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QUASI-PERMUTATION REPRESENTATIONS OF $SL(2, q)$ AND $PSL(2, q)$

HOUSHANG BEHRAVESH
Department of Mathematics, University of Urmia, Urmia, Iran

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1. Introduction. By a quasi-permutation matrix we mean a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Thus every permutation matrix over $\mathbb{C}$ is a quasi-permutation matrix. For a given finite group G, let $p(G)$ denote the minimal degree of a faithful permutation representation of G (or a faithful representation of G by permutation matrices), let $q(G)$ denote the minimal degree of a faithful representation of G by quasi-permutation matrices over the rational field $\mathbb{Q}$, and let $c(G)$ be the minimal degree of a faithful representation of G by complex quasi-permutation matrices. See [**1**].

By a rational valued character we mean a character χ corresponding to a complex representation of G such that $\chi(g) \in \mathbb{Q}$ for all $g \in G$. As the values of the character of a complex representation are algebraic numbers, a rational valued character is in fact integer valued. A quasi-permutation representation of G is then simply a complex representation of G whose character values are rational and non-negative. The module of such a representation will be called a quasi-permutation module. We will call a homomorphism from G to $GL(n, \mathbb{Q})$ a rational representation of G and its corresponding character will be called a rational character of G. Let $r(G)$ denote the minimal degree of a faithful rational valued character of G. It is easy to see that

$$r(G) \leq c(G) \leq q(G) \leq p(G)$$

where G is a finite group.

Let $SL(m, q)$ denote the group of all $m \times m$ matrices with determinant 1 over the field of q elements where q is a power of a prime p and $PSL(m, q) \cong G/Z(G)$ where $G = SL(m, q)$. We will apply the algorithms we developed in [**1**] to the groups $SL(2, q)$ and $PSL(2, q)$. We will show that $\lim_{q\to\infty} \frac{c(G)}{r(G)} = 1$, where $G = PSL(2, q)$. The quantities $p(G)$ for the finite simple groups are known and can be found in [**5**].

2. Algorithm for $p(G)$, $c(G)$ and $q(G)$.

LEMMA 2.1. *Let G be a finite group with a unique minimal normal subgroup. Then $p(G)$ is the smallest index of a subgroup with trivial core (that is, containing no non-trivial normal subgroup).*

Proof. See [**1**, Corollary 2.4].

DEFINITION 2.2. Let χ be a character of G such that, for all $g \in G$, $\chi(g) \in \mathbb{Q}$ and $\chi(g) \geq 0$. Then we say that χ is a *non-negative rational valued character*.

NOTATION. Let $\Gamma(\chi)$ be the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$.

DEFINITION 2.3 Let G be a finite group. Let χ be an irreducible complex character of G. Then define

(1) $d(\chi) = |\Gamma(\chi)|\chi(1)$,

(2) $m(\chi) = \begin{cases} 0 & \text{if } \chi = 1_G \\ |\min\{\Sigma_{\alpha\in\Gamma(\chi)}\chi^\alpha(g) : g \in G\}| & \text{otherwise,} \end{cases}$

(3) $c(\chi) = \Sigma_{\alpha\in\Gamma(\chi)}\chi^\alpha + m(\chi)1_G$.

COROLLARY 2.4. *Let* $\chi \in \mathrm{Irr}(G)$. *Then* $\Sigma_{\alpha\in\Gamma(\chi)}\chi^\alpha$ *is a rational valued character of* G. *Moreover* $c(\chi)$ *is a non-negative rational valued character of* G *and* $c(\chi)(1) = d(\chi) + m(\chi)$.

Proof. See [**1**, Corollary 3.7].

Now we will give algorithms for calculating $c(G)$ and $q(G)$ where G is a finite group with a unique minimal normal subgroup.

LEMMA 2.5. *Let G be a finite group with a unique minimal normal subgroup. Then*

(1) $c(G) = \min\{c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$;

(2) $q(G) = \min\{m_{\mathbb{Q}}(\chi)c(\chi)(1) : \chi$ is a faithful irreducible complex character of $G\}$.

Proof. See [**1**, Corollary 3.11].

LEMMA 2.6. *Let* $\chi \in \mathrm{Irr}(G)$, $\chi \neq 1_G$. *Then* $c(\chi)(1) \geq d(\chi) + 1 \geq \chi(1) + 1$.

Proof. From Definition 2.3 it follows that $c(\chi)(1)$ is a non-negative rational valued character of G so, by [**1**, Lemma 3.2], $m(\chi) \geq 1$. Now the result follows from Definition 2.3.

LEMMA 2.7. *Let* $\chi \in \mathrm{Irr}(G)$. *Then*

(1) $c(\chi)(1) \geq d(\chi) \geq \chi(1)$;

(2) $c(\chi)(1) \leq 2d(\chi)$.

Equality occurs if and only if $Z(\chi)/\ker\chi$ *is of even order*.

Proof. (1) follows from the definition of $c(\chi)(1)$ and $d(\chi)$.
(2) See [**1**, Lemma 3.13].

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LEMMA 2.8. *Let G be a finite group. If the Schur index of each non-principal irreducible character is equal to m, then* $q(G) = mc(G)$.

Proof. See [**1**, Corollary 3.15].

3. Permutation representations.

THEOREM 3.1. *Let* $G = PSL(2, q)$, *where* $q = p^n$. *Then G contains only the following subgroups:*

(1) *elementary abelian p-groups of each order dividing q*;

(2) *cyclic groups of each order l with* $l \,\big|\, \frac{q \pm 1}{k}$ *where* $k = (q - 1, 2)$;

(3) *dihedral groups of each order 2l with l as in (2)*;

(4) *alternating group* A_4 *for* $p > 2$ *or* $p = 2$ *and* $n \equiv 0 \pmod{2}$;

(5) *symmetric group* S_4 *for* $q^2 - 1 \equiv 0 \pmod{16}$;

(6) *alternating group* A_5 *for* $p = 5$ *or* $q^2 - 1 \equiv 0 \pmod{5}$;

(7) *semidirect products of an elementary abelian group of order* p^m *and a cyclic group of order t for each* $m, 1 \leq m \leq n$, *and each t such that* $t \big| p^m - 1$ *and* $t \big| q - 1$;

(8) *the groups* $PSL(2, p^m)$ *for any m such that* $m|n$ *and* $PGL(2, p^m)$ *for any m such that* $2m|n$.

