The Weyl Problem With Nonnegative Gauss Curvature In Hyperbolic Space

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Abstract. In this paper, we discuss the isometric embedding problem in hyperbolic space with nonnegative extrinsic curvature. We prove *a priori* bounds for the trace of the second fundamental form H and extend the result to n-dimensions. We also obtain an estimate for the gradient of the smaller principal curvature in 2 dimensions.

1 Introduction

In 1916, H. Weyl posed the following problem: Consider a two-sphere S^2 and suppose g is a Riemannian metric on S^2 whose Gauss curvature is everywhere positive. Does there exist a global C^2 isometric embedding $X: (S^2, g) \to (\mathbb{R}^3, \sigma)$, where σ is the standard flat metric in $\mathbb{R}^3$?

The first attempt to solve the problem was made by Weyl himself. He used the continuity method to obtain *a priori* estimates up to the second derivatives. Later, H. Lewy [Le35] solved the problem in the case when g is analytic. In 1953, L. Nirenberg [Ni53] gave a complete solution under the very mild hypothesis that the metric g has continuous fourth derivative.

In 1964, A. V. Pogorelov [Po64] obtained several important refinements to the differential geometry of 2-dimensional submanifolds F of smooth 3-dimensional Riemannian manifolds R. The proof is based on derivative estimates that refine and complete his earlier work with A. D. Aleksandrov [AP50].

The Weyl estimate was later generalized to the case of nonnegative curvature in Euclidean space by J. A. Iaia [Ia92] and P. Guan, Y. Li in [GL94]. They obtained a $C^{1,1}$ embedding result for metrics of nonnegative Gauss curvature; see also [HZ95] for a different approach to the $C^{1,1}$ embedding result. Later in 1999, Y. Li and G. Weinstein [LW99] extended the estimates obtained in [GL94] to n-dimensions.

In this paper, we discuss the isometric embedding problem in hyperbolic space with nonnegative extrinsic Gauss curvature. As in the Euclidian case, the image of such an embedding, if it exists, bounds a convex body. However, the loss of strict positivity of K leads to degenerate Monge–Ampere equations which cause difficulties.

By combining A. V. Pogorelov's method with the results obtained by B. Guan, J. Spruck and M. Szapiel [GSS09] and [GS11], we can also obtain *a priori* bounds for principal curvatures of any strictly convex closed hypersurface with positive sectional curvature in $\mathbb{H}^{n+1}$. Moreover, we proved that when the sectional curvature is equal to -1 at finitely many points, the above statement is still true.

For the case of strictly positive Gauss curvature, it is well known that the regularity exists, because the corresponding PDE is uniformly elliptic. Thus, if the given metric is C^∞, then the resulting embedding is C^∞. However, this does not hold in the case under consideration. So, another natural question to ask is the regularity of the embedding.

In this paper, we will denote the intrinsic Gauss curvature by K and extrinsic Gauss curvature by $\widetilde{K}$. The main theorems of this paper are the following.

Theorem 1.1   *Let $g \in C^4$ be a Riemannian metric on S^2 with Gauss curvature satisfying*

(i)   *$K(P_i) = -1$, $1 \le i \le n$;*
(ii)   *$K(Q) > -1$ for any $Q \ne P_i$, where $\{P_i\} \in S^2$ are finite isolated points.*

Then there exists a $C^{1,1}$ isometric embedding $X \colon (S^2, g) \to (\mathbb{H}^3, h)$ where

$$h = \frac{4}{(1 - |x|^2)^2}(dx_1^2 + dx_2^2 + dx_3^2).$$

By studying the regularity of the embedding, we also prove the following theorem.

Theorem 1.2   *If $g \in C^5$, under the hypothesis of Theorem 1.1 and also assume that $\liminf_{Q \to P_i} H(Q) = c > 0$, then $\kappa_1 \in C^{0,1}$ in $B_r(P_i)$, where κ_1 is the smaller of the two principal curvatures and $r > 0$ sufficiently small.*

We extend the results of Theorem 1.1 to higher dimensions by bounding the mean curvature H in terms of the scalar curvature and its Laplacian. This is a direct generalization of the Weyl estimate.

Theorem 1.3   *Let g be a C^4 metric with sectional curvature ≥ -1 and let $X \colon (S^n, g) \to \mathbb{H}^{n+1}$ be a C^4 isometric embedding. Suppose the sectional curvature S of g satisfies*

(i)   *$S(P_i, \chi) = -1$ for some $\chi \in \bigwedge^2 T_{P_i}M$; $1 \le i \le n$;*
(ii)   *$S(Q) > -1$, for any $Q \ne P_i$, where $\{P_i\} \in S^n$ are finite isolated points.*

Let H be the trace of the second fundamental form of X, and let R be the extrinsic scalar curvature of g. Then the following inequality holds:

$$(1.1) \qquad\qquad H^2 \le C_1|\Delta R| + C_2(R^2 + R)$$

where C_1, C_2 depends only on the metric g and dimension n.

An outline of the contents of the paper is as follows. Section 2 contains the estimate of mean curvature and proves Theorem 1.1. Section 3 introduces the relations between different models of hyperbolic space, and also states how to transform the coordinates of a small neighborhood to simplify the calculations that will be used in Section 4 to prove Theorem 1.2 and also in Section 5 to prove Theorem 1.3. The use of the transform map between two different models is unusual, but seems to be necessary in proving the partial third derivative estimates.

2 A Mean Curvature Estimate

Theorem 2.1 *Let $g \in C^4$ be a Riemannian metric on S^2 with Gauss curvature satisfying*

(i) $K(P_i) = -1, 1 \leq i \leq n$;

(ii) $K(Q) > -1$ for any $Q \neq P_i$, where $\{P_i\} \in S^2$ are finite isolated points.

Then there exists a $C^{1,1}$ isometric embedding $X \colon (S^2, g) \to (\mathbb{H}^3, h)$ where

$$h = \frac{4}{(1 - |x|^2)^2}(dx_1^2 + dx_2^2 + dx_3^2).$$

Proof Following [GL94] we first approximate g^0 in C^4 by a sequence of C^∞ metrics g^ϵ with corresponding intrinsic Gauss curvature $\{K^\epsilon\}$, such that $K^\epsilon > -1$ everywhere. Then we can apply the result from [Po64] to g^ϵ and therefore obtain a sequence of C^∞ isometric embeddings

$$X^\epsilon \colon (S^2, g^0) \to (\mathbb{H}^3, h).$$

It is not difficult to see that there exists constants $\alpha, \beta > 0$ (independent of ϵ), such that $-1 < K^\epsilon < \alpha$ and $\operatorname{diam}(X^\epsilon) < \beta$ for all $\epsilon > 0$. We immediately have that $\|X^\epsilon\|_{C^0} \leq C$.

In local coordinates,

$$g^0 = E^0 du^2 + 2F^0 dudv + G^0 dv^2,$$
$$g^\epsilon = E^\epsilon du^2 + 2F^\epsilon dudv + G^\epsilon dv^2.$$

We already know that $X^\epsilon \colon (S^2, g^\epsilon) \to (\mathbb{H}^3, h)$ is an isometric embedding, so we have

$$\langle X_u^\epsilon, X_u^\epsilon \rangle_{\mathbb{H}^3} = E^\epsilon, \quad \langle X_u^\epsilon, X_v^\epsilon \rangle_{\mathbb{H}^3} = F^\epsilon, \quad \langle X_v^\epsilon, X_v^\epsilon \rangle_{\mathbb{H}^3} = G^\epsilon.$$

It follows easily that $\|\nabla_{g^0} X^\epsilon\|_{C^0} \leq C$, where C is independent of ϵ.

The following will be devoted to establishing a bound on $\|\nabla_{g^0}^2 X^\epsilon\|_{C^0}$, which is independent of ϵ. Once we obtain such a bound, the limit of X^ϵ as $\epsilon \to 0$ will be a $C^{1,1}$ isometric embedding of g^0. For convenience, in the following we drop the dependence on ϵ in our notation.

Next we will prove $\max_{S^2} H \leq C$, where C is a constant independent of ϵ.

In hyperbolic space we have

$$(2.1) \qquad \frac{4}{(1 - |x|^2)^2}\langle X_u, X_u \rangle_{\mathbb{R}^3} = \langle X_u, X_u \rangle_{\mathbb{H}^3} = E = g_{11},$$

$$(2.2) \qquad \frac{4}{(1 - |x|^2)^2}\langle X_u, X_v \rangle_{\mathbb{R}^3} = \langle X_u, X_v \rangle_{\mathbb{H}^3} = F = g_{12},$$

$$(2.3) \qquad \frac{4}{(1 - |x|^2)^2}\langle X_v, X_v \rangle_{\mathbb{R}^3} = \langle X_v, X_v \rangle_{\mathbb{H}^3} = G = g_{22}.$$

Let the orientation be chosen so that the inner unit normal is given by

$$(2.4) \qquad \overline{X} = \frac{X_u \times X_v}{\sqrt{EG - F^2}} = \frac{2}{1 - |x|^2}\overline{X}_E,$$

where $\overline{X}_E$ is the Euclidean unit normal. The second fundamental form is then given by

$$\mathrm{II} = L du^2 + 2M dudv + N dv^2$$

where

$$(2.5) \qquad L = -\langle X_u, \overline{X}_u \rangle_{\mathbb{H}^3} = \langle \nabla_{X_u} X_u, \overline{X} \rangle_{\mathbb{H}^3},$$

$$(2.6) \qquad M = -\langle X_v, \overline{X}_u \rangle_{\mathbb{H}^3} = \langle \nabla_{X_u} X_v, \overline{X} \rangle_{\mathbb{H}^3},$$

$$(2.7) \qquad N = -\langle X_v, \overline{X}_v \rangle_{\mathbb{H}^3} = \langle \nabla_{X_v} X_v, \overline{X} \rangle_{\mathbb{H}^3}.$$

Hence the intrinsic Gauss and mean curvature are

$$(2.8) \qquad K = -1 + \frac{LN - M^2}{EG - F^2}$$

and

$$(2.9) \qquad H = \frac{1}{2} \frac{GL - 2FM + EN}{EG - F^2}.$$

The Gauss equation takes the form

$$(2.10) \qquad \nabla_{X_u} X_u = \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v + L\overline{X},$$

$$(2.11) \qquad \nabla_{X_u} X_v = \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v + M\overline{X},$$

$$(2.12) \qquad \nabla_{X_v} X_v = \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v + N\overline{X},$$

where $\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})$ with $\partial_1 = \partial_u$ and $\partial_2 = \partial_v$. The Weingarten equations take the form

$$(2.13) \qquad -\overline{X}_u = L_1^1 X_u + L_1^2 X_v,$$

$$(2.14) \qquad -\overline{X}_v = L_2^1 X_u + L_2^2 X_v,$$

where $\{L_j^i\}$ are expressions involving L, M, N and E, F, G. The Mainardi–Codazzi equations take the form

$$(2.15) \qquad L_v - M_u = L\Gamma_{12}^1 + M(\Gamma_{12}^2 - \Gamma_{11}^1) - N\Gamma_{11}^2,$$

$$(2.16) \qquad M_v - N_u = L\Gamma_{22}^1 + M(\Gamma_{22}^2 - \Gamma_{12}^1) - N\Gamma_{12}^2.$$

Let $\rho = \frac{2}{1-|x|^2}$. We consider the function

$$f = e^{\alpha\rho} H$$

on S^2, with $\alpha > 0$ to be determined later. Without loss of generality, we assume there is only one singular point $P_0 \in S^2$ such that $\lim_{\epsilon \to 0} K^\epsilon(P_0) = -1$. Let $\delta = \inf_{S^2/B_r(P_0)} K > -1$ and be independent of ϵ. Then, by a theorem in [Po64, p. 216], we have

$$(2.17) \qquad H \leq C_0, \quad \text{for } \forall P \in S^2 \backslash B_r(P_0),$$

where C_0 depends only upon the metric of $S^2 \backslash B_r(P_0)$ and the metric of the space.

