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ON THE LAVRENTIEV PHENOMENON

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ABSTRACT. A new look at Manià's classical example of the Lavrentiev Phenomenon leads to several pertinent observations.

The tension between existence theory and the development of necessary conditions has long been a spur to research in the calculus of variations. The conflict concerns which class of functions X to use when studying the basic problem

$$\inf \{\Lambda[x] := \int_a^b L(t, x(t), x'(t))\, dt : x(a) = A,\ x(b) = B,\ x \in X\}. \qquad \text{(P)}$$

For the derivation of necessary conditions, a class of (at least piecewise) smooth functions is the usual choice, whereas existence theorems routinely require the much larger class of absolutely continuous functions.

M. Lavrentiev [10] focused attention on the conflict by constructing an example of (P) in which the minimum value over the absolutely continuous functions was strictly less than the infimum over the smooth functions. Thus he showed that the passage from smooth to absolutely continuous functions was no small step: some problems have solutions in the latter class for which no smooth function — in fact, no $x(t)$ with $x' \in L^\infty[a, b]$ — can accurately approximate both the optimal arc and its value. Problems where

$$\inf_{x \in AC[a,b]} \Lambda[x] < \inf_{x \in AC^\infty[a,b]} \Lambda[x]$$

are now said to exhibit *the Lavrentiev phenomenon* (*LP*). (We write $AC^p[a,b] = \{x \in AC[a, b] : x' \in L^p[a, b]\}$, so $AC^1 = AC$ and AC^∞ is the class of functions Lipschitz on $[a, b]$.)

The possibility of LP prompts the question, "Just how bad can a minimizer in problem (P) be?" Ball and Mizel ([2], [3]) have studied several examples bearing directly on this question. Taking a more positive approach, Clarke and Vinter ([7], [8], [9]) have recently made significant progress in proving certain regularity properties of solutions under minimal hypotheses. Angell [1] and Cesari [4], [5] are also working in this field.

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Several years after Lavrentiev's example appeared, B. Manià found a much simpler problem displaying LP; his work is reported in [4], Section 18.5. The current paper presents a new analysis of Manià's example which is both simpler and more illuminating than its predecessors. It also discusses modifications of the problem which show LP's tenacity.

1. **An Auxiliary Problem**. Before turning to Maniá's example, let us consider the following version of (P):

$$\min \left\{ \int_{\tau}^{T} t^2[x'(t)]^6 \, dt : x(\tau) = 0,\ x(T) = \alpha T^{1/3} \right\},$$

where $0 \leq \tau < T$ and $\alpha > 0$ are given. Since $L(t, x, v) = t^2 v^6$ is jointly convex in (x, v), any extremal arc actually provides the problem's unique solution. The Euler-Lagrange equation gives $x(t) = 5/3ct^{3/5} + d$, where the constants c, d are determined by the end conditions. The result is

$$x(t) = \alpha T^{1/3}(T^{3/5} - \tau^{3/5})^{-1}(t^{3/5} - \tau^{3/5}).$$

Consequently the problem's minimum value is

$$\int_{\tau}^{T} t^2[x'(t)]^6 \, dt = (3/5)^5 \alpha^6 T^2/(T^{3/5} - \tau^{3/5})^5. \tag{1}$$

Note that for fixed α, the right side tends to $+\infty$ as $T - \tau \rightarrow 0$. This observation is central to the developments below.

2. **Maniá's Example**. Let us now consider the basic problem

$$\min \left\{ \Lambda[x] := \int_{-1}^{1} L(t, x(t), x'(t)) \, dt : x(-1) = -1,\ x(1) = 1 \right\}$$

with $L(t, x, v) = (x^3 - t)^2 v^6$. This Lagrangian, which was introduced by Maniá [11], is clearly nonnegative, so the absolutely continuous function $\hat{x}(t) = t^{1/3}$ minimizes Λ by giving $\Lambda[\hat{x}] = 0$. We will show that Λ carries $AC^\infty[-1, 1]$ into a set of positive numbers bounded away from zero, and thus verify LP.

