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Proceedings of the Royal Society of Edinburgh, page 1 of 26
DOI:10.1017/prm.2023.124

Various topological complexities of small covers and real Bott manifolds

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(Received 1 June 2023; accepted 27 November 2023)

In this paper, we compute the LS-category and equivariant LS-category of a small cover and its real moment angle manifold. We calculate a tight lower bound for the topological complexity of many small covers over a product of simplices. Then we compute symmetric topological complexity of several small covers over a product of simplices. We calculate the LS one-category of real Bott manifolds and infinitely many small covers.

Keywords: topological complexity; small cover; real bott manifold; $\mathcal{D}$-topological complexity; symmetric topological complexity

2020 *Mathematics Subject Classification:* 55M30; 57S12

1. Introduction

The topological complexity of a space is a numerical homotopy invariant introduced by Michael Farber in [11], which connects motion planning problems in robotics. Briefly, given a mechanical system $\mathcal{M}$, a motion planning algorithm for $\mathcal{M}$ is a function that associates to any pair of states (a, b) of $\mathcal{M}$ to a continuous motion of the system starting at a and ending at b. Interestingly, the topological complexity is a particular case of another homotopy invariant called the 'sectional category' of a map $p\colon E \to B$ where E and B are path connected spaces. The sectional category of p, denoted by $\mathrm{secat}(p)$, is the least integer k such that there is an open cover $\{U_1, \dots, U_k\}$ of B, and there is a local section $s_i\colon U_i \to E$ of p for each i satisfying $p \circ s_i = id_{U_i}\colon U_i \hookrightarrow B$ where id_{U_i} denotes the inclusion. We remark that the genus of a fibration was introduced by Schwarz [25]. However, James [20] used 'sectional category' instead of 'genus'.

Let Y be the space of all possible configurations of a mechanical system. We assume that Y is a Hausdorff path-connected topological space. Let PY be the space of all continuous paths $\gamma\colon [0, 1] \to Y$ in Y equipped with the compact-open

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https://doi.org/10.1017/prm.2023.124 Published online by Cambridge University Press

topology. Consider the path fibration

$$\pi\colon PY \to Y \times Y \tag{1.1}$$

defined by $\pi(\gamma) = (\gamma(0), \gamma(1))$. A motion planning algorithm of Y is defined by a section $s\colon Y \times Y \to PY$ of the fibration π. This section exists if and only if Y is contractible. Interestingly in general, almost all configuration spaces are non-contractible. To compute the complexity of motion-planning algorithm for a non-contractible space Y, Farber defined the topological complexity of Y by the sectional category of π. The survey [**12**] contains several introductory results related to motion planning.

A symmetric version of the topological complexity arises when one restricts the local planners for which the motion from a to b is the reverse of the motion from b to a and the motion from a to a is constant. In notation, consider a map $s\colon Y \times Y \to PY$ (not necessarily continuous) such that $\pi \circ s = \mathrm{Id}_{Y \times Y}$ and $s(a, a)(t) = a$, $s(a, b)(t) = s(b, a)(1 - t)$ for all $a, b \in Y$ and $t \in [0, 1]$. This motivates the notion of symmetric topological complexity, given by Farber and Grant in [**13**]. Some developments in symmetric topological complexity can be found in [**17**–**19**].

Consider a continuous partial section $s\colon U \to PY$ of the fibration π over an open subset $U \subseteq Y \times Y$. The map s can be described as a homotopy $h\colon U \times [0, 1] \to Y$ defined by $h(u, t) = s(u)(t)$ for $u \in U$, $t \in [0, 1]$. Let $p_1\colon Y \times Y \to Y$ and $p_2\colon Y \times Y \to Y$ denote the projections onto the first and the second factor, respectively. Since s is a section, the homotopy h connects $h(u, 0) = p_1(u)$ and $h(u, 1) = p_2(u)$. Therefore, the open sets $U_i \subseteq Y \times Y$, which appear in the definition of topological complexity, can be equivalently characterized by the property that their two projections $U_i \to Y$ on the first and the second factors are homotopic. For an aspherical space Y, a connected subspace U of Y which is homotopy equivalent to a cell complex, the set of homotopy classes of maps $U \to Y$ is in a one-to-one correspondence with the set of conjugacy classes of homomorphisms $\pi_1(U, u_0) \to \pi_1(Y, y_0)$. Using this idea, Farber *et al.* introduced $\mathbf{TC}^{\mathcal{D}}(Y)$, the $\mathcal{D}$-topological complexity for a path-connected topological space, see [**14**]. Here the letter '$\mathcal{D}$' in the notation $\mathbf{TC}^{\mathcal{D}}(Y)$ stands for the 'diagonal'. In [**15**], Farber *et al.* introduced some properties of $\mathcal{D}$-topological complexity. Note that symmetric topological complexity is not homotopy invariant but $\mathcal{D}$-topological complexity is homotopy invariant. Some related results can be found in [**9**].