Next we will focus on estimating the curvature inside $B_r(P_0)$. Since we use the ball model for hyperbolic space, we can always choose our origin very close to $X(P_0)$ such that for any point $P \in X(B_r(P_0) \cap S^2)$ we have $|P|_{\mathbb{R}^3}^2 = p_1^2 + p_2^2 + p_3^2 \leq \frac{1}{100}$. (We

always assume r is small.) Restricting the function f on $X\big(B_r(P_0) \cap S^2\big)$, we can see that if $M = \max_{P \in X(B_r(P_0) \cap S^2)} f$ were achieved on the boundary, then we would have

$$e^{\alpha\rho}H \leq e^{3\alpha}C_0 \quad \text{for } \forall P \in X\big(B_r(P_0) \cap S^2\big).$$

Therefore, we assume M is achieved at an interior point Q. Let us write the metric $g = g^\epsilon$ near Q in conformal coordinates:

$$g = e^{2h}(du^2 + dv^2),$$

where $(u, v) = (0, 0)$ corresponds to Q and

$$h = \partial_u h = \partial_v h = 0 \quad \text{at } (0, 0).$$

The intrinsic Gauss and mean curvatures become

$$(2.18) \qquad\qquad K = -1 + \frac{LN - M^2}{e^{4h}},$$

$$(2.19) \qquad\qquad H = \frac{L + N}{2e^{2h}},$$

$$(2.20) \qquad\qquad K = -\frac{\widetilde{\triangle}h}{e^{2h}},$$

where $\widetilde{\triangle} = \partial_{11} + \partial_{22}$, also, the Mainardi–Codazzi equations are

$$(2.21) \qquad\qquad L_2 - M_1 = h_2(L + N) = 2He^{2h}h_2,$$

$$(2.22) \qquad\qquad M_2 - N_1 = -h_1(L + N) = -2He^{2h}h_1.$$

Clearly $\triangle_g K = e^{-2h}\widetilde{\triangle}K$. Differentiating (2.18), we have

$$(2.23) \qquad K_1 = \frac{1}{e^{4h}}(L_1 N + LN_1 - 2MM_1) - 4h_1(K + 1),$$

$$(2.24) \qquad K_{11} = \frac{1}{e^{4h}}(L_{11}N + LN_{11} + 2N_1L_1 - 2M_1^2 - 2MM_{11})$$

$$(2.25) \qquad\qquad - 8h_1 K_1 - 4h_{11}(K + 1) - 16h_1^2(K + 1),$$

$$(2.26) \qquad K_2 = \frac{1}{e^{4h}}(L_2 N + LN_2 - 2MM_2) - 4h_2(K + 1)$$

$$(2.27) \qquad K_{22} = \frac{1}{e^{4h}}(L_{22}N + LN_{22} + 2N_2L_2 - 2M_2^2 - 2MM_{22})$$

$$(2.28) \qquad\qquad - 8h_2 K_2 - 4h_{22}(K + 1) - 16h_2^2(K + 1).$$

Applying ∂_2 to (2.21), ∂_1 to (2.22), and adding, we have

$$(2.29) \qquad\qquad (-L_2 + 2He^{2h}h_2)_2 = (-N_1 + 2He^{2h}h_1)_1$$

which yields

$$(2.30) \quad N_{11} = L_{22} + 2H_1 h_1 e^{2h} - 2H_2 h_2 e^{2h} + 2(h_{11} - h_{22})He^{2h} + 4(h_1^2 - h_2^2)He^{2h}.$$

Differentiating (2.19) gives

$$(2.31) \qquad 2H_1 = \frac{L_1 + N_1}{e^{2h}} - 4h_1 H,$$

$$(2.32) \qquad 2H_{11} = \frac{L_{11} + N_{11}}{e^{2h}} - 8h_1 H_1 - 8h_1^2 H - 4h_{11} H,$$

$$(2.33) \qquad 2H_2 = \frac{L_2 + N_2}{e^{2h}} - 4h_2 H,$$

$$(2.34) \qquad 2H_{12} = \frac{L_{12} + N_{12}}{e^{2h}} - 4h_2 H_1 - 4h_{12} H - 4h_1 H_2 - 8h_1 h_2 H,$$

$$(2.35) \qquad 2H_{22} = \frac{L_{22} + N_{22}}{e^{2h}} - 8h_2 H_2 - 8h_2^2 H - 4h_{22} H.$$

Applying ∂_1 to (2.21) and ∂_2 to (2.22), we obtain

$$(2.36) \qquad M_{11} = L_{12} - 2H_1 e^{2h} h_2 - 2H e^{2h} h_{12} - 4H h_1 h_2 e^{2h},$$

$$(2.37) \qquad M_{22} = N_{12} - 2H_2 e^{2h} h_1 - 2H e^{2h} h_{12} - 4H h_1 h_2 e^{2h}.$$

So we have

(2.38)
$$\begin{aligned}
e^{6h} \triangle_g K &= e^{4h} \widetilde{\triangle} K \\
&= N(L_{11} + L_{22}) + L(N_{11} + N_{22}) - 2M(M_{11} + M_{22}) \\
&\quad + 2(L_1 N_1 + L_2 N_2 - M_1^2 - M_2^2) \\
&\quad - e^{4h} \big[8h_1 K_1 + 8h_2 K_2 + 4(h_{11} + h_{22})(K+1) + 16(h_1^2 + h_2^2)(K+1) \big].
\end{aligned}$$

Substituting (2.30), (2.36), and (2.37) into (2.38) we get

(2.39)
$$\begin{aligned}
e^{6h} \triangle_g K &= N[L_{11} + N_{11} - 2H_1 h_1 e^{2h} + 2H_2 h_2 e^{2h} - 2(h_{11} - h_{22})H e^{2h} \\
&\qquad\quad - 4(h_1^2 - h_2^2)H e^{2h}] \\
&\quad + L[L_{22} + N_{22} + 2H_1 h_1 e^{2h} - 2H_2 h_2 e^{2h} + 2(h_{11} - h_{22})H e^{2h} \\
&\qquad\quad + 4(h_1^2 - h_2^2)H e^{2h}] \\
&\quad - 2M[L_{12} + N_{12} - 2(H_1 h_2 + H_2 h_1)e^{2h} - 4H e^{2h} h_{12} - 8H h_1 h_2 e^{2h}] \\
&\quad + 2(L_1 N_1 + L_2 N_2 - M_1^2 - M_2^2) \\
&\quad - e^{4h}[8h_1 K_1 + 8h_2 K_2 + 4(h_{11} + h_{22})(K+1) + 16(h_1^2 + h_2^2)(K+1)].
\end{aligned}$$

Plugging in (2.32), (2.34), and (2.35) we obtain

(2.40)
$$\begin{aligned}
e^{6h} \triangle_g K &= N[2e^{2h} H_{11} + 6H_1 h_1 e^{2h} + 2H_2 h_2 e^{2h} + 2(h_{11} + h_{22})H e^{2h} + 4(h_1^2 + h_2^2)H e^{2h}] \\
&\quad + L[2e^{2h} H_{22} + 6h_2 H_2 e^{2h} + 2H_1 h_1 e^{2h} + 2(h_{11} + h_{22})H e^{2h} \\
&\qquad\quad + 4(h_1^2 + h_2^2)H e^{2h}] \\
&\quad - 2M[2e^{2h} H_{12} + 2(H_1 h_2 + H_2 h_1)e^{2h}] \\
&\quad - e^{4h}[8h_1 K_1 + 8h_2 K_2 + 4(h_{11} + h_{22})(K+1) + 16(h_1^2 + h_2^2)(K+1)].
\end{aligned}$$

Regrouping the terms and using (2.20), we get

(2.41)
$$e^{6h}\triangle_g K = 2e^{2h}(NH_{11} - 2MH_{12} + LH_{22}) + 2(L_1N_1 + L_2N_2 - M_1^2 - M_2^2)$$
$$+ N[6H_1h_1e^{2h} + 2H_2h_2e^{2h} + 2(h_{11} + h_{22})He^{2h} + 4(h_1^2 + h_2^2)He^{2h}]$$
$$+ L[6H_2h_2e^{2h} + 2H_1h_1e^{2h} + 2(h_{11} + h_{22})He^{2h} + 4(h_1^2 + h_2^2)He^{2h}]$$
$$- 4M(H_1h_2 + H_2h_1)e^{2h}$$
$$- e^{4h}[8h_1K_1 + 8h_2K_2 - 4K(K + 1)e^{-2h} + 16(h_1^2 + h_2^2)(K + 1)].$$

From (2.31) and (2.33) we derive that

(2.42)
$$L_1N_1 \leq \frac{1}{4}(L_1 + N_1)^2 = e^{4h}(H_1 + 2h_1H)^2,$$

(2.43)
$$L_2N_2 \leq \frac{1}{4}(L_2 + N_2)^2 = e^{4h}(H_2 + 2h_2H)^2.$$

By (2.19), (2.20), (2.42), and (2.43) and noting that $h = h_1 = h_2 = 0$ at point Q, we have

(2.44) $\triangle_g K \leq 2(NH_{11} - 2MH_{12} + LH_{22}) + 2(H_1^2 + H_2^2) - 4KH^2 + 4K(K + 1).$

By assumption, we have $f_i(Q) = 0$, $i = 1, 2$. Thus $f_i = \alpha e^{\alpha\rho}H\rho_i + e^{\alpha\rho}H_i = 0$ and

(2.45)
$$f_{ij} = e^{\alpha\rho}(\alpha^2 H\rho_i\rho_j + \alpha H_i\rho_j + \alpha H_j\rho_i + \alpha H\rho_{ij} + H_{ij}),$$

where $i, j = 1, 2$. Therefore, the following hold at Q,

(2.46) $\qquad\qquad H_1 = -\alpha\rho_1 H, \quad H_2 = -\alpha\rho_2 H,$

(2.47) $\qquad\qquad H_{11} = -\alpha H\rho_{11} + \alpha^2\rho_1^2 H + f_{11}e^{-\alpha\rho},$

(2.48) $\qquad\qquad H_{12} = -\alpha H\rho_{12} + \alpha^2\rho_1\rho_2 H + f_{12}e^{-\alpha\rho},$

(2.49) $\qquad\qquad H_{22} = -\alpha H\rho_{22} + \alpha^2\rho_2^2 H + f_{22}e^{-\alpha\rho}.$

Since f achieves a local maximum at Q, we also have

$$\{f_{ij}(Q)\} \leq 0, \quad 1 \leq i, j \leq 2.$$

Thus

(2.50) $\qquad NH_{11} - 2MH_{12} + LH_{22}$
$$\leq H[L(-\alpha\rho_{22} + \alpha^2\rho_2^2) - 2M(-\alpha\rho_{12} + \alpha^2\rho_1\rho_2)$$
$$+ N(-\alpha\rho_{11} + \alpha^2\rho_1^2)]$$

and

(2.51) $\qquad\qquad\qquad (H_1^2 + H_2^2) = H^2\alpha^2(\rho_1^2 + \rho_2^2).$

Combining (2.44), (2.50), and (2.51) we obtain

(2.52)
$$\triangle_g K \leq -2\alpha H(L\rho_{22} - 2M\rho_{12} + N\rho_{11}) + 2\alpha^2 H(L + N)(\rho_1^2 + \rho_2^2)$$
$$+ 2H^2\alpha^2(\rho_1^2 + \rho_2^2) + 4H^2 + 4K(K + 1)$$
$$= -2\alpha H(L\rho_{22} - 2M\rho_{12} + N\rho_{11}) + 6H^2\alpha^2(\rho_1^2 + \rho_2^2) + 4H^2 + 4K(K + 1).$$