Proof. See [**3**, p. 213].

LEMMA 3.2. *Every proper normal subgroup of* $G = SL(m, K)$ *is in* $Z(G)$ *except when* $m = 2$ *and* $|K| = 2$ *or* 3.

Proof. Let $N \lhd G$, let $Z = Z(G)$ and let $N \nleq Z$. Since $G/Z \cong PSL(n, K)$, so G/Z is a simple group by [**3**, p. 182].

Now consider NZ. It is a normal subgroup of G and $1 \neq NZ/Z \lhd G/Z$. Since G/Z is simple, $NZ = G$. And $G/N = NZ/N \cong Z/Z \cap N$, so G/N is abelian. Hence $N \geq G'$ and by [**3**, p. 181] we have $G' = G$ except when $m = 2$ and $|K| = 2$ or 3. Therefore $N = G$. Hence the result follows.

LEMMA 3.3. *Let* $G = SL(2, K)$ *and* $\mathrm{char}(K) \neq 2$. *Then G has a unique involution.*

Proof. The proof is easy.

COROLLARY 3.4. *Let* $G = SL(2, K)$ *and* $\mathrm{char}(K) \neq 2$. *Then* $Z(G) = \{\pm I_2\}$ *and* $\big|Z(G)\big| = 2$. *Moreover* $Z(G)$ *is the unique minimal normal subgroup of G and the core of any subgroup of even order is non-trivial.*

Proof. By [**3**, p. 181] we know that $Z(G) = \{\pm I_2\}$. Since G has a unique involution so by Lemma 3.2 when $q \neq 3$ the unique minimal normal subgroup of G is $Z(G)$.

Now let $q = 3$. Since in this case the order of G is 24, any non-trivial subgroup of G has order 3 or even order. If its order is 3, then in the notation of [**2**, 38.1] we have two different classes in which the elements have order 3 (namely c and d). Since $\langle c \rangle = \langle d \rangle$ and also c and d are not conjugate, the subgroups of order 3 are not normal. When its order is even it contains an element of order two. Since G has a unique involution, $Z(G)$ is contained in such a subgroup. Therefore $Z(G)$ is the unique minimal normal subgroup of G.

Lemma 3.5. *Let $G = SL(2, q)$ where $q = p^n$ is odd. Then the odd order subgroups of G are as follows*:

(1) *cyclic subgroups of each odd order dividing* $q \pm 1$;

(2) *subgroups of odd order of* $T(2, q) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in F_q, a \neq 0 \right\}$, *where* F_q *is the finite field of* q *elements (note that* $|T(2, q)| = (q-1)q$*).*

Proof. Let $H \leq G$ and let $Z = Z(G)$. Let $|H|$ be odd. We know that $ZH/Z \cong H/Z \cap H$. Since $|H|$ is odd so $Z \cap H = \{1\}$. But $ZH/Z \leq G/Z$. So odd order subgroups of G are isomorphic to odd order subgroups of $PSL(2, q)$, and by Theorem 3.1 the odd order subgroups are of type (1), (2) and (7). Since p is odd, in Theorem 3.1 part (2), we have $k = 2$ and $l \,\Big|\, \frac{q\pm1}{2}$. Hence $l \,|\, q \pm 1$. So G has cyclic subgroups of each odd order dividing $q \pm 1$.

Now we want to prove that each odd order subgroup of type (7) in Theorem 3.1 is isomorphic to a subgroup of $T = T(2, q)$. In fact we will show that it is conjugate to a subgroup of T.

Let H be an odd order subgroup of $PSL(2, q)$ of type (7). Then $H = L/Z$ where $L \leq G$. Since the order of H is odd so $(|L/Z|, |Z|) = 1$. So by Schur–Zassenhaus [7, Theorem 10.30] we have $L = Z \rtimes H_1$ where $H_1 \leq L$ and $L/Z \cong H_1$. So $H \cong H_1$. Hence $H_1 = B \rtimes A$ where B is an elememtary abelian group p^m and A is a cyclic subgroup of order t such that $t \,|\, p^m - 1$ and $t \,|\, p^n - 1$.

Let $U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in F_q \right\}$. Then U is a Sylow p-subgroup of G. By the Sylow Theorem [7, 5.9] there exists $g \in G$ such that $B^g \leq U$. So $H_1^g = B^g \rtimes A^g$. Now we have to show that $H_1^g \leq T$. Hence it is enough to prove that $A^g = A_1 \leq T$. Let $\xi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A_1$ and $\eta = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in B^g$ and $\lambda \neq 0$, Then $\xi\eta\xi^{-1} \in B^g$. But $\xi\eta\xi^{-1} = \begin{pmatrix} 1 - ca\lambda & a^2\lambda \\ -c^2\lambda & 1 + ca\lambda \end{pmatrix}$. So $c^2\lambda = 0$. Therefore $c = 0$, and $\xi \in T$.

Case (1) is similar to (7).

Theorem 3.6. *Let $G = SL(2, q)$ where q is odd. Then*

$$p(G) = (q-1)_2(q+1).$$

Proof. By Lemma 2.1 we have to find a subgroup of G with maximal order and trivial core, say H. If $|H|$ be even then by Corollary 3.4 its core is not trivial. So $|H|$ is odd. Conversely by Corollary 3.4 every subgroup of odd order has trivial core.

We will use Lemma 3.5 frequently . Let $q \equiv 3 \pmod 4$, that is, $\frac{q-1}{2} \equiv 1 \pmod 2$. By Lemma 3.5 we have $|H| = q\left(\frac{q-1}{2}\right)$ and $p(G) = 2(q+1)$.

Let $q \equiv 1 \pmod 4$, that is, $\frac{q-1}{2} \equiv 0 \pmod 2$ and $\frac{q+1}{2} \equiv 1 \pmod 2$. But $q > \frac{q+1}{2} > \frac{q-1}{2}$ (as $q \geq 3$). Thus, the Sylow p-subgroup of G has order exceeding that of any odd order subgroup of type (1). On the other hand, if

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$H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a, b \in F_q, a^l = 1 \right\}$, where $q - 1 = (q-1)_2 l$, then H is of type (2) and of order ql which is maximal. Hence $p(G) = (q-1)_2(q+1)$.