A small cover of dimension n is an n-dimensional closed smooth manifold with a locally standard $\mathbb{Z}_2^n$-action whose orbit space is a simple polytope. It was introduced in the pioneering paper [**8**] as a generalization of real projective toric varieties. An n-dimensional toric variety is an algebraic normal variety that admits an action of $(\mathbb{C}^*)^n$ with an open dense orbit. A non-singular complete toric variety is simply called a toric manifold. The real locus of a toric manifold is called a real toric manifold. A real Bott tower is a sequence of smooth complete real toric varieties, see subsection 2.2. In this paper, we compute lower and upper bounds for the topological complexity, symmetric topological complexity and LS one-category of a class of small covers and real Bott manifolds.

The paper is organized as follows. In § 2, we study the definition of small cover over a simple polytope, generalized real Bott manifold and the relation between them. We modify the cohomology ring of a small cover over a product of simplices

https://doi.org/10.1017/prm.2023.124 Published online by Cambridge University Press

$\prod_{j=1}^{m} \Delta^{n_j}$ with $\mathbb{Z}_2$ coefficients as $\mathbb{Z}_2[y_1, y_2, \ldots, y_m]/I$ where I is given in (2.14). We prove $y_j^{n_j} \neq 0$ in the cohomology ring $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ as in (2.13), see lemma 2.6. We also recall the notion of real moment angle manifolds and complexes.

In § 3, we recall the definition and some properties of LS-category and equivariant LS-category of a topological space. We compute the LS-category and equivariant LS-category of a small cover. We calculate the LS-category of the real moment angle manifold for r-gon and the equivariant LS-category of a real moment angle complex.

In § 4, we give a tight lower bounds to the topological complexity of a small cover over a product of two simplices. We compute the topological complexity for some classes of real Bott manifolds.

In § 5, we rewrite the definition and some basic properties of symmetric topological complexity and give bounds for the symmetric topological complexity of several small covers over a product of simplices.

Finally in § 6, we recall the definition and some basic properties of the LS one-category and $\mathcal{D}$-topological complexity. We calculate the exact value of LS one-category of a simple polytope when its real moment angle manifold is simply connected and orientable. We calculate LS one-category of a small cover over a product of simplices, and give bounds of $\mathcal{D}$-topological complexity for a small cover over a product of simplices.

2. Cohomology rings of small covers, generalized real Bott manifolds and real moment angle complexes

In this section, we recall simple polytopes and the constructive definition of a small cover over a simple polytope using [**8**]. We also review the definition of a (generalized) real Bott manifold and then discuss its relation with a small cover over a finite product of simplices. We give a presentation of the cohomology ring of a generalized real Bott manifold. Later, we study real moment angle manifolds and complexes.

2.1. Small covers and its cohomology ring

In this subsection, we recall the definition of small cover and its cohomology ring with $\mathbb{Z}_2$-coefficients following [**8**].

A convex polytope is a convex hull of finitely many points in $\mathbb{R}^n$ for some $n \in \mathbb{Z}_{\geqslant 0}$. The face of dimension 0 and $(n-1)$ in a convex polytope of dimension n are called the vertex and the facet of the polytope, respectively. The vertex set and the facet set of a convex polytope P are denoted by $V(P)$ and $\mathcal{F}(P)$, respectively. An n-dimensional convex polytope is called simple if at each vertex exactly n many facets intersect. Throughout this paper, we denote an n-dimensional simple polytope by P.