Note first that $\Lambda[\cdot]$ is symmetric, in the sense that if an arc $x(\cdot)$ satisfies the end conditions and $\Lambda[x] < +\infty$ then the same is true of the reflected arc $y(t) := -x(-t)$. In fact, $\Lambda[y] = \Lambda[x]$. So we lose no generality in studying only the objective values of those arcs $x(\cdot)$ with $x(0) \leq 0$.

Now for each $\alpha \in (0, 1)$, define the set $R_\alpha := \{(t, x) : t \in [0, 1],\ 0 \leq x \leq \alpha t^{1/3}\}$. In R_α, one has $L(t, x, v) \geq ((\alpha t^{1/3})^3 - t)^2 v^6 = (1 - \alpha^3)^2 t^2 v^6$. For any fixed α, any admissible arc $x \in AC^\infty$ with $x(0) \leq 0$ must spend some time in R_α. More precisely, one has $1 > T > \tau \geq 0$, where

$$\tau = \sup\{t \geq 0 : x(t) = 0\} \quad \text{and} \quad T = \inf\{t \geq 0 : x(t) = \alpha t^{1/3}\}.$$

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It follows from the analysis of the auxiliary problem above that

$$\begin{aligned}\Lambda[x] &\geq (1-\alpha^3)^2 \int_\tau^T t^2[x'(t)]^6\, dt \\ &\geq (3/5)^5(1-\alpha^3)^2\alpha^6 T^2/(T^{3/5}-\tau^{3/5})^5 \qquad (2)\\ &\geq (3/5)^5(1-\alpha^3)^2\alpha^6.\end{aligned}$$

(The last inequality holds because $T = 1$ gives the previous expression its minimum over $T \in (\tau, 1)$, and then $1 - \tau^{3/5} \leq 1$.) The maximum value of the right side occurs when $\alpha = 2^{-1/3}$: we deduce that for every admissible $x \in AC^\infty$, $\Lambda[x] \geq (1/2)^4(3/5)^5$. This confirms LP.

Note that the key observation above is that any $x(\cdot) \in AC^\infty$ spends time in R_α. Hence the same lower bound on $\Lambda[x]$ remains valid for any arc x spending time in R_α — in particular, $\Lambda[x] \geq (1/2)^4(3/5)^5$ for any $x \in AC^p[-1, 1]$ for $p \geq 3/2$. One could say that the singularity displayed by Λ is much worse than that defining LP, since the latter requires such a property only for $p = +\infty$. The computations of this section are the basis for all the developments to follow, which can therefore be refined to replace AC^∞ by AC^p for $p \geq 3/2$. In this general discussion, however, we consider the results in AC^∞ sufficiently compelling to leave their refinement implicit.

3. **Lipschitz Approximation**. It is well known tht any absolutely continuous function is uniformly approximable by Lipschitz functions, in the following sense.

THEOREM 1. *Let any $\hat{x} \in AC[a, b]$ be given. The there is a sequence $\{x_k\} \in AC^\infty$ such that*

(i) $x_k(a) = \hat{x}(a)\ \forall k$, $x_k(b) = \hat{x}(b)\ \forall k$, *and* $\|x_k - \hat{x}\|_\infty \to 0$ *as* $k \to \infty$,

(ii) $\|x_k'\|_\infty \leq k\ \forall k$,

(iii) $\forall \epsilon > 0\ \exists M > 0$ *s.t.* $\displaystyle\int_{\{t:|x_k'(t)|\geq M\}} |x_k'(t)|\, dt \leq \epsilon\ \forall k$. (*i.e., the derivative sequence is "uniformly integrable".*)

The presence of LP in the example discussed above makes it clear that no matter how well a sequence of Lipschitz arcs x_k approximates $\hat{x}(t) := t^{1/3}$, the values $\Lambda[x_k]$ cannot approach $\Lambda[\hat{x}] = 0$. So what happens to this sequence of values? Surprisingly, *any sequence $\{x_k\}$ of admissible Lipschitz arcs obeying $\|x_k - \hat{x}\|_\infty \to 0$ also has* $\Lambda[x_k] \to +\infty$! To see this, note that for any admissible $x \in AC^\infty$ obeying $\|x - \hat{x}\|_\infty < \epsilon$ (and $x(0) \leq 0$), the interval $[\tau, T]$ defined above cannot be very large. Indeed, one must have $0 \leq \tau < T \leq (\epsilon/(1-\alpha))^3$. So instead of the lower bound computed in (2), we have

$$\Lambda[x] \geq (3/5)^5(1-\alpha^3)^2\alpha^6 T^2/(T^{3/5}-\tau^{3/5})^5 \geq (3/5)^5(1-\alpha^3)^2\alpha^6(1-\alpha)^3/\epsilon^3.$$