LEMMA 3.7. *Let $G = SL(2, q)$ where $q = 2^n$. Then $SL(2, q)$ is a simple group when $n \neq 1$, and when $n = 1$ it has a unique minimal normal subgroup, which has order 3.*

Proof. See [**3**, p. 182].

THEOREM 3.8. *Let $G = SL(2, q)$ where $q = 2^n$. Then $p(G) = q + 1$.*

Proof. We show that every proper subgroup H of G has order less than or equal to $q(q-1)$. Let $p^m = q$ and $t = q - 1$. Then by Theorem 3.1 a subgroup of type (7) exists whose order is equal to $q(q-1)$.

Let $n = 1$. Then $|G| = 6$ and it has a subgroup of order 2 with trivial core and a normal subgroup of order 3. So $p(G) = \frac{6}{2} = 3$.

Now let $n \neq 1$. Note that $\left|SL(2,4)\right| = 60$ and $SL(2,4) \cong A_5$. So subgroups of type (6) cannot be considered when $n = 2$. We will use Theorem 3.1 frequently.

Subgroups of type (1), (2), (3), (7). By Theorem 3.1 part (1), (2), (3), (7) the orders of such subgroups of G are less than or equal to $q, q \pm 1, 2(q \pm 1)$ and $q(q-1)$ respectively. But $2(q+1) < q(q-1)$ because $q^2 - 3q - 2 > 0$ when $q \geq 4$. So among these subgroups of G the maximal order is $q(q-1)$.

Subgroup of type (4). Let $n = 2k$, that is, $q = 4^k$. Then G has a subgroup of order 12 by Theorem 3.1 part (4). But $q(q-1) \geq 12$ (as $k \geq 1$ and $q \geq 4$).

Subgroup of type (5). As q is a power of 2, $16 \nmid q^2 - 1$. So S_4 is not a subgroup of G.

Subgroup of type (6). Let $2^{2n} \equiv 1 \pmod 5$. Then by an earlier remark, we may assume that $n \geq 3$. Further, if $n = 3$, $2^6 = 64 \equiv -1 \pmod 5$ so that we may assume that $n \geq 4$. Now $q \geq 2^4 = 16$ and $q(q-1) \geq 16 \times 15 > |A_5| = 60$.

Subgroup of type (8). We will consider two different cases.

Let $m|n$ and $2m \nmid n$, that is, $n = m(2k+1)$. Theorem 3.1 part (8) implies that $PSL(2, 2^m)$ is a subgroup of G, and $\left|PSL(2, 2^m)\right| = (2^m - 1)2^m(2^m + 1)$. We have

$$(2^m - 1)(2^m + 1) \leq (2^{mk} - 1)(2^{mk} + 1) = 2^{2mk} - 1 \leq 2^{m(2k+1)-1}$$

so

$$(2^m - 1)2^m(2^m + 1) \leq 2^m(2^{m(2k+1)} - 1) \leq 2^{m(2k+1)}(2^{m(2k+1)} - 1) = q(q-1).$$

Now let $2m \mid n$. Then $n = 2mk$. We know that $\left|PGL(2, 2^m)\right| = (2^m - 1)2^m(2^m + 1)$ and $(2^m - 1)(2^m + 1) \leq 2^{2mk} - 1$ so

$$(2^m - 1)2^m(2^m + 1) \leq 2^m(2^{2mk} - 1) \leq 2^{2mk}(2^{2mk} - 1) = q(q-1).$$

Therefore in both cases $(2^m - 1)2^m(2^m + 1) \leq q(q-1)$. Hence $p(G) = q + 1$.

THEOREM 3.9. *Let $G = PSL(2, q)$ where q is odd. Then $p(G) = q + 1$ except when $q = 5, 7, 9, 11$ and in these cases $p(G) = 5, 7, 6, 11$ respectively.*

Proof. When $q \geq 5$, the result follows from [**3**, II.8.27 and II.8.28] because G is simple so that every non-trivial permutation representation is faithful.

When $q = 3$, G is isomorphic to the alternating group A_4 of degree 4 in which a Sylow 3-subgroup is core-free and of minimal index among such subgroups.

4. Quasi-permutation representations. We begin with a brief summary of facts relevant to our treatment of the special linear and projective special linear groups.

THEOREM 4.1. *Let F be the finite field of $q = p^n$ elements, p an odd prime, and let v be a generator of the cyclic group of $F^* = F - \{0\}$. Let*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, z = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, d = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}, a = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

in $G = SL(2, F)$. G contains an element b of order $q + 1$.

For any $x \in G$, let (x) denote the conjugacy class of G containing x. Then G has exactly $q + 4$ conjugacy classes (1), (z), (c), (d), (zc), (zd), (a), (a^2),...,$(a^{\frac{q-3}{2}})$, (b), (b^2),...,$(b^{\frac{q-1}{2}})$, satisfying

Table of Conjugacy Classes of $SL(2,p^n)$

x	1	z	c	d	zc	zd	a^l	b^m
$\lvert(x)\rvert$	1	1	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$\frac{1}{2}(q^2-1)$	$q(q+1)$	$q(q-1)$

for $1 \leq l \leq (q-3)/2$, $1 \leq m \leq (q-1)/2$.