Definition 2.1. *A function $\lambda\colon \mathcal{F}(P) \to \mathbb{Z}_2^n$ is called a characteristic function if the submodule of $\mathbb{Z}_2^n$ generated by $\{\lambda(F_{i_1}), \ldots, \lambda(F_{i_\ell})\}$ is an ℓ-dimensional direct summand of $\mathbb{Z}_2^n$ whenever $F_{i_1} \cap \cdots \cap F_{i_\ell} \neq \emptyset$. The vector $\lambda_i := \lambda(F_i)$ is called the characteristic vector associated with the facet F_i for $i = 1, \ldots, r$, and the pair (P, λ) is called a characteristic pair.*

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We recall the construction of a small cover from a characteristic pair (P, λ). For each point $p \in P$, let $F(p)$ be the unique face of P, which contains p in its relative interior. Let $F(p) = F_{i_1} \cap \cdots \cap F_{i_k}$ for some unique facets $F_{i_1}, \ldots, F_{i_k}$. Define $G_{F(p)}$ as a subgroup of $\mathbb{Z}_2^n$ generated by $\lambda(F_{i_1}), \ldots, \lambda(F_{i_k})$. We define an equivalence relation on $P \times \mathbb{Z}_2^n$ as follows:

$$(p, g) \backsim (q, h) \Leftrightarrow p = q, g^{-1}h \in G_{F(p)}.$$

The identification space $M^n(P, \lambda) := (P \times \mathbb{Z}_2^n)/\sim$ has an n-dimensional manifold structure with a natural $\mathbb{Z}_2^n$-action induced by the group operation on the second factor of $P \times \mathbb{Z}_2^n$. The projection onto the first factor gives the orbit map

$$\rho\colon M^n(P, \lambda) \to P \text{ defined by } [p, g]_\sim \mapsto p,$$

where $[p, g]_\sim$ is the equivalence class of (p, g). The manifold $M^n(P, \lambda)$ is called a small cover over P with the characteristic function λ, see [**8**] for details.

Let $\{F_1, \ldots, F_r\}$ be the facets of P and the indeterminates $v_1, \ldots, v_r$ correspond bijectively to the facets $F_1, \ldots, F_r$ respectively.

PROPOSITION 2.2 [**8**, Theorem 4.14]. *Let $\rho\colon M^n(P, \lambda) \to P$ be a small cover over a simple polytope P with $|\mathcal{F}(P)| = r$. Then*

$$H^*(M^n(P, \lambda), \mathbb{Z}_2) \cong \mathbb{Z}_2[v_1, \ldots, v_r]/(\tilde{I} + \tilde{J}),$$

where the ideal $\tilde{I}$ is generated by the monomials $v_{s_1} \cdots v_{s_\ell}$, if $F_{s_1} \cap \cdots \cap F_{s_\ell} = \emptyset$, and the ideal $\tilde{J}$ is generated by the n coordinates of the vector $\Lambda_{\tilde{J}}$ where $\Lambda_{\tilde{J}} = \sum_{i=1}^{r} \lambda_i v_i$.

EXAMPLE 2.3. The n-dimensional real projective space $\mathbb{RP}^n$ is an example of a small cover over the n-dimensional simplex Δ^n. A finite product of $\mathbb{RP}^n$'s is also a small cover.

2.2. Generalized real Bott manifolds and its cohomology ring

In this subsection, we study generalized real Bott manifolds and give a nice presentation of its cohomology ring with $\mathbb{Z}_2$-coefficients.

A generalized real Bott tower of height m is a sequence

$$B_m \xrightarrow{\pi_m} B_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\mathrm{pt}\} \tag{2.1}$$

of manifolds $B_j = \mathbb{P}(\underline{\mathbb{R}} \oplus E_j^{(1)} \oplus \cdots \oplus E_j^{(n_j)})$, where $\underline{\mathbb{R}}$ is the trivial line bundle over B_{j-1}, $E_j^{(i)}$ is a real line bundle over B_{j-1} for $i = 1, \ldots, n_j$, and $j = 1, \ldots, m$. Here $\mathbb{P}(\cdot)$ denotes the projectivization. The space B_j is called a j-th stage generalized real Bott manifold. In this case, when $n_j = 1$ for every j, B_j is called a real Bott manifold.

PROPOSITION 2.4 [**21**, Corollary 4.6]. *The j-th stage generalized real Bott manifold B_j of the tower (2.1) is a small cover over $\prod_{i=1}^{j} \Delta^{n_i}$ where Δ^{n_i} is the n_i-simplex.*

The converse statement also holds by the following proposition.