Now let $\vec{r} = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}$. Then we have

$$(2.53) \qquad \rho_u = \langle \vec{r}, X_u \rangle_{\mathbb{H}^3}, \quad \rho_v = \langle \vec{r}, X_v \rangle_{\mathbb{H}^3},$$

$$(2.54) \qquad \rho_u^2 + \rho_v^2 \leq \langle \vec{r}, \vec{r} \rangle_{\mathbb{H}^3},$$

$$(2.55) \qquad \rho_{uu} = \frac{1+|x|^2}{1-|x|^2} E + \bar{\Gamma}_{11}^1 \left\langle \frac{\partial}{\partial u}, \vec{r} \right\rangle_{\mathbb{H}^3} + \bar{\Gamma}_{11}^2 \left\langle \frac{\partial}{\partial v}, \vec{r} \right\rangle_{\mathbb{H}^3} + L \langle \overline{X}, \vec{r} \rangle_{\mathbb{H}^3},$$

$$(2.56) \qquad \rho_{uv} = \frac{1+|x|^2}{1-|x|^2} F + \bar{\Gamma}_{12}^1 \left\langle \frac{\partial}{\partial u}, \vec{r} \right\rangle_{\mathbb{H}^3} + \bar{\Gamma}_{12}^2 \left\langle \frac{\partial}{\partial v}, \vec{r} \right\rangle_{\mathbb{H}^3} + M \langle \overline{X}, \vec{r} \rangle_{\mathbb{H}^3},$$

and

$$(2.57) \qquad \rho_{vv} = \frac{1+|x|^2}{1-|x|^2} G + \bar{\Gamma}_{22}^1 \left\langle \frac{\partial}{\partial u}, \vec{r} \right\rangle_{\mathbb{H}^3} + \bar{\Gamma}_{22}^2 \left\langle \frac{\partial}{\partial v}, \vec{r} \right\rangle_{\mathbb{H}^3} + N \langle \overline{X}, \vec{r} \rangle_{\mathbb{H}^3}.$$

Therefore, at point Q we have

$$(2.58) \quad \Delta_g K \leq -2\alpha H(L\rho_{22} + N\rho_{11}) + 6H^2\alpha^2(\rho_1^2 + \rho_2^2) + 4H^2 + 4K(K+1)$$

$$\leq -2\alpha H \left[L\left(\frac{1+|x|^2}{1-|x|^2} G + N \langle \overline{X}, \vec{r} \rangle_{\mathbb{H}^3} \right) + N\left(\frac{1+|x|^2}{1-|x|^2} E + L \langle \overline{X}, \vec{r} \rangle_{\mathbb{H}^3} \right) \right]$$

$$+ 6\alpha^2 H^2 \langle \vec{r}, \vec{r} \rangle_{H^3} + 4H^2 + 4K(K+1)$$

$$\leq -4\alpha H^2 \frac{1+|x|^2}{1-|x|^2} + 6\alpha^2 H^2 \frac{4|x|^2}{(1-|x|^2)^2} + 4H^2 + 4K(K+1)$$

$$= 4\left(1 - \alpha \frac{1+|x|^2}{1-|x|^2} + 6\frac{\alpha^2|x|^2}{(1-|x|^2)^2} \right) H^2 + C(K).$$

Now choose $\alpha = 2$; by our assumption, at Q, $|x|^2 < \frac{1}{100}$. Hence, we get

$$(2.59) \qquad \Delta_g K \leq -H^2 + C(K),$$

which implies $H^2 \leq C(K) - \Delta_g K$. Therefore,

$$(2.60) \qquad \max_{\overline{X(B_r(P_0) \cap S^2)}} f \leq e^{2\rho(Q)} H(Q) \leq C,$$

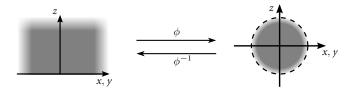from which we conclude

$$\max_{S^2} H \leq C,$$

where C is independent of ϵ. This completes the proof. $\blacksquare$

Remark 2.2 From equation (2.58), we can see that if $|x|^2 < \frac{\alpha-1}{6\alpha^2+\alpha-1}$ for some $\alpha > 1$ (*i.e.*, diam(X^ϵ) is small in the hyperbolic space), then the existence of a $C^{1,1}$ embedding is true as long as $K \geq -1$ on S^2.

3 Different Models of Hyperbolic Space

3.1 Ball Model and Upper Half-space Model

Some times it is easier to do calculations in the upper half-space model instead of the ball model. Therefore, we can use the following transformation function between the

coordinates:

$$\phi(x, y, z) = \frac{1}{x^2 + y^2 + (z+1)^2}(2x, 2y, x^2 + y^2 + z^2 - 1)$$

$$\phi^{-1}(x, y, z) = \frac{1}{x^2 + y^2 + (z-1)^2}\left(2x, 2y, 1 - (x^2 + y^2 + z^2)\right).$$

In the next section, we are going to get estimates around singular points. So it will be helpful if we can transform coordinates of a neighborhood of the singular point such that it can be represented as a graph in the upper half-space model.

For any point P belonging to our submanifold, we want to force a neighborhood of P to be graphical over the xy-plane in the upper half-space model. We can achieve this by the following procedure.

Step 1. Using the inward normal vector N of P, we can get a geodesic $\gamma(t)$ starting from P with tangent N.

Step 2. Rotate P such that N is parallel to the $\frac{\partial}{\partial z}$ direction. Then choose some $d > 0$, such that $\gamma(d)$ is the origin, and $\gamma([0, d])$ lies on the z-axis in the unit ball coordinates.

Step 3. Use the transformation ϕ^{-1} above to get P on the z-axis in the upper half-space coordinates, with the normal N parallel to $\frac{\partial}{\partial z}$. Therefore, there is a neighborhood of P that can be written as the graph of a function $z = u(x, y)$ over the xy-plane. Moreover, $\nabla z = 0$ at $(0, 0)$.

Note that the above procedure is valid for n-dimensions.

3.2 Vertical Graph in Upper Half-space Model

We will use the half-space model

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$$

equipped with the hyperbolic metric

(3.1) $$ds^2 = \frac{\sum_{i+1}^{n+1} dx_i^2}{x_{n+1}^2}.$$

If Σ is the graph of a function $u(x)$ and $x \in \Omega \subset R^n \times \{0\}$,

$$\Sigma = \{(x, x_{n+1}) : x \in \Omega, x_{n+1} = u(x)\},$$

then the coordinate vector fields and downward unit normal are given by

$$X_i = e_i + u_i e_{n+1}, \quad \mathbf{n} = u\nu = u\frac{u_i e_i - e_{n+1}}{w},$$

where $w = \sqrt{1 + |\nabla u|^2}$ and ν is the Euclidean downward unit normal to Σ. The first fundamental form g_{ij} is then given by

$$(3.2) \qquad g_{ij} = \langle X_i, X_j \rangle_{\mathbb{H}^{n+1}} = \frac{1}{u^2}(\delta_{ij} + u_i u_j) = \frac{g_{ij}^e}{u^2}.$$

To compute the second fundamental form h_{ij}, we use

$$(3.3) \qquad \Gamma_{ij}^k = \frac{1}{x_{n+1}}(-\delta_{jk}\delta_{in+1} - \delta_{ik}\delta_{jn+1} + \delta_{ij}\delta_{kn+1})$$

to obtain

$$(3.4) \qquad \nabla_{X_i} X_j = \left(\frac{\delta_{ij}}{x_{n+1}} + u_{ij} - \frac{u_i u_j}{x_{n+1}} \right) e_{n+1} - \frac{u_j e_i + u_i e_j}{x_{n+1}}.$$

Then

$$(3.5) \qquad h_{ij} = \langle \nabla_{X_i} X_j, u\nu \rangle_{\mathbb{H}^{n+1}} = \frac{-1}{u^2 w}(\delta_{ij} + u_i u_j + u u_{ij}).$$

4 Proof of Theorem 1.2

4.1 Partial Third Derivative Estimates

In this subsection, we will establish the following lemma. The argument is based on the argument originally given by E. Calabi [Ca58], see also [CNS84] and [Ia92].

Lemma 4.1   *In a neighborhood of* $(0, 0)$,

$$(4.1) \qquad L\sigma \geq \frac{\sigma^2}{2} - \frac{C}{\widetilde{K}^4}$$

holds, where

$$(4.2) \qquad L\sigma = \rho^{ij}\sigma_{ij} + \frac{4u_i \sigma_i}{(1 + |\nabla u|^2)u},$$

$$\sigma = \rho^{kl}\rho^{pq}\rho^{rs}\rho_{kpr}\rho_{lqs},$$

$$\rho = -\frac{x^2 + u^2}{2},$$

and C is a constant depending on the maximum of $\frac{1}{w}$ near $(0,0)$, the maximum of $|\widetilde{K}|_{C^2}$, the maximum of $\widetilde{K}^{1/2}|\nabla^3\widetilde{K}|$, the maximum of $|\nabla\widetilde{K}|^2/\widetilde{K}$, and the maximum of mean curvature H.

Remark 4.2   The reason for the presence of the term $|\nabla\widetilde{K}|^2/\widetilde{K}$ instead of $1/\widetilde{K}$ is that for a large class of functions satisfying $\widetilde{K}(0,0) = 0$ and $\widetilde{K}(Q) \neq 0$ when $Q \neq (0,0)$, the quantity $|\nabla\widetilde{K}|^2/\widetilde{K}$ is bounded while $1/\widetilde{K}$ is not.

Proof   In the following let $\rho = -\frac{x^2+u^2}{2}$. By (3.5), we have

$$(4.3) \qquad u^2 w h_{ij} = \rho_{ij}, \quad \text{where } w = \sqrt{1 + |\nabla u|^2}.$$

Hence,

$$(4.4) \qquad \frac{\det \rho_{ij}}{\det g} = \frac{u^4 w^2 \det h_{ij}}{\det g} = u^4 w^2 \widetilde{K}.$$

Since $g_{ij} = \frac{1}{u^2}(\delta_{ij} + u_i u_j)$,

$$\det g = \frac{1}{u^4} \begin{vmatrix} 1 + u_1^2 & u_1 u_2 \\ u_2 u_1 & 1 + u_2^2 \end{vmatrix} = \frac{w^2}{u^4},$$

$$(4.5) \qquad \det \rho_{ij} = w^4 \widetilde{K} = \widetilde{K}(1 + |\nabla u|^2)^2.$$

Let $m = \widetilde{K}(1 + |\nabla u|^2)^2$, then we have

$$\log(\det \rho_{ij}) = \log m.$$

Differentiating three times gives

$$(4.6) \qquad \rho^{ij} \rho_{ijk} = (\log m)_k,$$

$$(4.7) \qquad \rho^{ij} \rho_{ijkp} = \rho^{ia} \rho_{abp} \rho^{bj} \rho_{ijk} + (\log m)_{kp},$$

$$(4.8) \qquad \rho^{ij} \rho_{ijkpr} = \rho^{ia} \rho_{abr} \rho^{bj} \rho_{ijkp} + \rho^{ia} \rho_{abpr} \rho^{bj} \rho_{ijk} + \rho^{ia} \rho_{abp} \rho^{bj} \rho_{ijkr}$$

$$(4.9) \qquad \qquad - 2\rho^{ic} \rho_{cdr} \rho^{da} \rho_{abp} \rho^{bj} \rho_{ijk} + (\log m)_{kpr}$$