Put $\varepsilon = (-1)^{(q-1)/2}$. Let $\rho \in \mathbb{C}$ be a primitive $(q-1)$-th root of 1, $\sigma \in \mathbb{C}$ a primitive $(q+1)$-th root of 1. Then the complex character table of G is

Character Table of $SL(2,p^n)$

	1	z	c	d	a^l	b^m
1_G	1	1	1	1	1	1
ψ	q	q	0	0	1	-1
χ_i	$q+1$	$(-1)^i(q+1)$	1	1	$\rho^{il}+\rho^{-il}$	0
θ_j	$q-1$	$(-1)^j(q-1)$	-1	-1	0	$-(\sigma^{jm}+\sigma^{-jm})$
ξ_1	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$(-1)^l$	0
ξ_2	$\frac{1}{2}(q+1)$	$\frac{1}{2}\varepsilon(q+1)$	$\frac{1}{2}(1-\sqrt{\varepsilon q})$	$\frac{1}{2}(1+\sqrt{\varepsilon q})$	$(-1)^l$	0
η_1	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	0	$(-1)^{m+1}$
η_2	$\frac{1}{2}(q-1)$	$-\frac{1}{2}\varepsilon(q-1)$	$\frac{1}{2}(-1-\sqrt{\varepsilon q})$	$\frac{1}{2}(-1+\sqrt{\varepsilon q})$	0	$(-1)^{m+1}$

for $1 \leq i \leq (q-3)/2$, $1 \leq j \leq (q-1)/2$, $1 \leq l \leq (q-3)/2$, $1 \leq m \leq (q-1)/2$. (The columns for the classes (zc) and (zd) are missing in this table. These values are obtained from the relations

$$\chi(zc) = \frac{\chi(z)}{\chi(1)}\chi(c),\ \chi(zd) = \frac{\chi(z)}{\chi(1)}\chi(d),$$

for all irreducible characters χ of G.)

Proof. See [**2**, 38.1].

THEOREM 4.2. *Let F be the finite field of $q = 2^n$ elements, and let v be a generator of the cyclic group $F^* = F - \{0\}$. Let*

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, a = \begin{pmatrix} v & 0 \\ 0 & v^{-1} \end{pmatrix}$$

in $G = SL(2, F)$. G contains an element b of order $q + 1$.

For any $x \in G$, let (x) denote the conjugacy class of G containing x. Then G has exactly $q + 1$ conjugacy classes $(1), (c), (a), (a^2), \ldots, (a^{(q-2)/2}), (b), (b^2), \ldots, (b^{q/2})$, where

Table of Conjugacy Classes of $SL(2,2^n)$

x	1	c	a^l	b^m
$\lvert(x)\rvert$	1	$(q^2 - 1)$	$q(q + 1)$	$q(q - 1)$

for $1 \le l \le (q - 2)/2$, $1 \le m \le q/2$.

Let $\rho \in \mathbb{C}$ be a primitive $(q - 1)$-th root of 1. The table of G over $\mathbb{C}$ is

Character Table of $SL(2,2^n)$

	1	c	a^l	b^m
1_G	1	1	1	1
ψ	q	0	1	-1
χ_i	$q + 1$	1	$\rho^{il} + \rho^{-il}$	0
θ_j	$q - 1$	-1	0	$-(\sigma^{jm} + \sigma^{-jm})$

for $1 \le i \le (q - 2)/2$, $1 \le j \le q/2$, $1 \le l \le (q - 2)/2$, $1 \le m \le q/2$.

Proof. See [**2**, 38.2].

THEOREM 4.3. *Let $G = SL(2, q)$. If q is a power of 2, then the Schur index of any irreducible character of G over the rational numbers $\mathbb{Q}$ is 1. If q is a power of an odd prime p, then the Schur indices of the irreducible characters of G over the rational numbers $\mathbb{Q}$ are as follows:*

Table of Schur Indices

	$q \equiv 1 \pmod 4$	$q \equiv 3 \pmod 4$
1_G	1	1
ψ	1	1
χ_i	2 (i odd)	2 (i odd)
	1(i even)	1 (i even)
θ_j	2 (j odd)	2 (j odd)
	1 (j even)	1 (j even)
ξ_1	1	1
ξ_2	1	1
η_1	2	1
η_2	2	1

Proof. See [**8**].

LEMMA 4.4. *Let G be a finite group and let $N \triangleleft G$.*
(1) *Let χ be a character of G. Define $\hat{\chi}(Ng) = \chi(g)$. Then $\hat{\chi}$ is a character of G/N.*
(2) $\chi \in \mathrm{Irr}(G/N)$ *if and only if* $\hat{\chi} \in \mathrm{Irr}(G/N)$.

Proof. See [**4**, 2.22].

Let χ be a character of G and N a normal subgroup of G. As $\hat{\chi}(Ng) = \chi(g)$ for all $g \in G$, it is convenient to use the notation χ in place of $\hat{\chi}$ for this character of G/N.

THEOREM 4.5. *All irreducible characters of $PSL(2, q)$ have Schur index 1 over $\mathbb{Q}$ The irreducible characters of $PSL(2, q)$ where q is odd are:*

$$(1)\ \ 1, \psi, \chi_2, \chi_4, \dots, \chi_{\frac{q-5}{2}}, \theta_2, \theta_4, \dots \theta_{\frac{q-1}{2}}, \xi_1, \xi_2 \ \text{ if } q \equiv 1 \pmod 4;$$

$$(2)\ \ 1, \psi, \chi_2, \chi_4, \dots, \chi_{\frac{q-3}{2}}, \theta_2, \theta_4, \dots, \theta_{\frac{q-3}{2}}, \eta_1, \eta_2 \ \text{ if } q \equiv 3 \pmod 4.$$

Proof. Since $PSL(2, q) \cong SL(2, q)/Z(SL(2, q))$, we can find the irreducible characters of $PSL(2, q)$ from the non-faithful irreducible characters of $SL(2, q)$ by using Lemma 4.4.

LEMMA 4.6. *If $G = SL(2, q)$ where q is odd, and if χ is a faithful irreducible character of G, then $m(\chi) = 2d(\chi)$. It follows that*

$$c(G) = 2\min\{d(\chi) : \chi \in \mathrm{Irr}(G), \chi \textit{ faithful}\};$$

$$q(G) = 2\min\{m_{\mathbb{Q}}(\chi)d(\chi) : \chi \in \mathrm{Irr}(G), \chi \textit{ faithful}\}.$$

Proof. As χ is faithful and $z^2 = 1$, $\chi(z) = -\chi(1)$. Thus $z \in Z(\chi)/\ker\chi$. Therefore $Z(\chi)/\ker\chi$ is of even order. Hence by Lemma 2.7, $m(\chi) = 2d(\chi)$.

As G has a unique minimal normal subgroup by Corollary 2.5, the result follows from Corollary 3.4.