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Proposition 2.5 [**10**, Proposition 2.7]. *Every small cover over a product of simplices is a generalized real Bott manifold.*

Now we discuss the cohomology ring of a small cover over a finite product of simplices. Let

$$P := \prod_{j=1}^{m} \Delta^{n_j}, \tag{2.2}$$

where Δ^{n_j} is a simplex of dimension n_j. Then, the dimension of P is $n := \sum_{j=1}^{m} n_j$. Let

$$\mathcal{N}_s := \sum_{j=1}^{s} n_j, \tag{2.3}$$

for $s = 1, \ldots, m$. Thus $\mathcal{N}_1 = n_1$ and $\mathcal{N}_m = n$. Let us assume $\mathcal{N}_0 := 0$.

Let $V(\Delta^{n_j}) := \{v_0^j, \ldots, v_{n_j}^j\}$ be the vertices of Δ^{n_j} for $j = 1, \ldots, m$. Then the vertex set of P is given by

$$V(P) := \{v_{\ell_1\ell_2\ldots\ell_m} := (v^1_{\ell_1}, v^2_{\ell_2}, \ldots, v^m_{\ell_m}) \mid 0 \leqslant \ell_j \leqslant n_j\}. \tag{2.4}$$

Let $\mathcal{F}(\Delta^{n_j}) := \{F_0^{\Delta_j}, \ldots, F_{n_j}^{\Delta_j}\}$ be the facets of Δ^{n_j} where the facet $F_{k_j}^{\Delta_j}$ does not contain the vertex $v^j_{k_j}$ for $j = 1, \ldots, m$. So, the facet set of P is

$$\mathcal{F}(P) := \{F^j_{k_j} \mid 0 \leqslant k_j \leqslant n_j, j = 1, \ldots, m\}, \tag{2.5}$$

where $F^j_{k_j} := \Delta^{n_1} \times \cdots \times \Delta^{n_{j-1}} \times F^{\Delta_j}_{k_j} \times \Delta^{n_{j+1}} \times \cdots \times \Delta^{n_m}$. Observe that the vertex $v_{\ell_1\ell_2\ldots\ell_m}$ is the unique intersection of the n-many facets of $\mathcal{F}(P) \setminus \{F^j_{\ell_j} \mid j = 1, \ldots, m\}$. In particular,

$$v_{0\ldots0} = F^1_1 \cap \cdots \cap F^1_{n_1} \cap \cdots \cap F^m_1 \cap \cdots \cap F^m_{n_m}. \tag{2.6}$$

Let

$$\lambda\colon \mathcal{F}(P) \to \mathbb{Z}_2^n \tag{2.7}$$

be a $\mathbb{Z}_2$-characteristic function on P where P is the product of simplices as in (2.2). Then from (2.6), we have $\{\lambda(F^1_1), \ldots, \lambda(F^1_{n_1}), \ldots, \lambda(F^m_1), \ldots, \lambda(F^m_{n_m})\}$ is a basis of $\mathbb{Z}_2^n$ over $\mathbb{Z}_2$. So, we may assume that these vectors are assigned with the standard basis vectors. Thus,

$$\lambda(F^j_1) = \boldsymbol{e}_{\boldsymbol{\mathcal{N}_{j-1}+1}}, \ldots, \lambda(F^j_{n_j}) = \boldsymbol{e}_{\boldsymbol{\mathcal{N}_j}},$$

for $j = 1, \ldots, m$. The remaining m facets $\{F^1_0, \ldots, F^m_0\}$ are assigned with the vectors as follows

$$\lambda(F^j_0) := \boldsymbol{\alpha_j} \in \mathbb{Z}_2^n \text{ for } j = 1, \ldots, m, \tag{2.8}$$

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so that the above assignment satisfies definition 2.1. This gives us vector matrices of order $(1\times m)$ and $(m\times m)$, and a scalar matrix of order $(n\times m)$ as following:

$$A := \begin{pmatrix}\boldsymbol{\alpha_1} & \boldsymbol{\alpha_2} & \dots & \boldsymbol{\alpha_m}\end{pmatrix}_{1\times m} = \begin{pmatrix}\boldsymbol{\alpha_1^1} & \dots & \boldsymbol{\alpha_m^1}\\ \vdots & \dots & \vdots \\ \boldsymbol{\alpha_1^m} & \dots & \boldsymbol{\alpha_m^m}\end{pmatrix}_{m\times m}$$

