcal{G}(\xi) \to k(\xi)^*$ in the sense of Galois cohomology.*

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Proof. Let $\tilde{c}_{\sigma,\tau}=s_\sigma s_\tau s_{\sigma\tau}^{-1}$ be a factor system associated with the extension $1\to Z^f\to G\to\mathscr{G}(\xi)\to 1$, let $\{e_g \mid g\in G\}$ be a canonical k-basis of (k, G, f) and let $\beta:(k, Z^f, f)\to\mathscr{A}(Z^f,\xi)$ be the k-algebra epimorphism associated with ξ. G operates on $\mathscr{A}(Z^f,\xi)$ according to $i_g(\beta(e_n))=\beta(e_g e_n e_g^{-1})$ for all $n\in Z^f$, $g\in G$. Consider the following algebra of rank $(k(\alpha):k)$ over $\mathscr{A}(Z^f,\xi)\cong\mathrm{Mat}_{\xi(1)}(k(\xi))$: $U=\bigoplus_{\sigma\in\mathscr{G}(\xi)}\mathrm{Mat}_{\xi(1)}(k(\xi))u_\sigma$ with multiplication rules $u_\sigma a u_\sigma^{-1}=i_{s_\sigma}(a)$, $a\in\mathrm{Mat}_{\xi(1)}(k(\xi))$, $u_\sigma u_\tau=\beta(e_{\tilde{c}_{\sigma,\tau}})u_{\sigma\tau}$. The map γ of (k, G, f) into U defined by $\gamma(e_{ns_\sigma})=f(n,s_\sigma)^{-1}\beta(e_n)u_\sigma$, for all $g=ns_\sigma\in G$, is an epimorphism of k-algebras. It is obvious that $U\cong\mathscr{A}(G,\chi)$. $\mathscr{G}(\xi)$ operates on $\mathrm{Mat}_{\xi(1)}(k(\xi))$ as it operates on the coefficients of a matrix, i.e. as it operates on ξ.

The map $\sigma^{-1}\circ i_{s_\sigma}:\mathrm{Mat}_{\xi(1)}(k(\xi))\to\mathrm{Mat}_{\xi(1)}(k(\xi))$ is a $k(\xi)$-automorphism by Lemma 1.3. By a well-known theorem of Skolem-Noether there is an invertible matrix $B_\sigma\in\mathrm{Mat}_{\xi(1)}(k(\xi))$ such that, for all $\alpha\in\mathrm{Mat}_{\xi(1)}(k(\xi))$, we have $(\sigma^{-1}\circ i_{s_\sigma})(a)=B_\sigma^{-1}aB_\sigma$. Set $C_\sigma=\sigma(B_\sigma)$. Then we have $i_{s_\sigma}(a)=C_\sigma^{-1}aC_\sigma$. Setting $v_\sigma=C_\sigma u_\sigma$, we have $v_\sigma a v_\sigma^{-1}=\sigma(a)$ and $v_\sigma v_\tau=c_{\sigma,\tau}v_{\sigma\tau}$ for a factor system c of $\mathscr{G}(\xi)\times\mathscr{G}(\xi)$ into $k(\xi)^*$ in the sense of Galois cohomology. The crossed product $\Gamma(k(\xi)/k, c)=\bigoplus_{\sigma\in\mathscr{G}(\xi)}k(\xi)v_\sigma$ with respect to c is a subalgebra of U with centralizer $\mathrm{Mat}_{\xi(1)}(k)$.

We shall call crossed product algebras of the type described in Lemma 1.4, *Kummer algebras.*

COROLLARY 1.5. *The index of $\mathscr{A}(G,\chi)$ is equal to the index of $\Gamma(k(\xi)/k, c)$.*

Now let the index m of $\mathscr{A}(Z^f,\xi)$ be greater than 1. Form the m-fold direct product $G^{(m)}=G\times\ldots\times G$, the (outer) product $f^{(m)}=f\times\ldots\times f$ and the (outer tensor) product $\xi^{(m)}=\xi\times\ldots\times\xi$. Then we have $\mathscr{A}(Z^{f(m)},\xi^{(m)})\cong\mathscr{A}(Z^f,\xi)\otimes_{k(\xi)}\cdots\otimes_{k(\xi)}\mathscr{A}(Z^f,\xi)$, $k(\xi^{(m)})=k(\xi)$, $\chi^{(m)}=\mathrm{Ind}_{Z^{f(m)}}^{G^{(m)}}(\xi^{(m)})$ and the index of $\mathscr{A}(Z^{f(m)},\xi^{(m)})$ is equal to 1. Hence there is a factor system c_m of $\mathscr{G}(\xi)\times\mathscr{G}(\xi)$ into $k(\xi)^*$ such that the index of $\mathscr{A}(G^{(m)},\chi^{(m)})$ is equal to the index κ of $\Gamma(k(\xi)/k, c_m)$. Hence the index of $\mathscr{A}(G,\chi)$ is equal to κm.