LEMMA 4.7. *Let ξ be a primitive nth root of unity. Then $\xi + \xi^{-1}$ is rational if and only if $n = 1, 2, 3, 4, 6$. The values which occur are as follows:*

n	1	2	3	4	6
$\xi + \xi^{-1}$	2	-2	-1	0	1

Proof. The result is clear for $n = 1$ or $n = 2$ so that we may assume that $n \geq 3$.

As $x^2 - (\xi + \xi^{-1})x + 1 = (x - \xi)(x - \xi^{-1})$, the index $\big(\mathbb{Q}(\xi) : \mathbb{Q}(\xi + \xi^{-1})\big) = 2$ unless $\xi \in \mathbb{Q}$, that is, unless $n = 1$ or 2. It follows that $\xi + \xi^{-1} \in \mathbb{Q}$ if and only if $\phi(n) = (\mathbb{Q}(\xi) : \mathbb{Q}) = 2$. Examination of the possibilities shows that $\phi(n) = 2$ if and only if $n = 3, 4$ or 6.

COROLLARY 4.8. *Let ξ be a primitive nth root of unity and $m \in \mathbb{Z}$. If $\xi + \xi^{-1} \in \mathbb{Q}$, then so is $\xi^m + \xi^{-m}$.*

Proof. This follows from Lemma 4.7.

COROLLARY 4.9. *Let $n = 2k$ and ξ be a primitive nth root of unity. Then $\xi + \xi^{-1}$ is rational if and only if $k = 1, 2, 3$.*

Proof. $2k = 1, 2, 3, 4, 6$ by Lemma 4.7. So $k = 1, 2, 3$.

COROLLARY 4.10. *Let ξ be a primitive nth root of unity. Let $1 \leq j \leq n$. Then $\xi^j + \xi^{-j}$ is rational if and only if $n = j, 2j, 3j, 4j, 6j, \frac{3}{2}j, \frac{4}{3}j, \frac{6}{5}j$.*

Proof. Let (j, n) denote the greatest common divisor of j and n. Write $j = a(j, n)$ and $n = b(j, n)$ so that a and b are coprime and $0 < \frac{a}{b} \leq 1$.

As ξ^j is a primitive bth root of unity, Lemma 4.7 shows that $\xi^j + \xi^{-j}$ is rational if and only if $b = 1, 2, 3, 4$ or 6. For these values of b, the corresponding possibilities for $\frac{a}{b}$ are $1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{6}$ and $\frac{5}{6}$. As $j = \frac{a}{b}n$, the result follows.

LEMMA 4.11. *Let σ be a primitive $(q+1)$th root of unity and let $q = p^n$ where p is an odd prime. Suppose that $q \equiv 7 (\mathrm{mod}\ 8)$ and that $j = 1, 3, \ldots, \frac{q-1}{2}$. Then $\sigma^j + \sigma^{-j}$ is not rational.*

Proof. Suppose that $\sigma^j + \sigma^{-j} \in \mathbb{Q}$. As $1 \leq j \leq \frac{q-1}{2}$, Corollary 4.10 implies that $j = \frac{q+1}{d}$ for $d = 3, 4$ or 6. By hypothesis, $8 \big| q + 1$ so that $\frac{q+1}{d}$ is even for $d = 3, 4$ or 6. This contradicts the assumption that j is odd.

LEMMA 4.12. *Let q be a power of an odd prime. Let ξ be a primitive $(q+1)$th root of unity. If $q \equiv 3 (\mathrm{mod}\ 8)$ and l is a positive integer, then $\xi^{\frac{q+1}{4}l} + \xi^{-\frac{q+1}{4}l}$ is rational.*

Proof. This follows from Corollary 4.10 and Corollary 4.8.

COROLLARY 4.13. *Let $G = SL(2, q)$ where q is odd. If $q \equiv 3 (\text{mod } 8)$ then $\theta_{\frac{q+1}{4}}$ is a faithful irreducible rational valued character.*

Proof. This follows from Lemma 4.12 and the character table of G.

THEOREM 4.14. *Let $G = SL(2, q)$ where $q = p^n$ is odd. If $q \equiv 1$ (mod 4) then*

$$q(G) = 2c(G) = \begin{cases} 2(q-1) & \text{if } n \text{ is even} \\ 4(q-1) & \text{otherwise} \end{cases}.$$

If $q \equiv 3$ (mod 4) then

$$c(G) = \begin{cases} 2(q+1) & \text{if } q \equiv 7 (\text{mod } 8) \\ 2(q-1) & \text{if } q \equiv 3 (\text{mod } 8) \end{cases}$$

and

$$q(G) = 2(q+1).$$

Proof. By Lemma 4.6 we need to look at each faithful irreducible character χ, say, and calculate $d(\chi)$.

By Lemma 2.7(1) we have

$$d(\chi_i) \geq q+1.$$

$d(\theta_j) = |\Gamma_j|(q-1) \geq q-1$ where $\Gamma_j = \Gamma(\mathbb{Q}(\theta_j) : \mathbb{Q})$. Hence $d(\theta_j) \geq q-1$. But by Lemma 4.11 we can sharpen this inequality when $q \equiv 7$ (mod 8) and $j = 1, 3, \ldots, \frac{q-1}{2}$ as $|\Gamma_j| \geq 2$. So in this case $d(\theta_j) \geq 2(q-1)$. Also, when $q \equiv 3$ (mod 8), then $\frac{q+1}{4}$ is odd and $1 \leq \frac{q+1}{4} \leq \frac{q-1}{2}$ so by Corollary 4.13 the character $\theta_{\frac{q+1}{4}}$ is an irreducible rational valued character. Therefore $\left|\Gamma_{\frac{q+1}{4}}\right| = 1$ and $d(\theta_{\frac{q+1}{4}}) = q-1$.

$$d(\xi_1) = d(\xi_2) = \tfrac{1}{2}|\Gamma_\xi|(q+1) \text{ where } \Gamma_\xi = \Gamma(\mathbb{Q}(\xi_1) : \mathbb{Q}) = \Gamma(\mathbb{Q}(\xi_2) : \mathbb{Q}).$$

$$d(\eta_1) = d(\eta_2) = \tfrac{1}{2}|\Gamma_\eta|(q-1) \text{ where } \Gamma_\eta = \Gamma(\mathbb{Q}(\eta_1) : \mathbb{Q}) = \Gamma(\mathbb{Q}(\eta_2) : \mathbb{Q}).$$

Moreover

$$|\Gamma_\xi| = |\Gamma_\eta| = \begin{cases} 1 & \text{if } n \text{ is even and } \varepsilon = 1 \\ 2 & \text{otherwise.} \end{cases}$$

First let $q \equiv 1 \bmod 4$. Then by [**2**, 38.1] we have $\varepsilon = 1$. Hence the faithful irreducible characters are $\eta_1, \eta_2, \chi_1, \chi_3, \ldots, \chi_{\frac{q-3}{2}}, \theta_1, \theta_3, \ldots, \theta_{\frac{q-3}{2}}$. Also by [**8**] the Schur index for each faithful irreducible character is equal to 2 so by Lemma 2.8 we have $q(G) = 2c(G)$.