Next, consider

$$(4.10) \qquad \sigma = \rho^{kl} \rho^{pq} \rho^{rs} \rho_{kpr} \rho_{lqs}.$$

Differentiating this with respect to x_i we get

$$(4.11) \qquad \sigma_i = (-\rho^{ka} \rho_{abi} \rho^{bl} \rho^{pq} \rho^{rs} - \rho^{kl} \rho^{pa} \rho_{abi} \rho^{bq} \rho^{rs} - \rho^{kl} \rho^{pq} \rho^{ra} \rho_{abi} \rho^{bs}) \rho_{kpr} \rho_{lqs}$$
$$\qquad + 2\rho^{kl} \rho^{pq} \rho^{rs} \rho_{kpri} \rho_{lqs}.$$

Now fix a point P and rotate about P so that $\{\rho_{ij}\}$ is diagonal at P. Thus, in these coordinates at the point P, we have

$$(4.12) \qquad \rho^{ij} \sigma_{ij} = 2 \frac{\rho_{kpr} \rho_{kprii}}{\rho_{ii} \rho_{kk} \rho_{pp} \rho_{rr}} + 2 \frac{\rho_{kpri}^2}{\rho_{ii} \rho_{kk} \rho_{pp} \rho_{rr}} - 8 \frac{\rho_{kli} \rho_{lpr} \rho_{kpri}}{\rho_{ii} \rho_{kk} \rho_{ll} \rho_{pp} \rho_{rr}}$$
$$\qquad - 4 \frac{\rho_{rsi} \rho_{kps} \rho_{kpri}}{\rho_{ii} \rho_{kk} \rho_{pp} \rho_{rr} \rho_{ss}} - 2 \frac{\rho_{kpr} \rho_{lpr} \rho_{klii}}{\rho_{ii} \rho_{kk} \rho_{ll} \rho_{pp} \rho_{rr}} - \frac{\rho_{kpr} \rho_{kps} \rho_{rsii}}{\rho_{ii} \rho_{kk} \rho_{pp} \rho_{rr} \rho_{ss}}$$
$$\qquad + 4 \frac{\rho_{ali} \rho_{kai} \rho_{kpr} \rho_{lpr}}{\rho_{aa} \rho_{ii} \rho_{kk} \rho_{ll} \rho_{pp} \rho_{rr}} + 2 \frac{\rho_{rai} \rho_{asi} \rho_{kpr} \rho_{kps}}{\rho_{aa} \rho_{ii} \rho_{kk} \rho_{pp} \rho_{rr} \rho_{ss}}$$
$$\qquad + 2 \frac{\rho_{kli} \rho_{pqi} \rho_{kpr} \rho_{lqr}}{\rho_{ii} \rho_{kk} \rho_{ll} \rho_{pp} \rho_{qq} \rho_{rr}} + 4 \frac{\rho_{kli} \rho_{rsi} \rho_{kpr} \rho_{lps}}{\rho_{ii} \rho_{kk} \rho_{ll} \rho_{pp} \rho_{rr} \rho_{ss}}.$$

Consider the first term in (4.12) and use (4.8) to get

$$(4.13) \qquad 2 \frac{\rho_{kpr} \rho_{kprii}}{\rho_{ii} \rho_{kk} \rho_{pp} \rho_{rr}} = 2 \frac{\rho_{kpr}}{\rho_{kk} \rho_{pp} \rho_{rr}} \frac{\rho_{iikpr}}{\rho_{ii}} = 2 \frac{\rho_{kpr}}{\rho_{kk} \rho_{pp} \rho_{rr}}$$
$$\qquad \times \left[\frac{\rho_{abr} \rho_{abkp}}{\rho_{aa} \rho_{bb}} \frac{\rho_{abpr} \rho_{abk}}{\rho_{aa} \rho_{bb}} + \frac{\rho_{abp} \rho_{abkr}}{\rho_{aa} \rho_{bb}} - 2 \frac{\rho_{cdr} \rho_{dbp} \rho_{cbk}}{\rho_{cc} \rho_{dd} \rho_{bb}} + (\log m)_{kpr} \right].$$

Thus,

$$(4.14) \quad \rho^{ij}\sigma_{ij} = 2\frac{\rho_{kpr}\rho_{abr}\rho_{abkp}}{\rho_{aa}\rho_{bb}\rho_{kk}\rho_{pp}\rho_{rr}} + 2\frac{\rho_{abk}\rho_{kpr}\rho_{abpr}}{\rho_{aa}\rho_{bb}\rho_{kk}\rho_{pp}\rho_{rr}} + 2\frac{\rho_{abp}\rho_{kpr}\rho_{abkr}}{\rho_{aa}\rho_{bb}\rho_{kk}\rho_{pp}\rho_{rr}}$$

$$- 4\frac{\rho_{cdr}\rho_{dbp}\rho_{cbk}\rho_{kpr}}{\rho_{bb}\rho_{cc}\rho_{dd}\rho_{kk}\rho_{pp}\rho_{rr}} + 2\frac{\rho_{kpri}^2}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}} - 8\frac{\rho_{kli}\rho_{lpr}\rho_{kpri}}{\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{rr}}$$

$$- 4\frac{\rho_{rsi}\rho_{kps}\rho_{kpri}}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ss}} - 2\frac{\rho_{kpr}\rho_{lpr}\rho_{klii}}{\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{rr}} - \frac{\rho_{kpr}\rho_{kps}\rho_{rsii}}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ss}}$$

$$+ 4\frac{\rho_{ali}\rho_{kai}\rho_{kpr}\rho_{lpr}}{\rho_{aa}\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{rr}} + 2\frac{\rho_{rai}\rho_{asi}\rho_{kpr}\rho_{kps}}{\rho_{aa}\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ss}}$$

$$+ 2\frac{\rho_{kli}\rho_{pqi}\rho_{kpr}\rho_{lqr}}{\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{qq}\rho_{rr}} + 4\frac{\rho_{kli}\rho_{rsi}\rho_{kpr}\rho_{lps}}{\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{rr}\rho_{ss}} + 2\frac{\rho_{kpr}(\log m)_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}.$$

Let

$$A = \frac{\rho_{kpr}\rho_{lqr}\rho_{kli}\rho_{pqi}}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{qq}\rho_{rr}\rho_{ll}}, \quad B = \frac{\rho_{ali}\rho_{kai}\rho_{kpr}\rho_{lpr}}{\rho_{aa}\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{rr}}.$$

We have

$$(4.15) \quad \rho^{ij}\sigma_{ij} = \left(2\frac{\rho_{kpr}\rho_{abr}\rho_{abkp}}{\rho_{aa}\rho_{bb}\rho_{kk}\rho_{pp}\rho_{rr}} - 8\frac{\rho_{kli}\rho_{lpr}\rho_{kpri}}{\rho_{ii}\rho_{kk}\rho_{ll}\rho_{pp}\rho_{rr}}\right) + 2\frac{\rho_{kpri}^2}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}}$$

$$+ 3B + 2A + 2\frac{\rho_{kpr}(\log m)_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} - 3\frac{\rho_{kpr}\rho_{lpr}}{\rho_{ll}\rho_{kk}\rho_{pp}\rho_{rr}}(\log m)_{kl}.$$

Thus,

$$(4.16) \quad \rho^{ij}\sigma_{ij} = \frac{2}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}}\left(\rho_{kpri}^2 - 3\frac{\rho_{kli}\rho_{lpr}\rho_{kpri}}{\rho_{ll}}\right) + 3B + 2A$$

$$+ 2\frac{\rho_{kpr}(\log m)_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} - 3\frac{\rho_{kpr}\rho_{lpr}(\log m)_{kl}}{\rho_{ll}\rho_{kk}\rho_{pp}\rho_{rr}}$$

$$= \frac{2}{\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}}\left[\rho_{kpri} - \sum_l \frac{(\rho_{kli}\rho_{lpr} + \rho_{pli}\rho_{klr} + \rho_{rli}\rho_{lpk})}{2\rho_{ll}}\right]^2$$

$$- \frac{1}{2\rho_{ii}\rho_{kk}\rho_{pp}\rho_{rr}}\left[\sum_l \frac{(\rho_{kli}\rho_{lpr} + \rho_{pli}\rho_{klr} + \rho_{rli}\rho_{lpk})}{\rho_{ll}}\right]^2$$

$$+ 3B + 2A + 2\frac{\rho_{kpr}(\log m)_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} - 3\frac{\rho_{kpr}\rho_{lpr}(\log m)_{kl}}{\rho_{ll}\rho_{kk}\rho_{pp}\rho_{rr}}$$

$$\geq 2B + (B - A) + 2\frac{\rho_{kpr}(\log m)_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} - 3\frac{\rho_{kpr}\rho_{lpr}(\log m)_{kl}}{\rho_{ll}\rho_{kk}\rho_{pp}\rho_{rr}}.$$

Now let $\nu_{kpr} = \rho_{kpr}/(\rho_{kk}\rho_{pp}\rho_{rr})^{1/2}$. Thus, $B = \nu_{kpr}\nu_{lpr}\nu_{kai}\nu_{lai}$ and $A = \nu_{kpr}\nu_{lqr}\nu_{kli}\nu_{pqi}$. Observe that

$$(4.17) \quad \frac{1}{2}\sigma^2 = \frac{1}{2}\left(\sum_{kpr}\nu_{kpr}^2\right)^2 \leq \sum_k \left(\sum_{pr}\nu_{kpr}^2\right)^2$$

$$\leq \sum_{kl}\left(\sum_{pr}\nu_{kpr}\nu_{lpr}\right)^2 = \nu_{kpr}\nu_{lpr}\nu_{kai}\nu_{lai} = B.$$

Next consider $B - A$,

$$(4.18) \qquad 0 \leq \left(\sum_r \nu_{jri}\nu_{rkl} + \nu_{jrk}\nu_{ril} - 2\nu_{jrl}\nu_{rik} \right)^2$$

$$= \left(\nu_{jri}\nu_{rkl} + \nu_{jrk}\nu_{ril} - 2\nu_{jrl}\nu_{rik} \right) \left(\nu_{jqi}\nu_{qkl} + \nu_{jqk}\nu_{qil} - 2\nu_{jql}\nu_{qik} \right)$$

$$= B - A.$$

Therefore,

$$\rho^{ij}\sigma_{ij} \geq \sigma^2 + 2\frac{\rho_{kpr}(\log m)_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} - 3\frac{\rho_{kpr}\rho_{lpr}(\log m)_{kl}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}.$$