If the kernel of the symplectic pairing of $Z(Z^f)\times Z(Z^f)$ into k^* determined by $f\,|_{Z(Z^f)}$ is not trivial, we repeat the whole process. After r steps we arrive at a group M with nondegenerate center with respect to a factor system f' of $M\times M$ into a certain field extension L of k. At each step we get a Kummer algebra of the type described above. More precisely, we have proved the following lemma.

LEMMA 1.6. *The index of $\mathscr{A}(G,\chi)$ is equal to the product of the index of a simple component $\mathscr{A}(M,\rho)$ of a twisted group algebra (L, M, f'), where the group M has a nondegenerate center with respect to f', with the indices of a certain number of Kummer algebras. The number of these Kummer algebras is equal to the number of reduction steps which are used to arrive at a group with nondegenerate center.*

2. Groups with nondegenerate center. Now let the finite nilpotent group have a nondegenerate center with respect to a factor system $f\in Z^2(G, k^*)$. Let φ be a simple character of $C\otimes(k, Z(G), f)$ which occurs as constituent of the restriction of a simple character χ of $C\otimes(k, G, f)$ to $Z(G)$. We assume $k(\chi)=k$. By [**13**, Proposition 4.1], the

k-algebra $(k, Z(G), f)$ is central simple, i.e. $\mathscr{A}(Z(G), \varphi) \cong (k, Z(G), f)$. Therefore $I(\varphi) = G$. Clifford's theory yields a factor system $\tilde{f} \in Z^2(G/Z(G), k^*)$ and a simple character $\tilde{\chi}$ of $C \otimes (k, G/Z(G), \tilde{f})$ such that $k(\tilde{\chi}) = k$ and $\mathscr{A}(G, \chi) \cong \mathscr{A}(Z(G), \varphi) \otimes_k \mathscr{A}(G/Z(G), \tilde{\chi})$ (see [**5**, §2]); the statements there refer to the ordinary group algebra but the proofs remain unchanged in the twisted case.

If $G/Z(G)$ is not abelian, we can repeat the whole reduction process from the beginning. Let us put our results together in the following lemma.

LEMMA 2.1. *Let G be a finite nilpotent group with nondegenerate center with respect to $f \in Z^2(G, k^*)$, let χ be a simple character of $C \otimes (k, G, f)$ with values in k and let φ be a simple character of $C \otimes (k, Z(G), f)$ which occurs as a constituent of $\mathrm{Res}^G_{Z(G)}(\chi)$. Then there is a (central) factor system $\tilde{f} \in Z^2(G/Z(G), k^*)$ and a simple character $\tilde{\chi}$ of $C \otimes (k, G/Z(G), f)$ such that the central simple k-algebras $\mathscr{A}(G, \chi)$ and $(k, Z(G), f) \otimes_k \mathscr{A}(G/Z(G), \tilde{\chi})$ are isomorphic.*

3. Central simple twisted group algebras of abelian groups. The preceding results, especially Lemmas 1.6 and 2.1, have shown the importance of the following two types of algebras: (I) central simple twisted group algebras of abelian groups, (II) crossed products of Kummer extensions with respect to factor systems in the sense of Galois cohomology.

LEMMA 3.1. *Every cyclic Kummer algebra is a central simple twisted group algebra of an abelian group.*

Proof. Let $\Gamma(K/k, c)$ be a Kummer algebra. Suppose that $K = k(\sqrt[n]{a})$ is a cyclic extension of k with Galois group $\mathscr{G} = \langle \sigma \rangle$. Then $\Gamma(K/k, c) \cong \Gamma(K/k, \sigma, b)$, $b = \prod_{j=1}^{n-1} c_{\sigma, \sigma^j}$. Choose a primitive nth root of unity ξ in k. Then we may describe $\Gamma(K/k, c)$ by generators $(\sqrt[n]{a})^i$, $(\sqrt[n]{b})^j$, $0 \le i, j \le n-1$, with relations $(\sqrt[n]{a})(\sqrt[n]{b}) = \xi(\sqrt[n]{b})(\sqrt[n]{a})$. $\mathscr{G} \times \mathscr{G}$ operates on $\Gamma(K/k, \sigma, b)$ in an obvious way as a group of k-algebra automorphisms. It fixes exactly the center k. So by [**13**, 7.4] there is a (central) factor system $f \in Z^2(\mathscr{G} \times \mathscr{G}, k^*)$ such that $\Gamma(K/k, \sigma, b) \cong (k, \mathscr{G} \times \mathscr{G}, f)$.