For n even we have $d(\eta_1) = d(\eta_2) = \frac{1}{2}(q-1)$ and this is the minimal value.

For n odd we have $d(\eta_1) = d(\eta_2) = q-1$.

Next let $q \equiv 3 \pmod 4$. Then by [**2**, 38.1] we have $\varepsilon = -1$. Hence the faithful irreducible characters are $\xi_1, \xi_2, \chi_1, \chi_3, \ldots, \chi_{\frac{q-5}{2}}, \theta_1, \theta_3, \ldots, \theta_{\frac{q-1}{2}}$.

In this case $d(\xi_1) = d(\xi_2) = q+1$ and $m_{\mathbb{Q}}(\xi_1) = m_{\mathbb{Q}}(\xi_2) = 1$.

Finally, note that, when $q \equiv 3 \pmod 8$, $\theta_{\frac{q+1}{4}}$ is rational valued and $d(\theta_{\frac{q+1}{4}}) = q-1$, the minimal value. When $q \equiv 7 \pmod 8$, then by Lemma 4.11, the minimal value is achieved by ξ_1 as $2(q-1) \geq q+1$.

An overall picture is provided by the tables, compiled using Lemma 4.6, [**2**, 38.1] for the Schur indices and the preceding arguments.

q	$\equiv 1 \pmod 4$		$\equiv 3 \pmod 4$	
q	n even	n odd	$\equiv 3 \pmod 8$	$\equiv 7 \pmod 8$
$d(\chi_i)$	$\geq q+1$	$\geq q+1$	$\geq q+1$	$\geq q+1$
$d(\theta_j)$	$\geq q-1$	$\geq q-1$	$\geq q-1$	$\geq 2(q-1)$
$d(\xi_1)$	not faithful	not faithful	$q+1$	$q+1$
$d(\eta_1)$	$\frac{1}{2}(q-1)$	$q-1$	not faithful	not faithful
$c(G)$	$q-1$	$2(q-1)$	$2(q-1)$	$2(q+1)$

q	$\equiv 1 \pmod 4$		$\equiv 3 \pmod 4$	
q	n even	n odd	$\equiv 3 \pmod 8$	$\equiv 7 \pmod 8$
$m_{\mathbb{Q}}(\chi_i)d(\chi_i)$	$\geq 2(q+1)$	$\geq 2(q+1)$	$\geq 2(q+1)$	$\geq 2(q+1)$
$m_{\mathbb{Q}}(\theta_j)d(\theta_j)$	$\geq 2(q-1)$	$\geq 2(q-1)$	$\geq 2(q-1)$	$\geq 4(q-1)$
$m_{\mathbb{Q}}(\xi_1)d(\xi_1)$	not faithful	not faithful	$q+1$	$q+1$
$m_{\mathbb{Q}}(\eta_1)d(\eta_1)$	$(q-1)$	$2(q-1)$	not faithful	not faithful
$q(G)$	$2(q-1)$	$4(q-1)$	$2(q+1)$	$2(q+1)$

LEMMA 4.15. *Let $G = SL(2,2)$. Then*

$$d(\psi) = 2;$$
$$c(\psi)(1) = 3;$$
$$q(G) = c(G) = 3.$$

Proof. From [**8**] the Schur index of each irreducible character is 1. So by Lemma 2.8 we have $c(G) = q(G)$.

Since the only faithful irreducible character of G is ψ, the result follows.

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Lemma 4.16. *Let $G = SL(2, q)$ where $q = 2^n$ and $n \geq 2$. Then for each j, $1 \leq j \leq \frac{q}{2}$*

(1) *θ_j is rational if and only if $q \equiv -1 (\text{mod } 3)$ and $j = \frac{q+1}{3}$*;

(2) *$d(\theta_j) \geq q - 1$, and equality holds if θ_j is rational*;

(3) *$c(\theta_j)(1) \geq q + 1$, and equality holds if θ_j is rational.*

Proof. As $1 \leq j \leq \frac{q}{2} < \frac{q+1}{2}$ and as σ is a primitive $(q+1)$th root of unity, Corollaries 4.10 and 4.8 imply that θ_j is rational if and only if $j = \frac{q+1}{6}, \frac{q+1}{4}, \frac{q+1}{3}$. Since $q + 1$ is odd, $\frac{q+1}{6}$ and $\frac{q+1}{4}$ are not integers. Thus, $\sigma^j + \sigma^{-j} \in \mathbb{Q}$ if and only if $3 \big| (q+1)$ and $j = \frac{q+1}{3}$. This proves (1).

If θ_j is not rational, then $|\Gamma| \geq 2$ where $\Gamma = \Gamma\big(\mathbb{Q}(\theta_j) : \mathbb{Q}\big)$ so that $c(\theta_j)(1) \geq d(\theta_j) \geq 2(q-1) > q+1$ by Lemma 2.7. On the other hand if $3 \big| (q+1)$, then $8 \leq q$ so that $3 \leq \frac{q}{2}$; but $\theta_{\frac{q+1}{3}}(b^3) = -2 \leq \theta_{\frac{q+1}{3}}(g)$ for all $g \in G$ so that $m\Big(\theta_{\frac{q+1}{3}}\Big) = 2$. Thus $d\Big(\theta_{\frac{q+1}{3}}\Big) = q - 1$ and $c\Big(\theta_{\frac{q+1}{3}}\Big)(1) = q + 1$. This completes the proofs of (2) and (3).