Recall that $m = \widetilde{K}(1 + |\nabla u|^2)^2$. We have

$$(4.19) \quad \rho^{ij}\sigma_{ij} \geq \sigma^2 + 2\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}[\log \widetilde{K} + 2\log(1 + |\nabla u|^2)]_{kpr}$$

$$- 3\frac{\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}[\log \widetilde{K} + 2\log(1 + |\nabla u|^2)]_{kl}$$

$$= \sigma^2 + 2\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}(\log \widetilde{K})_{kpr} + 4\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}[\log(1 + |\nabla u|^2)]_{kpr}$$

$$- 3\frac{\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}(\log \widetilde{K})_{kl} - 6\frac{\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}[\log(1 + |\nabla u|^2)]_{kl}$$

$$= \sigma^2 + 2\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}\left(\frac{\widetilde{K}_{kpr}}{\widetilde{K}} - \frac{\widetilde{K}_{kp}\widetilde{K}_r + \widetilde{K}_{kr}\widetilde{K}_p + \widetilde{K}_{pr}\widetilde{K}_k}{\widetilde{K}^2} + 2\frac{\widetilde{K}_k\widetilde{K}_p\widetilde{K}_r}{\widetilde{K}^3} \right)$$

$$- 3\frac{\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}\left(\frac{\widetilde{K}_{kl}}{\widetilde{K}} - \frac{\widetilde{K}_k\widetilde{K}_l}{\widetilde{K}^2} \right) + 8\frac{\rho_{kpr}u_iu_{ikpr}}{\rho_{kk}\rho_{pp}\rho_{rr}(1 + |\nabla u|^2)} + S_1,$$

where

$$(4.20) \quad S_1 = 4\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}\left\{ 2\frac{(u_{ipr}u_{ik} + u_{ip}u_{ikr} + u_{ir}u_{ikp})}{1 + |\nabla u|^2} \right.$$

$$- 4\frac{u_qu_{qr}(u_{ip}u_{ik} + u_iu_{ikp})}{(1 + |\nabla u|^2)^2} - 4\frac{u_lu_{lp}(u_{ir}u_{ik} + u_iu_{ikr})}{(1 + |\nabla u|^2)^2}$$

$$\left. - 4\frac{u_iu_{ik}(u_{lr}u_{lp} + u_lu_{lpr})}{(1 + |\nabla u|^2)^2} + 8\frac{u_{ik}u_{lp}u_{qr}u_iu_lu_q}{(1 + |\nabla u|^2)^3} \right\}$$

$$- 6\frac{\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}\left\{ 2\frac{u_{il}u_{ik} + u_iu_{ikl}}{1 + |\nabla u|^2} - 4\frac{u_iu_{ik}u_qu_{ql}}{(1 + |\nabla u|^2)^2} \right\}.$$

Note that at the point P we have

$$(4.21) \qquad \sigma_i = 2\frac{\rho_{ikpr}\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} - 3\frac{\rho_{kli}\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}},$$

$$(4.22) \qquad \rho_{pr} = -(uu_{pr} + u_pu_r + \delta_{pr}),$$

$$(4.23) \qquad \rho_{kpr} = -(u_ku_{pr} + uu_{kpr} + u_{kp}u_r + u_pu_{kr}),$$

$$(4.24) \qquad \rho_{ikpr} = -(u_{ik}u_{pr} + u_ku_{ipr} + u_iu_{kpr} + uu_{ikpr} + u_{ikp}u_r$$

$$(4.25) \qquad\qquad + u_{kp}u_{ir} + u_{ip}u_{kr} + u_pu_{ikr}),$$

and

$$(4.26) \qquad 4\frac{u_i\sigma_i}{(1+|\nabla u|^2)u} = -8\frac{u_i u_{ikpr}}{(1+|\nabla u|^2)}\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} + S_2,$$

where

$$(4.27)$$

$$S_2 = 4\frac{u_i\sigma_i}{(1+|\nabla u|^2)u} + 8\frac{u_i u_{ikpr}\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}(1+|\nabla u|^2)}$$

$$= 4\frac{u_i}{(1+|\nabla u|^2)u}\Bigg\{ -2\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}[u_{ik}u_{pr} + u_k u_{ipr} + u_i u_{kpr} + u_r u_{ikp} + u_{kp}u_{ir}$$

$$+ u_{ip}u_{kr} + u_p u_{ikr}] - 3\frac{\rho_{kli}\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}\Bigg\}.$$

Therefore,

$$(4.28) \qquad L\sigma = \rho^{ij}\sigma_{ij} + 4\frac{u_i\sigma_i}{(1+|\nabla u|^2)u}$$

$$\geq \sigma^2 + 2\frac{\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}}\left(\frac{\widetilde{K}_{kpr}}{\widetilde{K}} - \frac{\widetilde{K}_{kp}\widetilde{K}_r + \widetilde{K}_{kr}\widetilde{K}_p + \widetilde{K}_k\widetilde{K}_{pr}}{\widetilde{K}^2} + 2\frac{\widetilde{K}_k\widetilde{K}_p\widetilde{K}_r}{\widetilde{K}^3} \right)$$

$$- 3\frac{\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{rr}\rho_{ll}}\left(\frac{\widetilde{K}_{kl}}{\widetilde{K}} - \frac{\widetilde{K}_k\widetilde{K}_l}{\widetilde{K}^2} \right) + S_1 + S_2.$$

Recall that a point P near $\big(0, 0, u(0,0)\big)$ has been fixed, and a rotation of coordinates has been performed about P so that $\{\rho_{ij}\}$ is diagonal at P. Consequently, $\{h_{ij}\}$ is diagonal at P as well. Thus,

$$(4.29) \qquad \rho_{11}\rho_{22} = w^4\widetilde{K} = \widetilde{K}(1+|\nabla u|^2)^2,$$

and

$$(4.30) \qquad \rho_{11} + \rho_{22} = u^2 w(h_{11} + h_{22}) = 2w\widetilde{H}.$$

We let $\widetilde{H} = \frac{u^2}{2}(h_{11} + h_{22})$.

Since

$$(4.31) \qquad 2H = g^{11}h_{11} + g^{22}h_{22}$$

$$= u^2\left(1 - \frac{u_1^2}{w^2}\right)h_{11} + u^2\left(1 - \frac{u_2^2}{w^2}\right)h_{22}$$

$$= u^2(h_{11} + h_{22}) - \frac{u_1^2}{w^2}u^2 h_{11} - \frac{u_2^2}{w^2}u^2 h_{22}$$

$$\leq 2\widetilde{H},$$

we see that

$$(4.32) \qquad 2H \geq \frac{2}{w^2}\widetilde{H}.$$

We have

$$(4.33) \qquad 0 < \rho_{11}, \quad \rho_{22} \leq 2w\widetilde{H} \leq 2w^3 H.$$

Thus,

$$(4.34) \qquad 0 < \frac{1}{\rho_{ii}} = \frac{\rho_{3-i3-i}}{\rho_{ii}\rho_{3-i3-i}} \leq \frac{2w^3 H}{w^4 \widetilde{K}} = \frac{2H}{\widetilde{K}W}.$$

Also, by assumption, $|\nabla u|(0,0) = 0$, so $|\nabla u| < 1$ near $(0,0)$. In the above coordinates at P,

$$|\nabla u|^2(P) = u_1^2 + u_2^2.$$

So, we have $|u_1| < 1$, $|u_2| < 1$. We may also assume $u \geq 1$ near $(0,0)$. Moreover, at P,

$$\sigma = \sum_{kpr} \frac{\rho_{kpr}^2}{\rho_{kk}\rho_{pp}\rho_{rr}}.$$

Thus for each fixed k, p, r, we have

$$\frac{\rho_{kpr}^2}{\rho_{kk}\rho_{pp}\rho_{rr}} \leq \sigma,$$

and

$$\frac{|\rho_{kpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}} \leq \sigma^{1/2}.$$

Now, at P,

$$|\nabla \widetilde{K}|^2 = \widetilde{K}_1^2 + \widetilde{K}_2^2,$$
$$|\nabla^2 \widetilde{K}|^2 = \widetilde{K}_{11}^2 + 2\widetilde{K}_{12}^2 + \widetilde{K}_{22}^2,$$
$$|\nabla^3 \widetilde{K}|^2 = \sum_{kpr} \widetilde{K}_{kpr}^2.$$

Therefore, from (4.28),

$$(4.35) \quad \left| \sum_{kpr} \frac{2\rho_{kpr}}{\rho_{kk}\rho_{pp}\rho_{rr}} \left(\frac{\widetilde{K}_{kpr}}{\widetilde{K}} - \frac{(\widetilde{K}_{kp}\widetilde{K}_r + \widetilde{K}_{kr}\widetilde{K}_p + \widetilde{K}_{pr}\widetilde{K}_k)}{\widetilde{K}^2} + \frac{2\widetilde{K}_k \widetilde{K}_p \widetilde{K}_r}{\widetilde{K}^3} \right) \right|$$

$$\leq 2 \sum_{kpr} \frac{|\rho_{kpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}} \frac{1}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}} \left(\frac{|\nabla^3 \widetilde{K}|}{\widetilde{K}} + \frac{3|\nabla^2 \widetilde{K}| \, |\nabla \widetilde{K}|}{\widetilde{K}^2} + \frac{2|\nabla \widetilde{K}|^3}{\widetilde{K}^3} \right)$$

$$\leq 2\sigma^{1/2} \left(\frac{|\nabla^3 \widetilde{K}|}{\widetilde{K}} + \frac{3|\nabla^2 \widetilde{K}| \, |\nabla \widetilde{K}|}{\widetilde{K}^2} + \frac{2|\nabla \widetilde{K}|^3}{\widetilde{K}^3} \right) \sum_{kpr} \frac{1}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}$$

$$\leq 2\sigma^{1/2} \left(\frac{|\nabla^3 \widetilde{K}|}{\widetilde{K}} + \frac{3|\nabla^2 \widetilde{K}| \, |\nabla \widetilde{K}|}{\widetilde{K}^2} + \frac{2|\nabla \widetilde{K}|^3}{\widetilde{K}^3} \right) \sum_{kpr} \frac{2^{3/2} H^{3/2}}{\widetilde{K}^{3/2} w^{3/2}}$$

$$= \frac{2^{11/2} H^{3/2}}{w^{3/2}} \left(\widetilde{K}^{1/2} |\nabla^3 \widetilde{K}| + \frac{3|\nabla^2 \widetilde{K}| \, |\nabla \widetilde{K}|}{\widetilde{K}^{1/2}} + \frac{2|\nabla \widetilde{K}|^3}{\widetilde{K}^{3/2}} \right) \frac{\sigma^{1/2}}{\widetilde{K}^3}$$

$$\leq \frac{C}{\widetilde{K}^3} \sigma^{1/2},$$

where C is a constant depending on the maximum of $\frac{1}{w}$ near $(0,0)$, $|\widetilde{K}|_{C^2}$, $\widetilde{K}^{1/2}|\nabla^3\widetilde{K}|$, $|\nabla\widetilde{K}|/\widetilde{K}^{1/2}$, and the maximum of H. Next, still from (4.28),

$$
\begin{aligned}
(4.36) \quad & \left| -\frac{3\rho_{kpr}\rho_{lpr}}{\rho_{kk}\rho_{pp}\rho_{ll}\rho_{rr}}\left(\frac{\widetilde{K}_{kl}}{\widetilde{K}} - \frac{\widetilde{K}_k\widetilde{K}_l}{\widetilde{K}^2}\right)\right| \\
& \quad \leq \left(\frac{|\nabla^2\widetilde{K}|}{\widetilde{K}} + \frac{|\nabla\widetilde{K}|^2}{\widetilde{K}^2}\right)\sum_{kplr}\frac{3|\rho_{kpr}||\rho_{lpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}\sqrt{\rho_{rr}\rho_{pp}\rho_{ll}}}\frac{1}{\sqrt{\rho_{kk}\rho_{ll}}} \\
& \quad \leq 3\sigma\left(\frac{|\nabla^2\widetilde{K}|}{\widetilde{K}} + \frac{|\nabla\widetilde{K}|^2}{\widetilde{K}^2}\right)\times 2^5\frac{H}{\widetilde{K}w} \\
& \quad \leq \frac{C}{\widetilde{K}^2}\sigma.
\end{aligned}
$$