We see that it is important to know the structure of central simple twisted group algebras of abelian groups. So let G be an abelian group, $f \in Z^2(G, k^*)$ such that the symplectic pairing $\omega : G \times G \to k^*$ associated with f is nondegenerate. We decompose the "symplectic space" (G, ω) into an orthogonal direct product of "hyperbolic planes" $(G, \omega) \cong \bigtimes_i (\mathbb{Z}(m_i) \times \mathbb{Z}(m_i), \omega_i)$, see [**14**, Theorem 3.8]. It follows that $(k, G, f) \cong \bigotimes_i (k, \mathbb{Z}(m_i) \times \mathbb{Z}(m_i), f)$. Let

$$\mathbb{Z}(m_i) \times \mathbb{Z}(m_i) \cong \langle x_i \rangle \times \langle y_i \rangle, \qquad c(x_i) = \prod_{\nu=1}^{m_i - 1} f(x_i, x_i^\nu), \qquad c(y_i) = \prod_{\mu=1}^{m_i - 1} f(y_i, y_i^\mu).$$

Then, choosing a primitive m_ith root of unity ξ_i in k, we may describe $(k, \mathbb{Z}(m_i) \times \mathbb{Z}(m_i), f)$ by generators

$$(\sqrt[m_i]{c(x_i)})^k, \qquad (\sqrt[m_i]{c(y_i)})^l, \qquad 0 \le k, l \le m_i - 1$$

with relations

$$(\sqrt[m_i]{c(x_i)})(\sqrt[m_i]{c(y_i)}) = \xi_i(\sqrt[m_i]{c(y_i)})(\sqrt[m_i]{c(x_i)}).$$

It follows that the index of this algebra divides m_i. m_i divides the order w of the group of roots of unity in k. Since k is a number field, the index equals the exponent in the Brauer group and we see that the wth exponent of the class of (k, G, f) in the Brauer group of k is trivial. Hence we have proved the next theorem.

THEOREM 3.3. *The index of a central simple twisted group algebra of an abelian group over a number field k divides the order of the group of roots of unity in k.*

Let $\mathrm{Br}_w(k)$ denote the subgroup of the Brauer group of k whose elements have orders dividing w and let W denote the group of roots of unity in k. It is proved in algebraic K-theory (Bass, Tate) that there is an isomorphism $h_1: K_2k/wK_2k \to W \otimes \mathrm{Br}_w(k)$. Hence (k, G, f) determines an element of K_2k/wK_2k which may be represented not just by a product of symbols but actually by a symbol, see [**7**]. Call this symbol $\{f_1, f_2\}_G$. The structure of (k, G, f), especially its splitting properties, is determined completely by $\{f_1, f_2\}_G$ mod wK_2k, which in turn may be "computed explicitly" with the aid of the local norm residue symbols $\left(\frac{f_1, f_2}{\mathfrak{p}}\right)$ for all primes $\mathfrak{p}$ of k, see [**6**].

4. An estimation for the Schur index. Now we are in a position to determine the index of a simple component of a twisted group algebra (k, G, f) of the finite nilpotent group G over a number field k, at least in principle, because by Lemmas 1.6, 2.1 and 3.1 this problem may be reduced to the case of a central simple twisted group algebra of an abelian group which is determined by a symbol. The formulation of a complete result is left to the reader. We content ourselves with some examples in Section 5 and the following simple estimation for the Schur index which follows directly from the above discussion.

THEOREM 4.1. *There is a radical extension L of k which is determined by the reduction process described in Lemmas* 1.6 *and* 2.1 *such that, for a simple component $\mathscr{A}(G, \chi)$ of (k, G, f), the algebra $L \otimes_{k(\chi)} \mathscr{A}(G, \chi)$ is similar to a tensor product of central simple twisted group algebras of abelian groups. In particular, by Theorem* 3.2, *the index of $\mathscr{A}(G, \chi)$ divides $(L:k)w_L$, where w_L denotes the order of the group of roots of unity in L.*

5. Examples. As first example we consider an abelian group G. Let N be the kernel of the symplectic pairing $\omega: G \times G \to k^*$ associated with $f \in Z^2(G, k^*)$. Let $\bar{G} = G/N$. Then ω defines a nondegenerate pairing from $\bar{G} \times \bar{G}$ to k^* which is also denoted by ω. Let $(k, N, f) \cong \bigtimes_i K_i$ be a decomposition into a direct sum of fields. By Reid's reduction process described in Section 1 there are factor systems $t_i: \bar{G} \times \bar{G} \to K_i^*$ such that $(k, G, f) \cong \bigtimes_i (K_i, \bar{G}, t_i)$. It follows from the explicit construction of the t_i that the pairing determined by them is ω. From this and Theorem 3.2 we conclude that the index of a simple component $(K_i, \bar{G}, t_i)$ divides w, the order of the group of roots of unity of k. Moreover, we get a norm criterion for the splitting of $(K_i, \bar{G}, t_i)$ see [**9**, 9.17].