Since $PSL(2, 2^n) \cong SL(2, 2^n)$, we will calculate $c(G)$ and $q(G)$ for $SL(2, 2^n)$.

Theorem 4.17. *Let $G = SL(2, q)$ where $q = 2^n$. Then*

$$c(G) = q(G) = q + 1.$$

Proof. From [8] the Schur index of each irreducible character is 1. So by Lemma 2.8 we have $c(G) = q(G)$.

(a) Let $q = 2$. Then by Lemma 4.15, $c(G) = q(G) = 3$.
(b) Lemma 2.7(1) shows that $d(\chi_i) \geq q + 1$, while Lemma 4.16 has dealt with θ_j.

The values are set out in the following tables.

Table (1)

q	2	$\equiv -1$ (mod 3)	otherwise
$d(\psi)$	2	q	q
$d(\chi_i)$	no χ_i exists	$\geq q+1$	$\geq q+1$
$d(\theta_j)(1)$	not faithful	$\geq q-1$	$> q-1$

q	2	$\equiv -1$ (mod 3)	otherwise
$c(\psi)(1)$	3	$q+1$	$q+1$
$c(\chi_i)(1)$	no χ_i exists	$\geq q+1$	$\geq q+1$
$c(\theta_j)(1)$	not faithful	$\geq q+1$	$> q+1$
$c(G)$	3	$q+1$	$q+1$

The next result concerns the groups $PSL(2, q)$ for q odd. Aside from the case $q = 3$, these groups are simple so that their non-trivial irreducible characters are faithful. As explained in Lemma 4.4, the characters of $PSL(2, q)$ are obtained from those of $SL(2, q)$ and we will use the names of its characters as given in [**2**, 38.1] in what follows.

LEMMA 4.18. *Let* $G = PSL(2, q)$ *where* $q = p^n$ *and* q *is odd. Let* n *be odd and* $q \notin C = \{3, 5, 7, 11\}$. *Then* $c(\theta_j)(1) \geq q + 1$ *for* $j, 0 \leq j \leq \frac{q-1}{2}$.

Proof. If θ_j is not rational valued, then $|\Gamma| \geq 2, \Gamma = \Gamma\big(\mathbb{Q}(\theta_j) : \mathbb{Q}\big)$, so that $c(\theta_j)(1) \geq d(\theta_j) = |\Gamma|\theta_j(1) \geq 2(q - 1) \geq q + 1$.

If it is rational valued, then, by Lemma 4.10, $j = \frac{q+1}{d}$ for $d = 3$, 4 or 6 and $\theta_j(\bar{b}^d) = -2$ where $\bar{b}$ denotes the image of b in $PSL(2, q)$. As $q > 11, b^d \neq z$ so that $m(\theta_j) = 2$ and $c(\theta_j)(1) = q - 1 + 2 = q + 1$.

THEOREM 4.19. *Let* $G = PSL(2, q)$ *where* $q = p^n$ *is odd. Then*

$$(1) \quad c(G) = q(G) = \begin{cases} \frac{1}{2}\big(q + \sqrt{q}\big) & \textit{if } n \textit{ is even}, \\ q + 1 & \textit{otherwise}, \end{cases}$$

if $q \notin \{5, 7, 11\}$;

$$(2) \quad c(G) = q(G) = 5, 7, 11 \textit{ if } q = 5, 7, 11, \quad \textit{respectively}.$$

Proof. From [**8**] the Schur index of each irreducible character is 1. So by Lemma 2.8 we have $c(G) = q(G)$.

By [**8**], ψ is an irreducible rational valued character of G. So

$$c(\psi)(1) = q + 1.$$

From Lemma 2.7(1), for all i,

$$c(\chi_i)(1) \geq q + 1.$$

That $c(\theta_j)(1) \geq q + 1$ for all j was shown in Lemma 4.18.

Let $q \notin \{3, 5, 7, 11\}$. If $q \equiv 1 \pmod 4$ then, by [**8**],

$$c(\xi_1)(1) = c(\xi_2)(1) = \begin{cases} \frac{q+1}{2} + \frac{\sqrt{q}-1}{2} = \frac{q+\sqrt{q}}{2} & \text{if } n \text{ even}, \\ q + 3 & \text{otherwise}. \end{cases}$$

If $q \equiv 3 \bmod 4$ then $\varepsilon = -1$ and, by [**8**],

$$c(\eta_1)(1) = c(\eta_2)(1) = q + 1.$$

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As $q+2 \geq \sqrt{q}$, $q+1 \geq \frac{q+\sqrt{q}}{2}$. This establishes (1) as can be seen in the summary tables which follow.

Table (2)

$q \notin \{3, 5, 7, 11\}$				
q	$\equiv 1 \pmod 4$		$\equiv 3 \pmod 4$	
q	n even	n odd	$\equiv 3 \pmod 8$	$\equiv 7 \pmod 8$
$d(\psi)$	q	q	q	q
$d(\chi_i)$	$\geq q+1$	$\geq q+1$	$\geq q+1$	$> q+1$
$d(\theta_j)$	$\geq q-1$	$\geq q-1$	$\geq q-1$	$\geq q-1$
$d(\xi_1)$	$\frac{1}{2}(q+1)$	$q+1$	no ξ_1 exsists	no ξ_1 exists
$d(\eta_1)$	no η_1 exists	no η_1 exists	$q-1$	$q-1$

$q \notin \{3, 5, 7, 11\}$				
q	$\equiv 1 \pmod 4$		$\equiv 3 \pmod 4$	
q	n even	n odd	$\equiv 3 \pmod 8$	$\equiv 7 \pmod 8$
$c(\psi)(1)$	$q+1$	$q+1$	$q+1$	$q+1$
$c(\chi_i)(1)$	$\geq q+1$	$\geq q+1$	$\geq q+1$	$\geq q+1$
$c(\theta_j)(1)$	$\geq q+1$	$\geq q+1$	$\geq q+1$	$\geq q+1$
$c(\xi_1)(1)$	$\frac{q+\sqrt{q}}{2}$	$q+3$	no ξ_1 exists	no ξ_1 exists
$c(\eta_1)(1)$	no η_1 exists	no η_1 exists	$q+1$	$q+1$
$c(G)$	$\frac{q+\sqrt{q}}{2}$	$q+1$	$q+1$	$q+1$

Now let $q \in \{3, 5, 7, 11\}$. We will show that when $q \in \{5, 7, 11\}$ then $c(\theta_j)(1) = q$ and this value is minimal. From Lemma 2.7(1) we have

Table (3)

q	3	5	7	11
$d(\psi)$	3	5	7	11
$d(\chi_i)$	no χ_i exists	no χ_i exists	≥ 8	≥ 12
$d(\theta_j)$	no θ_j exists	4	6	10
$d(\xi_1)$	no ξ_1 exists	6	no ξ_1 exists	no ξ_1 exists
$d(\eta_1)$	2	no η_1 exists	6	10

Let $q = 3$. Then ψ, η_1 and η_2 are the faithful irreducible characters of G. Note that $d(\eta_1) = d(\eta_2) = 2$ and $m(\eta_1) = m(\eta_2) = 2$. Therefore $c(G) = 4$.