Finally, estimates are to be obtained for $|S_1|$ and $|S_2|$. From (4.22), (4.23), (4.24), and the assumption that $u \geq 1$ near $(0,0)$, we know at P that

$$
\begin{aligned}
(4.37) \quad & |uu_{pr}| \leq |\rho_{pr}| + 2, \\
& |u_{pr}| \leq |\rho_{pr}| + 2 \leq 2w^3H + 2,
\end{aligned}
$$

and

$$
\begin{aligned}
(4.38) \quad & uu_{kpr} = -\rho_{kpr} - u_ku_{pr} - u_ru_{kp} - u_pu_{kr}, \\
& |u_{kpr}| \leq |uu_{kpr}| \leq |\rho_{kpr}| + 3\sum_{pr}(|\rho_{pr}| + 2) \\
& \qquad \leq |\rho_{kpr}| + 24w^3H + 24 \\
& \qquad \leq \sigma^{1/2}\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}} + 24w^3 + 24 \\
& \qquad \leq \sigma^{1/2}\times 2^{9/2}w^{9/2}H^{3/2} + 24w^3H + 24.
\end{aligned}
$$

Now for $|S_1|$ we have,

$$
\begin{aligned}
(4.39) \quad |S_1| \leq{}& \frac{8|\rho_{kpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\frac{1}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\sum_{ikpr}(|u_{ipr}u_{ik}| + |u_{ip}u_{ikr}| + |u_{ir}u_{ikp}|) \\
& + \frac{48|\rho_{kpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{ll}}}\frac{1}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\sum_{ikpqr}|u_qu_{qr}|(|u_{ip}u_{ik}| + |u_iu_{ikp}|) \\
& + \frac{32|\rho_{kpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\frac{1}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\sum_{iklpqr}|u_{ik}u_{lp}u_{qr}u_iu_lu_q| \\
& + \frac{12|\rho_{kpr}\rho_{lpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}\sqrt{\rho_{rr}\rho_{pp}\rho_{ll}}}\frac{1}{\sqrt{\rho_{kk}\rho_{ll}}}\sum_{ikl}(|u_{il}u_{ik}| + |u_iu_{ikl}|) \\
& + \frac{24|\rho_{kpr}\rho_{lpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}\sqrt{\rho_{rr}\rho_{pp}\rho_{ll}}}\frac{1}{\sqrt{\rho_{kk}\rho_{ll}}}\sum_{ikql}|u_iu_qu_{ik}u_{ql}| \\
={}& I + II + III + IV + V.
\end{aligned}
$$

$$I \leq 8\sigma^{1/2} \times 2^3 \times \frac{2^{3/2}H^{3/2}}{\widetilde{K}^{3/2}w^{3/2}} \times 2^4$$

$$\times 3(2w^3H + 2)\left(\sum |\rho_{ipr}| + 24w^3H + 24\right)$$

(4.40)
$$\leq \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2}\sum_{ipr}|\rho_{ipr}| + \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2}$$

$$\leq \frac{C}{\widetilde{K}^{3/2}}\sigma + \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2},$$

$$II \leq 48\sigma^{1/2} \times 2^3 \times \frac{2^{3/2}H^{3/2}}{\widetilde{K}^{3/2}w^{3/2}}$$

(4.41)
$$\times 2^5[(2w^3H + 2)^3 + (2w^3H + 2)(\sigma^{1/2}2^{9/2}w^{9/2}H^{3/2} + 24w^3H + 24)]$$

$$\leq \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2} + \frac{C}{\widetilde{K}^{3/2}}\sigma,$$

$$III \leq 32\sigma^{1/2} \times 2^3 \times \frac{2^{3/2}H^{3/2}}{\widetilde{K}^{3/2}w^{3/2}} \times 2^6(2w^3H + 2)$$

(4.42)
$$\leq \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2},$$

$$IV \leq 12\sigma \times 8\frac{H}{\widetilde{K}w}$$

(4.43)
$$\times 2^3[(2w^3H + 2)^2 + 2^{9/2}w^{9/2}H^{3/2}\sigma^{1/2} + 24w^3H + 24]$$

$$\leq \frac{C}{\widetilde{K}}\sigma + \frac{C}{\widetilde{K}}\sigma^{3/2},$$

(4.44) $V \leq 24\sigma \times \dfrac{8H}{\widetilde{K}w} \times 2^4(2w^3H + 2)^2 \leq \dfrac{C}{\widetilde{K}}\sigma.$

Combining equations (4.39)–(4.44), we get

(4.45)
$$|S_1| \leq \frac{C}{\widetilde{K}^{3/2}}\sigma + \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2} + \frac{C}{\widetilde{K}}\sigma + \frac{C}{\widetilde{K}}\sigma^{3/2}.$$

Similarly, we have

(4.46)
$$|S_2| \leq \frac{8|\rho_{kpr}|}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\frac{1}{\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}}\sum_{ikpr}[3|u_{ik}u_{pr}| + 4|u_k u_{ipr}|]$$

$$+ \sum_{klipr}\frac{12|\rho_{kli}\rho_{kpr}\rho_{lpr}|}{\sqrt{\rho_{kk}\rho_{ii}\rho_{ll}}\sqrt{\rho_{kk}\rho_{pp}\rho_{rr}}\sqrt{\rho_{rr}\rho_{pp}\rho_{ll}}}\sqrt{\rho_{ii}}$$

$$\leq 8\sigma^{1/2} \times 2^3 \times \frac{2^{3/2}H^{3/2}}{\widetilde{K}^{3/2}w^{3/2}}$$

$$\times 2^4[3(2w^2H + 2)^2 + 4(\sigma^{1/2}2^{9/2}w^{9/2}H^{3/2} + 24w^3H + 24)]$$

$$+ 12\sigma^{3/2} \times 2^5 \times 2^{1/2}w^{3/2}H^{1/2}$$

$$\leq \frac{C}{\widetilde{K}^{3/2}}\sigma^{1/2} + \frac{C}{\widetilde{K}^{3/2}}\sigma + C\sigma^{3/2}.$$

Therefore,

$$(4.47) \qquad L\sigma \geq \sigma^2 - \frac{C}{\widetilde{K}^3}\sigma^{1/2} - \frac{C}{\widetilde{K}^2}\sigma - \frac{C}{\widetilde{K}^{3/2}}\sigma$$
$$- \frac{C}{\widetilde{\widetilde{K}}^{3/2}}\sigma^{1/2} - \frac{C}{\widetilde{\widetilde{K}}}\sigma - \frac{C}{\widetilde{\widetilde{K}}}\sigma^{3/2} - C\sigma^{3/2}.$$

Moreover,

$$\frac{\sigma^{1/2}}{\widetilde{K}^3} \leq \frac{\delta^4}{4}\sigma^2 + \frac{3}{4\delta^{4/3}} \times \frac{4}{\widetilde{K}^4},$$

$$\frac{\sigma}{\widetilde{K}^2} \leq \frac{\delta^2}{2}\sigma^2 + \frac{1}{2\delta^2} \times \frac{1}{\widetilde{K}^4},$$

$$\frac{\sigma}{\widetilde{K}^{3/2}} \leq \frac{\sigma^{1/2}}{\widetilde{K}^3} + \sigma^{3/2},$$

$$\sigma^{3/2} \leq \delta\sigma^2 + \frac{\sigma}{\delta\widetilde{K}^2},$$

$$\frac{\sigma}{\widetilde{\widetilde{K}}} \leq \delta\sigma^2 + \frac{1}{\delta\widetilde{K}^2},$$

$$\frac{\sigma^{1/2}}{\widetilde{\widetilde{K}}^{3/2}} \leq \frac{\sigma}{\widetilde{\widetilde{K}}} + \frac{1}{\widetilde{K}^2},$$

$$\frac{\sigma^{3/2}}{\widetilde{\widetilde{K}}} \leq \delta\sigma^2 + \frac{\sigma}{\delta\widetilde{K}^2}.$$

Therefore, choosing $\delta > 0$ small, we have

$$(4.48) \qquad L\sigma \geq \frac{\sigma^2}{2} - \frac{C}{\widetilde{\widetilde{K}}^4}. \qquad \blacksquare$$

Following the idea of [Ia92], we obtain an estimate for $\widetilde{K}^2\sigma$ and $\widetilde{K}|\nabla H|$ near $(0,0)$ and finally prove Theorem 1.2.

Lemma 4.3 *Let* $X\colon (S^2, g) \to (\mathbb{H}^3, h)$ *be a* C^2 *isometric embedding, and* $g \in C^5$ *a metric with positive Gauss curvature* $\widetilde{K}$. *Then*

$$\widetilde{K}^2\sigma \leq C, \quad \widetilde{K}|\nabla H| \leq C,$$

where C *is a constant depending only on the maximum of* $\frac{1}{w}$ *near* $(0,0)$, *the maximum of* $|\widetilde{K}|_{C^2}$, *the maximum of* $\widetilde{K}^{1/2}|\nabla^3\widetilde{K}|$, *the maximum of* $|\nabla^2\widetilde{K}|/\widetilde{K}$, *and the maximum of mean curvature* H.

Proof Recall that

$$(4.49) \qquad \frac{\det \mathrm{Hess}\,\rho}{\det g} = u^4 w^2 \widetilde{K}.$$

Without loss of generality, it is assumed that $|\nabla u|(0,0) = 0$ and that the inner normal of embedding is chosen so that the mean curvature H is positive. Now let $0 \leq f = \widetilde{K}^2\sigma$. If f has its maximum away from $(0,0)$, then the estimate follows

since the equation (4.49) is uniformly elliptic there. So assume f achieves its maximum at Q, where Q is in some small ball centered at $(0,0)$. Then

$$f_i(Q) = 0, \quad \{f_{ij}(Q)\} \le 0$$

holds at Q. Since $\{\rho^{ij}\} > 0$, $\rho^{ij} f_{ij}(Q) \le 0$. Computing, we have

$$(4.50) \qquad f_i = 2\widetilde{K}\widetilde{K}_i\sigma + \widetilde{K}^2\sigma_i,$$

$$(4.51) \qquad f_{ij} = 2\widetilde{K}_i\widetilde{K}_j\sigma + 2\widetilde{K}\widetilde{K}_{ij}\sigma + 2\widetilde{K}\widetilde{K}_i\sigma_j + 2\widetilde{K}\widetilde{K}_j\sigma_i + \widetilde{K}^2\sigma_{ij}.$$

Combining (4.50) and (4.51) we get

$$(4.52) \qquad f_{ij} = 2\widetilde{K}\widetilde{K}_{ij}\sigma - 6\widetilde{K}_i\widetilde{K}_j\sigma + \widetilde{K}^2\sigma_{ij}.$$

Therefore,

$$(4.53) \qquad 0 \ge \rho^{ij}f_{ij} = 2\rho^{ij}\widetilde{K}_{ij}\widetilde{K}\sigma - 6\rho^{ij}\widetilde{K}_i\widetilde{K}_j\sigma + \widetilde{K}^2\rho^{ij}\sigma_{ij},$$

and

$$(4.54) \qquad \widetilde{K}^4\rho^{ij}\sigma_{ij} \le 6\rho^{ij}\widetilde{K}_i\widetilde{K}_jf - 2\rho^{ij}\widetilde{K}_{ij}\widetilde{K}f.$$

Next, (4.1) can be rewritten as

$$(4.55) \qquad \widetilde{K}^4 L\sigma = \widetilde{K}^4\rho^{ij}\sigma_{ij} + \frac{4\widetilde{K}^4 u_i\sigma_i}{(1+|\nabla u|^2)u} \ge \frac{1}{2}f^2 - C.$$

Thus,

$$(4.56) \qquad f^2 \le 2C + 2\widetilde{K}^4\rho^{ij}\sigma_{ij} + \frac{8\widetilde{K}^4 u_i\sigma_i}{(1+|\nabla u|^2)u}$$

$$\le 2C + 12\rho^{ij}\widetilde{K}_i\widetilde{K}_jf - 4\rho^{ij}\widetilde{K}_{ij}\widetilde{K}f - \frac{16\widetilde{K}^3 u_i\widetilde{K}_i}{(1+|\nabla u|^2)u}.$$

Since $|\nabla u| < 1$ near $(0,0)$, we have $|u_i\widetilde{K}_i| \le |\nabla\widetilde{K}|$. Moreover, at Q,

$$(4.57) \qquad 0 \le \rho^{ij}\widetilde{K}_i\widetilde{K}_j = \rho^{11}\widetilde{K}_1^2 + \rho^{22}\widetilde{K}_2^2$$

$$= \frac{\widetilde{K}_1^2}{\rho_{11}} + \frac{\widetilde{K}_2^2}{\rho_{22}} \le \frac{2H}{\widetilde{K}w}(\widetilde{K}_1^2 + \widetilde{K}_2^2)$$

$$= \frac{2H}{w}\frac{|\nabla\widetilde{K}|^2}{\widetilde{K}}.$$

Similarly,

$$(4.58) \qquad |\rho^{ij}\widetilde{K}_{ij}\widetilde{K}| = \frac{\widetilde{K}_{11}\widetilde{K}}{\rho_{11}} + \frac{\widetilde{K}_{22}\widetilde{K}}{\rho_{22}}$$

$$\le \frac{2H}{w}|\nabla^2\widetilde{K}|.$$

So, we have $f^2(Q) \le C + Cf(Q)$. This shows that $\widetilde{K}^2\sigma$ is bounded.