$$= \begin{pmatrix}\alpha_{11}^1 & \dots & \alpha_{m1}^1\\ \vdots & \dots & \vdots\\ \alpha_{1n_1}^1 & \dots & \alpha_{mn_1}^1\\ \vdots & \dots & \vdots\\ \alpha_{11}^m & \dots & \alpha_{m1}^m\\ \vdots & \dots & \vdots\\ \alpha_{1n_m}^m & \dots & \alpha_{mn_m}^m\end{pmatrix}_{n\times m},$$

where $\boldsymbol{\alpha_j}\in\mathbb{Z}_2^n$ is the j-th column vector of A, $\boldsymbol{\alpha_j^k}\in\mathbb{Z}_2^{n_k}$ is the $(k,\,j)$-th entry of the $m\times m$ vector matrix and $\alpha_{ji}^k\in\mathbb{Z}_2$ is the $(\mathcal{N}_{k-1}+i,\,j)$-th entry of the $n\times m$ scalar matrix. Throughout this paper, the vectors $\boldsymbol{e_i}$ and $\boldsymbol{\alpha_j}$ of $\mathbb{Z}_2^n$ are considered as the column entries of the matrices for $i=1,\,\dots,\,n$, and $j=1,\,\dots,\,m$.

Now we calculate the cohomology ring of the small cover $M^n(P,\,\lambda)$ when P is a product of simplices as in (2.2) and the characteristic function λ on P is given by (2.7). Let us assign the indeterminate x_i to the facet $F_{k_j}^j$ where

$$i = \left(\sum_{s=1}^{j-1} n_s\right) + k_j = \mathcal{N}_{j-1}+k_j,$$

for $1\leqslant k_j\leqslant n_j$, $j=1,\,\dots,\,m$. Therefore, $i\in\{1,\,\dots,\,n\}$. We also assign the indeterminate x_i to the facet F_0^j where $i=n+j$ for $j=1,\,\dots,\,m$. Note that $F_1^j\cap\cdots\cap F_{n_j}^j\cap F_0^j=\emptyset$. Then, from proposition 2.2, we have

$$H^*(M^n(P,\lambda);\mathbb{Z}_2)\cong\mathbb{Z}_2[x_1,\dots,x_{n+m}]/\tilde{I}+\tilde{J}, \tag{2.9}$$

where the ideals $\tilde{I}$ and $\tilde{J}$ are as follows. The ideal $\tilde{I}$ is given by

$$\tilde{I} = \big\langle\{x_{\mathcal{N}_{j-1}+1}x_{\mathcal{N}_{j-1}+2}\cdots x_{\mathcal{N}_j}x_{n+j}\mid j=1,\dots,m\}\big\rangle, \tag{2.10}$$

where $\mathcal{N}_j$ is defined in (2.3). The ideal $\tilde{J}$ is generated by the coordinates of

$$\Lambda_{\tilde{J}} = \begin{pmatrix}\lambda(F_1)^t & \lambda(F_2)^t & \dots & \lambda(F_{n+m})^t\end{pmatrix}_{(n\times(n+m))}\cdot\begin{pmatrix}x_1 & x_2 & \dots & x_{n+m}\end{pmatrix}^t_{(n+m)\times 1}. \tag{2.11}$$

In (2.11); for $i=1,\,\dots,\,n$, we denote $F_i=F_{k_j}^j$ with $i=\mathcal{N}_{j-1}+k_j$ for $1\leqslant k_j\leqslant n_j$, $j=1,\,\dots,\,m$ and for $i=n+1,\,\dots,\,n+m$, we denote $F_i=F_0^j$ with $i=n+j$ where $j=1,\,\dots,\,m$.

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Note that $\Lambda_{\tilde{J}}$ is an n tuple. The i-th coordinate of $\Lambda_{\tilde{J}}$ is

$$x_i + \alpha^j_{1k_j} x_{n+1} + \alpha^j_{2k_j} x_{n+2} + \cdots + \alpha^j_{mk_j} x_{n+m},$$

where $i = \mathcal{N}_{j-1} + k_j$; $k_j = 1, \ldots, n_j$ and $j = 1, \ldots, m$. Thus, any x_i can be written as a $\mathbb{Z}_2$-linear combination of $x_{n+1}, \ldots, x_{n+m}$ for $i = 1, \ldots, n$. For simplicity, we denote the indeterminate x_{n+j} by y_j for $j = 1, \ldots, m$. Thus,