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In short, there is a constant $c > 0$ such that

$$\|x - \hat{x}\|_\infty < \epsilon,\ x \in AC^\infty \Rightarrow \Lambda[x] \geq c\epsilon^{-3}. \tag{3}$$

This confirms the surprisingly bad behaviour of $\Lambda[\cdot]$ asserted above. In fact, the last paragraph of Section 2 explains why, in (3), AC^∞ may be replaced by AC^p for any $p \geq 3/2$.

4. **LP Persists**. Although the formulation of the example in Section 2 may seem contrived, the Lavrentiev Phenomenon is also observed in many related examples. For example, certain changes to the Lagrangian and modifications of the end conditions fail to eliminate LP.

To see how the Lagrangian can be modified, suppose that LP is observed in some problem of the form (P). More specifically, suppose that $\Lambda[\hat{x}] = 0$ for some $\hat{x} \in AC[a, b]$, while $\Lambda[x] \geq \eta > 0\ \forall x \in AC^\infty[a, b]$. Then consider any functional $M: AC[a, b] \to [0, +\infty)$ obeying $M[\hat{x}] < +\infty$. Upon choosing any $0 \leq \epsilon \leq \eta/(2M[\hat{x}])$, we find that

$$(\Lambda + \epsilon M)[\hat{x}] \leq 0 + \eta/2$$

$$(\Lambda + \epsilon M)[x] \geq \eta + 0 \qquad \forall x \in AC^\infty[a, b].$$

Thus LP persists for the modified functional $\Lambda + \epsilon M$. A significant application of this simple observation to the example of Section 2 involves the functional

$$M[x] := \int_{-1}^{1} |x'(t)|^{5/4}\, dt.$$

For $\hat{x}(t) = t^{1/3}$, one has $M[\hat{x}] < +\infty$, so it follows that for all $\epsilon > 0$ sufficiently small, LP is observed in the problem

$$\min\left\{\int_{-1}^{1} ((x^3 - t)^2 x'^6 + \epsilon |x'|^{5/4})\, dt : x(-1) = -1,\ x(1) = 1\right\}.$$

This is significant because the Lagrangian in this problem is strictly convex in the velocity variable, in which it moreover displays uniform superlinear growth. Thus the modified problem satisfies the hypotheses under which Tonelli showed the existence of a minimum over the class $AC[-1, 1]$. Ball and Mizel [3] were the first to produce examples showing that these conditions do not exclude LP, but their examples are considerably more complicated than ours. (On the other hand, their examples have $|x'|^2$ instead of $|x'|^{5/4}$.)

Let us now consider endpoint variations, denoting by $P(x_{-1}, x_1)$ the problem

$$\min\left\{\int_{-1}^{3} (x^3 - t)^2 x'^6\, dt : x(-1) = x_{-1},\ x(1) = x_1\right\}.$$

We have shown that $P(-1, 1)$ displays LP. In fact, LP clearly persists for any $x_{-1} \in [-1, 0)$ and $x_1 \in (0, 1]$. For then the optimal arc is

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$$\hat{x}(t) = \begin{cases} x_{-1} & \text{if} \quad -1 \le t^{1/3} < x_{-1}, \\ t^{1/3} & \text{if} \quad x_{-1} < t^{1/3} < x_1, \\ x_1 & \text{if} \quad x_1 \le t^{1/3} \le 1. \end{cases}$$