Next, we are going to estimate $\widetilde{K}|\nabla H|$. We know that

$$(4.59) \qquad 2H = \frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\rho_{ij}.$$

Differentiating and multiplying by $\widetilde{K}$ we get

$$(4.60) \qquad 2\widetilde{K}\nabla H = \widetilde{K}\nabla\left(\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\right)\rho_{ij} + \frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\widetilde{K}\rho_{ijk}.$$

Since the first term is bounded by $|X|_{C^2}$, we only need to find a bound for the second term. We have

$$(4.61)$$
$$\left|\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\widetilde{K}\rho_{ijk}\right|^2 \leq \max_{i,j}\left|\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\right|^2\left(\sum_{i,j,k}|\widetilde{K}\rho_{ijk}|\right)^2$$

$$\leq \max_{i,j}\left|\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\right|^2 8\left(\sum_{i,j,k}\widetilde{K}^2\rho_{ijk}^2\right)$$

$$= \max_{i,j}\left|\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\right|^2 8\left(\sum_{i,j,k}\widetilde{K}^2\frac{\rho_{ijk}^2}{\rho_{ii}\rho_{jj}\rho_{kk}}\rho_{ii}\rho_{jj}\rho_{kk}\right)$$

$$\leq \max_{i,j}\left|\frac{1}{w}\left(\delta_{ij} - \frac{u_i u_j}{w^2}\right)\right|^2 8\widetilde{K}^2\sigma \times 8w^9 H^3.$$

We can see that every term on the right-hand side is bounded. Therefore, we have $\widetilde{K}|\nabla H|$ is bounded. ∎

4.2 Proof of Theorem 1.2

Now we are ready to prove Theorem 1.2. The proof is the same as in the Euclidean case (see [Ia92]). For completeness, we include it here.

From Theorem 1.1, we know that there exists $X_0 \in C^{1,1}$ realizing g^0. Now choose a sequence of C^∞ isometric embedding $\{X_\epsilon\}$ and corresponding metrics $\{g^\epsilon\}$ such that $\|g^\epsilon - g^0\|_{C^5(g^0)} \to 0$, $\|X_\epsilon - X_0\|_{C^{1,1}(g^0)} \to 0$, the extrinsic Gauss curvature $\widetilde{K}^\epsilon > 0$, and $H^\epsilon \geq \frac{c}{2}$, where by assumption $0 < c = \liminf_{Q\to P_i} H(Q)$. Then from Lemma 4.3,

$$(4.62) \qquad\qquad\qquad \widetilde{K}^\epsilon|\nabla_{g^\epsilon}H^\epsilon|_{g^\epsilon} \leq C$$

on $B_{r_i}(P_i)$ where r_i and C are independent of ϵ. Also, on $S^2\backslash B_{r_i}(P_i)$, the $\{\widetilde{K}^\epsilon\}$ are bounded from below. Therefore, by standard elliptic theory, $\widetilde{K}^\epsilon|\nabla_{g^\epsilon}H^\epsilon|_{g^\epsilon} \leq C$ on $S^2\backslash B_{r_i}(P_i)$ and C is independent of ϵ. Hence,

$$(4.63) \qquad\qquad\qquad \widetilde{K}^\epsilon|\nabla_{g^\epsilon}H^\epsilon|_{g^\epsilon} \leq C$$

on S^2.

Now, since $H^\epsilon \geq c/2 > 0$, there exists $r_2 > 0$ independent of ϵ such that in $B_{r_i}(P_i)$

$$(4.64) \qquad\qquad\qquad (H^\epsilon)^2 - \widetilde{K}^\epsilon \geq c_0 > 0.$$

Hence,

$$\kappa_{1,\epsilon} = H^\epsilon - \sqrt{(H^\epsilon)^2 - \widetilde{K}^\epsilon}$$

is differentiable, and

$$\nabla_{g^\epsilon} \kappa_{1,\epsilon} = \frac{-\widetilde{K}^\epsilon \nabla_{g^\epsilon} H^\epsilon}{\sqrt{(H^\epsilon)^2 - \widetilde{K}^\epsilon}(H^\epsilon + \sqrt{(H^\epsilon)^2 - \widetilde{K}^\epsilon})} + \frac{\nabla_{g^\epsilon} \widetilde{K}^\epsilon}{2\sqrt{(H^\epsilon)^2 - \widetilde{K}^\epsilon}}.$$

From (4.63) and (4.64), it is easy to see that $|\nabla_{g^\epsilon} \kappa_{1,\epsilon}| \leq C$, where C is independent of ϵ.

5 A Priori Bounds in n-dimensions

5.1 Formulas on Hypersurfaces and Some Basic Identities

In this section we recall some basic identities on a hypersurface that were derived in [GS11] by comparing the induced hyperbolic and Euclidean metrics. In the following, we identify $\mathbb{H}^{n+1}$ with the upper half-space model.

Let Σ be a hypersurface in $\mathbb{H}^{n+1}$. We shall use g and ∇ to denote the induced hyperbolic metric and Levi–Civita connection on Σ, respectively. As Σ is also a submanifold of $\mathbb{R}^{n+1}$, we shall usually distinguish a geometric quantity with respect to the Euclidean metric by adding a 'tilde' over the corresponding hyperbolic quantity. For instance, $\widetilde{g}$ denotes the induced metric on Σ from $\mathbb{R}^{n+1}$, and $\widetilde{\nabla}$ is its Levi–Civita connection.

Let $\mathbf{x}$ be the position vector of Σ in $\mathbb{R}^{n+1}$, and set $u = \mathbf{x} \cdot \mathbf{e}$ where $\mathbf{e}$ is the unit vector in the positive x_{n+1} direction in $\mathbb{R}^{n+1}$, and '·' denotes the Euclidean inner product in $\mathbb{R}^{n+1}$. We refer to u as the *height function* of Σ.

We assume that Σ is orientable and let $\mathbf{n}$ be a (global) unit normal vector field to Σ with respect to the hyperbolic metric. This also determines a unit normal ν to Σ with respect to the Euclidean metric by the relation

$$\nu = \frac{\mathbf{n}}{u}.$$

We denote $\nu^{n+1} = \mathbf{e} \cdot \nu$.

Let $(z_1, \ldots, z_n)$ be local coordinates and $\tau_i = \frac{\partial}{\partial z_i}, \quad i = 1, \ldots, n$. The hyperbolic and Euclidean metrics of Σ are given by

$$g_{ij} = \langle \tau_i, \tau_j \rangle, \quad \widetilde{g}_{ij} = \tau_i \cdot \tau_j = u^2 g_{ij}.$$

The second fundamental forms are

(5.1)
$$h_{ij} = \langle D_{\tau_i} \tau_j, \mathbf{n} \rangle = -\langle D_{\tau_i} \mathbf{n}, \tau_j \rangle,$$
$$\widetilde{h}_{ij} = \nu \cdot \widetilde{D}_{\tau_i} \tau_j = -\tau_j \cdot \widetilde{D}_{\tau_i} \nu,$$

where D and $\widetilde{D}$ denote the Levi–Civita connections of $\mathbb{H}^{n+1}$ and $\mathbb{R}^{n+1}$, respectively. The following relations are well known (see (3.5)):

(5.2)
$$h_{ij} = \frac{1}{u}\widetilde{h}_{ij} + \frac{\nu^{n+1}}{u^2}\widetilde{g}_{ij}$$

and

(5.3)
$$\kappa_i = u\widetilde{\kappa}_i + \nu^{n+1}, \quad i = 1, \ldots, n$$

where $\kappa_1, \ldots, \kappa_n$ and $\widetilde{\kappa}_1, \ldots, \widetilde{\kappa}_n$ are the hyperbolic and Euclidean principal curvatures, respectively. The Christoffel symbols are related by the formula

$$(5.4) \qquad \Gamma^k_{ij} = \widetilde{\Gamma}^k_{ij} - \frac{1}{u}(u_i \delta_{kj} + u_j \delta_{ik} - \widetilde{g}^{kl} u_l \widetilde{g}_{ij}).$$

It follows that for $v \in C^2(\Sigma)$,

$$(5.5) \qquad \nabla_{ij} v = v_{ij} - \Gamma^k_{ij} v_k = \widetilde{\nabla}_{ij} v + \frac{1}{u}(u_i v_j + u_j v_i - \widetilde{g}^{kl} u_l v_k \widetilde{g}_{ij}),$$

where

$$v_i = \frac{\partial v}{\partial z_i}, \quad v_{ij} = \frac{\partial^2 v}{\partial z_i z_j}, \quad etc.$$

In particular,

$$(5.6) \qquad \nabla_{ij} u = \widetilde{\nabla}_{ij} u + \frac{2u_i u_j}{u} - \frac{1}{u} \widetilde{g}^{kl} u_k u_l \widetilde{g}_{ij}$$

and

$$(5.7) \qquad \nabla_{ij} \frac{1}{u} = -\frac{1}{u^2} \widetilde{\nabla}_{ij} u + \frac{1}{u^3} \widetilde{g}^{kl} u_k u_l \widetilde{g}_{ij}.$$

Moreover,

$$(5.8) \qquad \nabla_{ij} \frac{v}{u} = v \nabla_{ij} \frac{1}{u} + \frac{1}{u} \widetilde{\nabla}_{ij} v - \frac{1}{u^2} \widetilde{g}^{kl} u_k v_l \widetilde{g}_{ij}.$$

In $\mathbb{R}^{n+1}$,

$$(5.9) \qquad \widetilde{g}^{kl} u_k u_l = |\widetilde{\nabla} u|^2 = 1 - (\nu^{n+1})^2, \quad \widetilde{\nabla}_{ij} u = \widetilde{h}_{ij} \nu^{n+1}.$$

Therefore, by (5.3) and (5.7),

$$(5.10) \qquad \begin{aligned} \nabla_{ij} \frac{1}{u} &= -\frac{\nu^{n+1}}{u^2} \widetilde{h}_{ij} + \frac{1}{u^3} \left(1 - (\nu^{n+1})^2\right) \widetilde{g}_{ij} \\ &= \frac{1}{u}(g_{ij} - \nu^{n+1} h_{ij}). \end{aligned}$$

We note that (5.8) and (5.10) still hold for general local frames $\tau_1, \ldots, \tau_n$.