$$x_i = \sum_{\ell=1}^{m} \alpha^j_{\ell k_j} y_\ell \quad \text{where} \quad i = \mathcal{N}_{j-1} + k_j, k_j = 1, \ldots, n_j \text{ and } j = 1, \ldots, m, \tag{2.12}$$

in $H^*(M^n(P, \lambda); \mathbb{Z}_2)$. Then the generators of the ideal $\tilde{I}$ in (2.10) can be described in terms of y_j's. Therefore, we have

$$H^*(M^n(P, \lambda); \mathbb{Z}_2) \cong \mathbb{Z}_2[y_1, y_2, \ldots, y_m]/I, \text{ where} \tag{2.13}$$

$$I = \left\langle \left\{ \prod_{k_j=1}^{n_j} \left(\sum_{\ell=1}^{m} \alpha^j_{\ell k_j} y_\ell \right) y_j \mid j = 1, \ldots, m \right\} \right\rangle. \tag{2.14}$$

We have the following observation on the cohomology ring.

Lemma 2.6. *Let $M^n(P, \lambda)$ be a small cover over a finite product of simplices with the characteristic function λ as in (2.7). Then $y_j^{n_j}$ is non-zero in the cohomology ring $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ for $j \in \{1, \ldots, m\}$, where y_j's are as in (2.13).*

Proof. Let $P := \prod_{j=1}^{m} \Delta^{n_j}$ be a product of m simplices. We know the cohomology ring of a small cover over a product of simplices from (2.13). The function λ determines the following $m \times m$ vector matrix A.

$$A := \begin{pmatrix} \boldsymbol{\alpha^1_1} & \boldsymbol{\alpha^1_2} & \ldots & \boldsymbol{\alpha^1_m} \\ \boldsymbol{\alpha^2_1} & \boldsymbol{\alpha^2_2} & \ldots & \boldsymbol{\alpha^2_m} \\ \vdots & \vdots & \ldots & \vdots \\ \boldsymbol{\alpha^m_1} & \boldsymbol{\alpha^m_2} & \ldots & \boldsymbol{\alpha^m_m} \end{pmatrix}_{m \times m}.$$

Therefore, by the arguments in [5, Proposition 5.1], A is conjugate to a unipotent lower triangular vector matrix of the following form:

$$\tilde{A} := \begin{pmatrix} \mathbf{1} & \mathbf{0} & \ldots & \mathbf{0} \\ \boldsymbol{\beta^2_1} & \mathbf{1} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ldots & \vdots \\ \boldsymbol{\beta^m_1} & \boldsymbol{\beta^m_2} & \ldots & \mathbf{1} \end{pmatrix}_{m \times m}, \tag{2.15}$$

where $\boldsymbol{\beta^k_j} = (\beta^k_{j1}, \beta^k_{j2}, \ldots, \beta^k_{jn_k})^t \in \mathbb{Z}_2^{n_k}$ and $\mathbf{1} = (1, \ldots, 1)^t \in \mathbb{Z}_2^{n_k}$ for $k = 1, \ldots, m$. The matrix $\tilde{A}$ is called the Bott matrix. Thus, the ideal $\tilde{J}$ is generated

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by the coordinates of the following matrix.

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \beta_{11}^2 & 1 & \cdots & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_{1n_2}^2 & 0 & \cdots & 1 & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_{11}^m & 0 & \cdots & \beta_{21}^m & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \beta_{1n_m}^m & 0 & \cdots & \beta_{2n_m}^m & \cdots & \cdots & 1 & 1 \end{pmatrix}_{n\times(n+m)} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\mathcal{N}_1} \\ y_1 \\ x_{\mathcal{N}_1+1} \\ \vdots \\ y_2 \\ \vdots \\ \vdots \\ \vdots \\ x_{\mathcal{N}_m} \\ y_m \end{pmatrix}_{(n+m)\times 1}.$$

Let $\alpha_j := x_{\mathcal{N}_{j-1}+1}x_{\mathcal{N}_{j-1}+2}\cdots x_{\mathcal{N}_j}y_j$ for $j = 1, \ldots, m$. Here α_j's are generators of the ideal I in (2.13). From the above matrix multiplication, the first n_1 elements are $x_1 + y_1 = 0$, $x_2 + y_1 = 0$,..., $x_{\mathcal{N}_1} + y_1 = 0$. Therefore, we get $x_1 = x_2 = \cdots = x_{\mathcal{N}_1} = y_1$.