(Note that $\Lambda[\hat{x}] = 0$.) However, by choosing $\alpha > 0$ so small that $\alpha < \min\{x_{-1}, x_1\}$, the arguments of Section 2 show that $\Lambda[x] \ge (3/5)^5(1 - \alpha^3)^2\alpha^6 > 0$ for all admissible $x \in AC^\infty$. Similarly, it is evident that for some $\delta > 0$, LP persists even for $x_{-1} \in (-1 - \delta, -1]$ and $x_1 \in [1, 1 + \delta)$. To see this, one must only modify the solution $\hat{x}(t) = t^{1/3}$ of $P(-1, 1)$ near its endpoints to obtain an admissible arc $\tilde{x}$ with $\Lambda[\tilde{x}] < 1/2(3/5)^5(1 - \alpha^3)^2\alpha^6$. Since all admissible Lipschitz functions x must still obey $\Lambda[x] > (3/5)^5(1 - \alpha^3)^2\alpha^6$, LP persists.

Since there is a well-defined neighbourhood of $(x_{-1}, x_1) = (-1, 1)$ in which $P(x_{-1}, x_1)$ exhibits LP, one can actually choose $K > 0$ so large that LP occurs in the following *free-endpoint* Bolza problem:

$$\min\left\{\Lambda'[x] := K|(x(-1), x(1)) - (-1, 1)|^2 + \int_{-1}^{1} (x^3 - t)^2 x'^6 \, dt\right\}.$$

The arguments of the previous paragraph give the same conclusion even if a term $\epsilon |x'|^{5/4}$ is added to the integrand. In that case we obtain a free-endpoint Bolza problem whose smooth Lagrangian exhibits strict convexity and superlinear growth in the velocity variable, but which still displays LP. This shows that LP can occur even in "calm" problems—see Clarke [6] for the definition and application of this constraint qualification.

5. **Excluding LP**. The striking singularity of problems displaying LP is cause not only for interest, but for concern. For example, numerical procedures for solving variational problems approximately are virtually guaranteed to fail when LP is present, because they use approximating trajectories in AC^∞. Thus one of the most important aspects of research into LP is the search for conditions excluding it. T. S. Angell has given the following result, much in the spirit of the classical theory. (Notation: $T(x; \eta) = \{(t, y) : |y - x(t)| < \eta\}$.)

THEOREM 2. *([1], see [4, Ch. 18]). Consider the functional* $\Lambda : AC[a, b] \to R$ *defined by* $\Lambda[x] = \int_a^b L(t, x(t), x'(t))\, dt$. *Suppose* $\hat{x} \in AC[a, b]$ *is an arc for which* $\Lambda[\hat{x}] < +\infty$. *Under the following hypotheses, the standard sequence* $\{x_k\}$ *of Thm. 1 obeys* $\Lambda[x_k] \to \Lambda[\hat{x}]$.

(H1) *There exists* $\eta > 0$ *such that for all* $\delta > 0$, *there is a closed set* $K \subseteq [a, b]$ *with* $m([a, b]\backslash K) < \delta$ *for which L is continuous on* $\{(t, y, v) : t \in K, |y - \hat{x}(t)| < \eta\}$ *and locally bounded on* $T(\hat{x}; \eta) \times R$. *(This hypothesis is automatic if L is continuous on* $T(\hat{x}; \eta) \times R$ *for some* $\eta > 0$.)

(H2) $\Delta_1^{(k)} := \int_a^b |L(t, x_k, x_k') - L(t, \hat{x}, x_k')| \, dt \to 0$ *as* $k \to \infty$.

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Angell's theorem is widely applicable because (H2) isolates and eliminates the key technical problem in proving approximation results. However, the same hypothesis that gives the theorem its generality also limits its usefulness, since one must know $\hat{x}$ *a priori* to apply it. Thus we seek verifiable conditions implying (H2). Let us reconsider Manià's example with this in mind. Along the solution $\hat{x}(t) = t^{1/3}$, the Lagrangian is identically zero while the Lipschitz rank of $L(t, \cdot, \hat{x}'(t))$ is unbounded. The following (new) result indicates that if the Lipschitz rank of $L(t, \cdot, v)$ is limited by the growth of L in a reasonable way, then LP cannot occur. Here ∂ denotes Clarke's generalized gradient.