5.2 Proof of Theorem 1.3

For simplicity, in this subsection we are going to do calculations in the upper half-space model.

Let g be a C^4 metric of nonnegative sectional curvature on S^n, and let $X \colon (S^n, g) \to \mathbb{H}^{n+1}$ be a C^4 isometric embedding into the hyperbolic space $\mathbb{H}^{n+1}$ (we assume that the critical points are finite and isolated). Then, as in [Po64], for any point $P \in X(S^n)$ we can always construct a convex cap ω, such that $P \in \omega$ and $\partial\omega$ is a $n - 1$ dimensional submanifold with positive curvatures. Using the techniques introduced in Section 3, we can always assume that ω can be represented as a graph over $\mathbb{R}^n \times \{0\}$ in the half-space model, and on $\partial\omega$ the height function satisfies $u \equiv c > 0$.

When ω is strictly convex, we can apply results of [GSS09] directly and obtain *a priori* bounds for the principal curvatures that only depend on the metric g on $X^{-1}(\omega) \subset S^n$.

When there exists a critical point, say $P_0 \in \omega$. By lifting

$$\omega = \left\{ \left(x, u(x) \right) \mid x \in \mathbb{R}^n \right\} \quad \text{to} \quad \omega^\epsilon = \left\{ \left(x, u(x) + \epsilon \right) \mid x \in \mathbb{R}^n \right\}, \quad \epsilon > 0$$

small, we obtain a sequence of $g^\epsilon \in C^4\left(X^{-1}(\omega) \right)$ with positive sectional curvatures which are isometrically embeddable in $\mathbb{H}^{n+1}$, and g^ϵ converges to g on $X^{-1}(\omega)$.

Now we are going to generalize the result we proved in Section 2 to n-dimensions ($n \geq 3$), and in the following, for convenience, we drop the dependence on ϵ in our notation.

Theorem 5.1 *Let g be a C^4 metric with sectional curvature ≥ -1 and let $X \colon (S^n, g) \to \mathbb{H}^{n+1}$ be a C^4 isometric embedding. Suppose the sectional curvature S of g satisfies*

(i)   *$S(P_i, \chi) = -1$ for some $\chi \in \bigwedge^2 T_{P_i} M$, $1 \leq i \leq n$;*
(ii) *$S(Q) > -1$ for any $Q \neq P_i$, where $\{P_i\} \in S^n$ are finite isolated points.*

Let H be the trace of the second fundamental form of X, and let R be the extrinsic scalar curvature of g. Then the following inequality holds:

(5.11) $$H^2 \leq C_1 |\Delta R| + C_2 (R^2 + R),$$

where C_1, C_2 only depends on the metric g and dimension n.

Proof   Similar to Section 2, we consider a function $f = e^{\frac{\alpha}{u}} H$ in $B_r\left(X(P_0) \right)$ with $\alpha > 0$ to be chosen later. By the procedure introduced in Section 3, we can assume $u\left(X(P_0) \right) > 1$ and $\nabla u\left(X(P_0) \right) = 0$ in the upper half-space model. Choosing $r > 0$ small enough, we have $(\nu^{n+1})^2 > 1 - (\min u / 4 \max u)^2$ in $B_r\left(X(P_0) \right)$. Assume f achieves its maximum at an interior point $Q \in B_r\left(X(P_0) \right)$. We choose a local orthonormal frame $\tau_1, \ldots, \tau_n$ around Q. For convenience we shall write $v_{ij} = \nabla_{ij} v$, $h_{ijk} = \nabla_k h_{ij}$, $h_{ijkl} = \nabla_{lk} h_{ij}$, etc. Then at this point we have

(5.12) $$f_i = \alpha \left(\frac{1}{u} \right)_i H + H_i = 0,$$

(5.13) $$f_{ij} = e^{\frac{\alpha}{u}} \left(\alpha H \left(\frac{1}{u} \right)_{ij} - \alpha^2 H \left(\frac{1}{u} \right)_i \left(\frac{1}{u} \right)_j + H_{ij} \right) \leq 0.$$

Since $R = H^2 - \operatorname{tr} A^2$, by (5.12), we have

(5.14) $$\Delta R = 2H \Delta H + 2|\nabla H|^2 - 2|\nabla A|^2 - 2h^{ij} \Delta h_{ij}$$
$$\leq 2H \Delta H + 2\alpha^2 \left| \nabla \left(\frac{1}{u} \right) \right|^2 H^2 - 2h^{ij} H_{ij} + 2h^{ij} (R_{jlik} h^{lk} + R^k_{lik} h^l_j).$$

Also,

(5.15)
$$2(\operatorname{tr} A^2)^2 - 2 \operatorname{tr} A^3 \operatorname{tr} A = 2(H^2 - R)^2 - 2 \operatorname{tr} A^3 \operatorname{tr} A$$
$$\leq 2(H^2 - R)^2 - \operatorname{tr} A \{ 2(\operatorname{tr} A)^3 - 3[(\operatorname{tr} A)^2 - \operatorname{tr} A^2] \operatorname{tr} A \}$$
$$\leq -H^2 R + 2R^2,$$

where we used $3(\sum x_i^2)(\sum x_i) \leq (\sum x_i)^3 + 2\sum x_i^3$, for $x_i > 0$, $1 \leq i \leq n$.

Let $d^2 = 1 - (\nu^{n+1})^2$. Combining (5.15) and (5.14), we get

(5.16)

$$
\Delta R \leq 2H\Delta H - 2h^{ij}H_{ij} + 2\frac{\alpha^2 d^2}{u^2}H^2 + 2h^{ij}[(h_{ji}h_{lk} - h_{jk}h_{li}) - (g_{ji}g_{lk} - g_{jk}g_{li})]h^{lk}
$$
$$
+ 2h^{ij}[(h_i^k h_{lk} - h_k^k h_{li}) - (g_i^k g_{lk} - g_k^k g_{li})]h_j^l
$$
$$
= 2H\Delta H - 2h^{ij}H_{ij} + 2\frac{\alpha^2 d^2}{u^2}H^2 + 2[(\operatorname{tr} A^2)^2 - \operatorname{tr} A^3 \operatorname{tr} A]
$$
$$
- 2h^{ij}g_{ij}g_{lk}h^{lk} + 2h^{ij}g_{jk}g_{li}h^{lk} - 2h^{ij}g_i^k g_{lk}h_j^l + 2nh^{ij}g_{li}h_j^l
$$
$$
\leq 2H\Delta H - 2h^{ij}H_{ij} + 2\frac{\alpha^2 d^2}{u^2}H^2 - RH^2 + 2R^2 - 2H^2 + 2n(H^2 - R)
$$
$$
\leq 2(Hg^{ij} - h^{ij})H_{ij} + 2\frac{\alpha^2 d^2}{u^2}H^2 - RH^2 + (2n-2)H^2 + C_2(R^2 + R),
$$

where we used (5.9).

At Q,

$$
(Hg^{ij} - h^{ij})f_{ij} \leq 0.
$$

We have

(5.17)

$$
(Hg^{ij} - h^{ij})H_{ij} \leq (Hg^{ij} - h^{ij})\left(-\alpha H\left(\frac{1}{u}\right)_{ij} + \alpha^2 H\left(\frac{1}{u}\right)_i\left(\frac{1}{u}\right)_j\right)
$$
$$
\leq -\alpha H(Hg^{ij} - h^{ij})\left(\frac{1}{u}g_{ij} - \frac{\nu^{n+1}}{u}h_{ij}\right) + \alpha^2 H^2 g^{ij}\left(\frac{1}{u}\right)_i\left(\frac{1}{u}\right)_j
$$
$$
\leq \frac{-\alpha H^2(n-1)}{u} + \frac{\alpha HR}{u}\nu^{n+1} + \frac{\alpha^2 d^2 H^2}{u^2}.
$$

Therefore,

(5.18)
$$
\Delta R \leq \frac{-2\alpha(n-1)}{u}H^2 + \frac{2\alpha HR}{u}\sqrt{1-d^2}
$$
$$
+ \frac{4\alpha^2 d^2 H^2}{u^2} - RH^2 + (2n-2)H^2 + C_2(R^2 + R).
$$

By

$$
RH(2\alpha u\sqrt{1-d^2} - H) \leq R\alpha^2 u^2(1-d^2),
$$

we have

(5.19)
$$
u^2 \Delta R \leq -2\alpha u(n-1)H^2 + 4\alpha^2 d^2 H^2 + (2n-2)u^2 H^2 + \alpha^2 u^2(1-d^2)R + C_2(R^2 + R).
$$

Choosing $\alpha = \frac{n-1}{\min_{X(S^n)} u}(\max_{X(S^n)} u)^2$, we have

(5.20)
$$
u^2 \Delta R \leq \frac{-1}{2}(n-1)^2 H^2 + C_2(R^2 + R).
$$

Thus

(5.21)
$$
H^2 \leq C_1|\Delta R| + C_2(R^2 + R). \qquad \blacksquare
$$

References

[AP50]   A. D. Aleksandrov and A. V. Pogorelov, *Uniqueness of convex surfaces of revolution.* Mat. Sb. (N.S.) **26(68)**(1950), 183–204.

[CNS84]   L. Caffarell, L. Nirenberg, and J. Spruck, *Monge–Ampere equations.* Comm. Pure Appl. Math. **37**(1984), 369–402.
          http://dx.doi.org/10.1002/cpa.3160370306

[Ca58]   E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem of K. Jordens.* Michigan Math. J. **5**(1958), 105–126.
          http://dx.doi.org/10.1307/mmj/1028998055

[GL94]   P. Guan and Y. Li, *The Weyl problem with nonnegative Gauss curvature.* J. Differential Geom. **39**(1994), 331–342.

[GS11]   B. Guan and J. Spruck, *Convex hypersurfaces of constant curvature in hyperbolic space.* In: Surveys in geometric analysis and relativity, Adv. Lect. Math. (ALM) **20**, Int. Press, Somerville, MA, 2011, 241–257.

[GSS09]   B. Guan, J. Spruck, and M. Szapiel, *Hypersurfaces of constant curvature in hyperbolic space I.* J. Geom. Anal. **19**(2009), 772–795.
          http://dx.doi.org/10.1007/s12220-009-9086-7

[HZ95]   J. Hong and C. Zuily, *Isometric embedding of the 2-sphere with nonegative curvature in R^3.* Math. Z. **219**(1995), 323–334.
          http://dx.doi.org/10.1007/BF02572368

[Ia92]   J. A. Iaia, *Isometric embeddings of surfaces with nonnegative curvature in R^3.* Duke Math. J. **67**(1992), 423–459.
          http://dx.doi.org/10.1215/S0012-7094-92-06717-2

[Le35]   H. Lewy, *A priori limitations for solutions of Monge–Ampere equations, I.* Trans. Amer. Math. Soc. **37**(1935), 417–434.

[LW99]   Y. Li and G. Weinstein, *A priori bounds for co-dimension one isometric embeddings.* Amer. J. Math. **121**(1999), 945–965.
          http://dx.doi.org/10.1353/ajm.1999.0035

[Ni53]   Louis Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large.* Comm. Pure Appl. Math. **6**(1953), 337–394.
          http://dx.doi.org/10.1002/cpa.3160060303

[Po64]   A. V. Pogorelov, *Some results on surface theory in the large.* Advances in Math. **1**(1964), 191–264.
          http://dx.doi.org/10.1016/0001-8708(65)90039-3

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