So,

$$\alpha_1 = x_1x_2\cdots x_{n_1}y_1 = y_1^{n_1+1}.$$

We have the following using (2.12).

$$\begin{aligned} \alpha_j &= x_{\mathcal{N}_{j-1}+1}x_{\mathcal{N}_{j-1}+2}\cdots x_{\mathcal{N}_j}y_j \\ &= (y_j + \beta_{11}^j y_1 + \beta_{21}^j y_2 + \cdots + \beta_{(j-1)1}^j y_{j-1})(y_j + \beta_{12}^j y_1 + \beta_{22}^j y_2 \\ &\quad + \cdots + \beta_{(j-1)2}^j y_{j-1})\cdots(y_j + \beta_{1n_j}^j y_1 + \beta_{2n_j}^j y_2 + \cdots + \beta_{(j-1)n_j}^j y_{j-1})y_j, \end{aligned}$$

for $j = 2, \ldots, m$. Now the least power of y_j in α_j is $n_j + 1$. Our claim is that $y_j^{n_j} \neq 0$. If not, let $y_j^{n_j} = 0$. Then $y_j^{n_j} \in I$. But the least power of y_j which appears as a term in a polynomial in the ideal I is $y_j^{n_j+1}$. This is a contradiction. Hence, $y_j^{n_j} \notin I$, i.e. $y_j^{n_j} \neq 0$ in $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ for $j = 1, 2, \ldots, m$. □

2.3. Real moment angle manifolds and complexes

We recall the notion of real moment angle complexes. Let r be a positive integer and K be a simplicial complex with vertex set $[r] = \{1, \ldots, r\}$. For each simplex $\sigma \in K$, we define

$$(D^1, S^0)^\sigma = \big\{(x_1, \ldots, x_r) \in (D^1)^r \mid x_i \in S^0 \text{ when } i \notin \sigma\big\}.$$

Then the set

$$\mathbb{R}\mathcal{Z}_K := \bigcup_{\sigma \in K} (D^1, S^0)^\sigma \subseteq (D^1)^r$$

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is called the real moment angle complex of K. The space $\mathbb{R}\mathcal{Z}_K$ has a natural $\mathbb{Z}_2^r$-action induced from the $\mathbb{Z}_2^r$-action on $(D^1)^r$.

Let P be a simple polytope with facets $\{F_1, \ldots, F_r\}$. Then the set

$$K_P := \big\{\sigma = \{i_1, \ldots, i_k\} \mid F_{i_1} \cap \cdots \cap F_{i_k} \neq \emptyset\big\}$$

is a simplicial complex on $\{1, \ldots, r\}$, see [3, Chapter 1]. The set K_P is called the dual of P, and $\mathbb{R}\mathcal{Z}_{K_P}$ has a manifold structure. The space $\mathbb{R}\mathcal{Z}_{K_P}$ is called the real moment angle manifold for P.

Proposition 2.7. *Let $M^n(P, \lambda)$ be a small cover over a simple polytope P. Then there is a subgroup $\mathbb{Z}_\lambda$ of $\mathbb{Z}_2^r$ of rank $r-n$ such that $\mathbb{Z}_\lambda$ acts on $\mathbb{R}\mathcal{Z}_{K_P}$ freely and $\mathbb{R}\mathcal{Z}_{K_P}/\mathbb{Z}_\lambda \cong M^n(P, \lambda)$.*

Proof. This is similar to the proof of [3, Proposition 6.5], and [24, Proposition 2.4]. □

We note that $\mathbb{R}\mathcal{Z}_{K_{\Delta^n}} = \mathbb{S}^n$ and $\mathbb{R}\mathcal{Z}_{K_{P_1 \times P_2}} = \mathbb{R}\mathcal{Z}_{K_{P_1}} \times \mathbb{R}\mathcal{Z}_{K_{P_2}}$. Let $M^n(P, \lambda)$ be a small cover over an n-dimensional polytope $\prod_{j=1}^m \Delta^{n_j}$ for $j = 1, \ldots, m$. Then, the number of facets of $\prod_{j=1}^m \Delta^{n_j}$ is $n+m$, and the real moment angle manifold $\mathbb{R}\mathcal{Z}_{K_P}$ is $\prod_{j=1}^m \mathbb{S}^{n_j}$. By proposition 2.7, $M^n(P, \lambda)$ can be realized as the orbit space of the moment angle manifold $\prod_{j=1}^m \mathbb{S}^{n_j}$ by a free $\mathbb{Z}_2^m$-action. More precisely, the action of $\mathbb{Z}_2^m$ on $\prod_{j=1}^m \mathbb{S}^{n_j}$ is given by