THEOREM 3. *Consider Λ as in Thm. 2, but assume that L is nonnegative, continuous in (t, x, v) and locally Lipschitz in x for each (t, v). Given $\hat{x} \in AC[a, b]$ such that $\Lambda[\hat{x}] < +\infty$, the following hypotheses assure that $\Lambda[x_k] \to \Lambda[\hat{x}]$ (with $\{x_k\}$ as in Thm. 1).*

(a) *$\{\Lambda[x_k]\}$ is a bounded set of real numbers.*

(b) *For some $\eta > 0$, there are constants k_0, k_1, and a function $f_0(t) \in L^1[a, b]$ such that*

$$\sup |\partial_x L(t, y, v)| \leq k_0 L(t, y, v) + k_1 |y| + f_0(t) \ \forall (t, y, v) \in T(\hat{x}; \eta) \times R.$$

PROOF. For all k sufficiently large, we have $\|x_k - \hat{x}\| < \eta$. It follows from [6, Lemma 1, p. 181] that

$$|L(t, x_k, x_k') - L(t, \hat{x}, x_k')| \leq K_\eta(k_0 L(t, x_k, x_k') + f_0(t) + k_1 \max\{|x_k|, |\hat{x}|\})|x_k - \hat{x}| \quad \forall t \in [a, b]$$

for $K_\eta = (\exp(k_0\eta) - 1)/(k_0\eta)$. Hence

$$|\Delta_1^{(k)}| = \int_a^b |L(t, x_k, x_k') - L(t, \hat{x}, x_k')| \, dt \leq K_\eta \|x_k - \hat{x}\| \Big(k_0 \Lambda[x_k] + k_1 \max\{\|x_k\|, \|\hat{x}\|\} + \int_a^b f_0(t) \, dt \Big).$$

By assumption (a), the quantity in parentheses is bounded. Consequently $\Delta_1^{(k)} \to 0$ as $k \to +\infty$ and (H2) of Thm. 2 is verified. The result follows. ////

Theorems 2 and 3 take a classical approach to LP. In their formidable work on the regularity of solutions to problems in the calculus of variations, Clarke and Vinter use completely different methods to identify (among other things) situations in which the infimum of $\Lambda[\cdot]$ over AC is actually attained by an arc in AC^∞. Of course, these situations are free of LP. The hypotheses used by Clarke and Vinter are very mild indeed: for example, after assuming only the convexity and coercivity conditions of Tonelli's existence theorem, they prove that *whenever L is independent of t, all solutions of* (P) *lie in AC^∞*. For a precise formulation of this result, along with many other conditions giving the same conclusion, see [7], [8], [9].

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REFERENCES

1. T. S. Angell. *A note on approximation of optimal solutions of free problems of the calculus of variations*. Rend. Circ. Math. Palermo (2) **28** (1979), 258–272.

2. J. Ball and V. Mizel. *Singular minimizers for regular one-dimensional problems in the calculus of variations*. Bull. AMS **11** (1984), 143–146.

3. J. Ball and V. Mizel. *One-dimensional variational problems whose minimizers do not satisfy the Euler-Lagrange equation*. Arch. Rat. Mech. Anal. **90** (1985), 325–388.

4. L. Cesari. *Optimization — Theory and Applications: Problems with Ordinary differential equations*. New York: Springer-Verlag, 1983.

5. L. Cesari and T. S. Angell. *On the Lavrentiev phenomenon*. Preprint, 1985.

6. Frank H. Clarke. *Optimization and Nonsmooth Analysis*. New York: Wiley-Interscience, 1983.

7. Frank H. Clarke and Richard B. Vinter. *Regularity properties of solutions to the basic problem in the calculus of variations*. Trans. AMS **289** (1985), 73–98.

8. Frank H. Clarke and Richard B. Vinter. *Existence and Regularity in the small in the calculus of variations*. J. Diff. Eqns. **59** (1985), 336–354.

9. Frank H. Clarke and Richard B. Vinter. *Regularity of solutions to variational problems with polynomial Lagrangians*. Bull. Pol. Acad. Sci., **34** (1986), 73–81.

10. M. Lavrentiev. *Sur quelques problèmes du calcul des variations*. Annali di Matematica, Pura et Applicata **4** (1926), 7–28.

11. B. Manià. *Sopra un esempio di Lavrentieff*. Boll. Un. Mat. Ital. **13** (1934), 147–153.

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