Let $q = 5$. Then the irreducible characters of G are ψ, θ_2, ξ_1 and ξ_2. Here θ_2 is rational valued. Also $m(\theta_2) = 1$ so $c(\theta_2)(1) = 5$. Therefore $c(G) = 5$.

Let $q = 7$. Then the irreducible characters of G are $\psi, \chi_2, \theta_2, \eta_1$ and η_2. But $m(\theta_2) = 1$ so $c(\theta_2)(1) = 7$. Also by Lemma 2.6 we have $c(\eta_1)(1) = c(\eta_2)(1) \geq 7$. Therefore $c(G) = 7$.

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Let $q = 11$. Then the irreducible characters of G are ψ, χ_1, χ_4, θ_2, θ_4, η_1 and η_2. But $m(\theta_2) = 1$ so $c(\theta_2)(1) = 11$. Also by Lemma 2.6 we have $c(\theta_4)(1) \geq 11$ and $c(\eta_1)(1) = c(\eta_2)(1) \geq 11$. Therefore $c(G) = 11$.

5. Rational valued characters.

LEMMA 5.1. *Let G be a finite group. Let G have a unique minimal normal subgroup. Then*

$$r(G) = \min\{d(\chi) : \chi \textit{ is a faithful irreducible character of } G\}.$$

Proof. Let $\chi \in \mathrm{Irr}(G)$. Then $\Sigma_{\alpha\in\Gamma}\chi^{\alpha}$, where $\Gamma = \Gamma(\mathbb{Q}(\chi) : \mathbb{Q})$ is an irreducible rational valued character by [**4**. Corollary 10.2].

Let ϕ be a faithful rational valued character such that $r(G) = \phi(1)$. Since G has a unique minimal normal subgroup, there exists a faithful irreducible character, say χ, such that $[\phi, \chi] \neq 0$. So $\phi = \Sigma_{\alpha\in\Gamma}\chi^{\alpha} + \psi$, for some rational valued character ψ. Hence $\phi(1) \geq \Sigma_{\alpha\in\Gamma}\chi^{\alpha}(1) = d(\chi)$. So $r(G) = d(\chi)$.

LEMMA 5.2. *Let* $G = SL(2, q)$ *where q is odd. Then* $\frac{c(G)}{r(G)} = 2$.

Proof. This follows from Corollary 4.6.

LEMMA 5.3. *Let* $G = SL(2, q) \cong PSL(2, q)$ *where* $q = 2^n$. *Then*

$$r(G) = \begin{cases} q - 1 & \textit{if } q \equiv -1 (\mathrm{mod}\ 3) \textit{ and } n > 1, \\ q & \textit{otherwise}. \end{cases}$$

Proof. This follows from Table (1) and Lemma 4.16.

LEMMA 5.4. *Let* $G = PSL(2, q)$ *where q is odd,* $q = p^n$.

(1) *If* $q \equiv 3(\mathrm{mod}\ 4)$, *then* $r(G) = q - 1$.
(2) *If* $q \equiv 1(\mathrm{mod}\ 4)$, *then*

$$r(G) = \begin{cases} \frac{1}{2}(q+1) & \textit{if n is even}, \\ q - 1 & \textit{if n is odd and } q \equiv -1(\mathrm{mod}\ 3), \\ q & \textit{otherwise}. \end{cases}$$

Proof. This follows from Tables (2) and (3) except for the case $q \equiv 1(\mathrm{mod}\ 4)$ and n odd. In this case, $d(\theta_j) \geq q - 1$ for $1 \leq j \leq \frac{q-1}{2}$, j even. Thus, using Corollaries 4.10 and 4.8, we see that $r(G) = q - 1$ precisely when one of $\frac{q+1}{d}$, $d = 3, 4$ or 6, is an even integer. As $q \equiv 1(\mathrm{mod}\ 4)$ neither $d = 4$ nor $d = 6$ is possible. But $\frac{q+1}{3}$ is an even integer if and only if $q \equiv -1(\mathrm{mod}\ 3)$.

THEOREM 5.5. *Let* $G = PSL(2, q)$. *Then*

$$\lim_{q\to\infty} \frac{c(G)}{r(G)} = 1$$

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Proof. Let $G = PSL(2, q)$ where $q = 2^n$. Then $G \cong SL(2, q)$. By Lemma 5.3 we have $q - 1 \leq r(G) \leq q$. Also by Theorem 4.17 we have $c(G) = q + 1$ for $q \neq 2$. Hence $\frac{q+1}{q} \leq \frac{c(G)}{r(G)} \leq \frac{q+1}{q-1}$.

Let $G = PSL(2, q)$ where q is odd. By Lemma 5.4 we have $r(G) = \frac{1}{2}(q - 1)$ if n is even; otherwise $q - 1 \leq r(G) \leq q$. By Theorem 4.19 we have $\frac{c(G)}{r(G)} = \frac{q+\sqrt{q}}{q-1}$ if n is even; otherwise $\frac{q+1}{q} \leq \frac{c(G)}{r(G)} \leq \frac{q+1}{q-1}$. Hence in all cases $\frac{q+1}{q} \leq \frac{c(\xi)}{r(\xi)} \leq \frac{q+\sqrt{q}}{q-1}$ and so $\lim_{q\to\infty} \frac{c(G)}{r(G)} = 1$.

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