Now let $G = P$ be an extra special p-group, i.e. the center $Z(P)$ of P has order p and

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coincides with the commutator subgroup P' of P and P/P' is isomorphic to a direct product of $2m$ copies of $\mathbb{Z}(p)$, $m \in \mathbb{N}$. Let $f \in Z^2(P, k^*)$. In studying the index of a simple component of (k, P, f) we shall make use of the results in [**10**]. We have either $Z(f) = P'$ or $Z(f) = \{e\}$.

In the first case we have, by Reid's reduction process, in the usual notation: $(k, P, f) \cong \underset{i}{\times}(K_i, P/P', t_i)$. The index of a simple component of (k, P, f) over k is equal to the index of a simple component of some $(K_i, P/P', t_i)$ over K_i. Since P/P' is abelian, we can apply the results of the abelian case.

If $Z(f) = \{e\}$ then, by [**10**, (1.2)], the group P is the nonabelian group of order p^3 and exponent p if $p \neq 2$, and the dihedral group of order 8 if $p = 2$. Therefore $Z_f \cong \mathbb{Z}(p)$, $Z^f \cong \mathbb{Z}(p) \times \mathbb{Z}(p)$. Let α be a simple character of $C \otimes (k, Z_f, f)$ which occurs as a constituent in the restriction of a simple character χ of $C \otimes (k, G, f)$ to Z_f. Let $k(\chi) = k$, $K = k(\alpha)$. Let ξ be a simple character of $C \otimes (k, Z^f, f)$ which induces χ and $\xi|_{Z_f} = l\alpha$, $l \in \mathbb{N}$. Since $Z^f/Z_f \cong \mathbb{Z}(p)$ is cyclic we have $l = 1$. Hence ξ is one-dimensional. Since $k(\xi) = k(\alpha)$, ξ is realisable over K and by Theorem 1.4 the simple component $\mathcal{A}(P, \chi)$ belonging to χ is similar to a cyclic Kummer algebra $\Gamma(K/k, c)$. The index of this algebra is either equal to 1 or to p because $\chi(1) = p$. A norm criterion for $\Gamma(K/k, c)$ determines which case actually occurs.

The final example is concerned with a metacyclic p-group P, $p \neq 2$. Let $\mathcal{A}(P, \chi)$ be a simple component of (k, P, f) with $k(\chi) = k$. We have $(k, P, f) \cong \underset{i}{\times}(K_i, \bar{P}, t_i)$, where $\bar{P}$ has a non-degenerate center with respect to t_i, see [**8**, (5.3)]. The index of $\mathcal{A}(P, \chi)$ is equal to the index of a simple component $\mathcal{A}(\bar{P}, \bar{\chi})$ of some $(K_i, \bar{P}, t_i)$. We have $K_i = k$ because of our assumption $k(\chi) = k$. By Lemma 2.1, we have $\mathcal{A}(\bar{P}, \bar{\chi}) \cong \mathcal{A}(Z(\bar{P}), \varphi) \otimes_k \mathcal{A}(\bar{P}/Z(\bar{P}), \bar{\chi})$ for some (central) factor system $\bar{f} \in Z^2(\bar{P}/Z(\bar{P}), k^*)$, a simple character $\bar{\chi}$ of $C \otimes (k, \bar{P}/Z(\bar{P}), \bar{f})$ and a simple character φ of $C \otimes (k, Z(\bar{P}), f)$ which occurs as constituent of the restriction of $\bar{\chi}$ to $Z(\bar{P})$. Continuing this process we get a norm criterion for the splitting of $\mathcal{A}(P, \chi)$ and as upper estimation for its index the order of the group of roots of unity of $k(\chi) = k$.

6. The general case. Let G be an arbitrary finite group and $f \in Z^2(G, k^*)$. To construct splitting fields for (k, G, f) we may proceed as follows: at first we choose an extension K of k such that every K-elementary subgroup of G is elementary (in particular nilpotent). We may take for K the field $k(\{\xi_p \mid p \mid |G|\})$, ξ_p a primitive pth root of unity. By [**9**, (9.2)] we have: a simple component of $K \otimes (k, G, f)$ has index 1 over $L \supseteq K$ if for every elementary subgroup H of G all simple components of $K \otimes (k, H, f)$ have index 1 over L. Probably the step from k to K is not "sharp enough". To answer these questions we have to compute the defect set belonging to (k, G, f) in the sense of axiomatic representation theory.

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