$$\begin{aligned}
&(g_1, g_2, \ldots, g_m)((x_0^1, \ldots, x_{n_1}^1), \ldots, (x_0^m, \ldots, x_{n_m}^m)) \\
&\quad = ((g_1 x_0^1, (g_1^{\alpha_{11}^1} \cdots g_m^{\alpha_{m1}^1}) \cdot x_1^1, \ldots, (g_1^{\alpha_{1n_1}^1} \cdots g_m^{\alpha_{mn_1}^1}) \cdot x_{n_1}^1), \ldots, \\
&\quad\quad (g_m \cdot x_0^m, (g_1^{\alpha_{11}^m} \cdots g_m^{\alpha_{m1}^m}) \cdot x_1^m, \ldots, (g_1^{\alpha_{1n_m}^m} \cdots g_m^{\alpha_{mn_m}^m}) \cdot x_{n_m}^m))
\end{aligned} \tag{2.16}$$

where $(g_1, g_2, \ldots, g_m) \in \mathbb{Z}_2^m$ and $(x_0^j, \ldots, x_{n_j}^j) \in \mathbb{S}^{n_j}$ for $j = 1, \ldots, m$, see [10, Remark 2.3]. This $\mathbb{Z}_2^m$-action on $\prod_{j=1}^m \mathbb{S}^{n_j}$ is free and one has $\prod_{j=1}^m \mathbb{S}^{n_j}/\mathbb{Z}_2^m \cong M^n(P, \lambda)$.

3. Equivariant LS-category of small covers

In this section, we recall some basics of LS-category following [6]. Then, we compute the LS-category and the equivariant LS-category of a small cover over a simple polytope. Next, we compute the LS-category of the real moment angle manifold for r-gon and the equivariant LS-category of the real moment angle complex.

Let G be a compact topological group acting continuously on a Hausdorff topological space Y. In this case, Y is called a G-space. A subset U of a G-space Y is called G-invariant if $GU \subseteq U$. The homotopy $H\colon U \times I \to Y$ is called G-homotopy if for any $g \in G$, $y \in U$ and $t \in I$, we have $gH(y, t) = H(gy, t)$. A G-invariant open subset U of Y is called G-categorical if there exists an equivariant homotopy $H\colon U \times I \to Y$ such that H_0 is the inclusion, and $H_1\colon U \to Y$ has the image in a single G-orbit. In particular, U is called categorical if G is trivial. Here we denote the orbit of an element $y \in Y$ by $\mathcal{O}(y)$.

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Definition 3.1. *The equivariant LS-category of a G-space Y, denoted by $cat_G(Y)$, is the least positive number of G-categorical invariant open sets required to cover Y. If no such covering exists, then $cat_G(Y) = \infty$.*

In particular, if G is trivial, then $cat_G(Y)$ is called the LS-category of Y, denoted by $cat(Y)$.

Let Y be a space and R be a commutative ring. The least integer n such that all $(n+1)$-fold cup products vanish in $H^*(Y; R)$ is called the cup-length of Y with coefficients in R, denoted by $\mathrm{cl}_R(Y)$. If no such n exists, we write $\mathrm{cl}_R(Y) = \infty$. The cup-length gives a lower bound for LS-category, as follows:

Proposition 3.2. *The cup-length of a topological space Y is less than the LS-category of Y, i.e. $cl_R(Y) + 1 \leqslant cat(Y)$, see [**6**, Proposition 1.5].*

Proposition 3.3. *If Y is a manifold, then $cat(Y) \leqslant dim(Y) + 1$, see [**6**, Theorem 1.7].*

Theorem 3.4. *Let $M^n(P, \lambda)$ be an n-dimensional small cover over a simple polytope P. Then $cat(M^n(P, \lambda)) = n + 1$.*

Proof. Since P is a simple polytope, at each vertex, exactly n many facets intersect. Let v be a vertex of P, and $v = F_{s_1} \cap \cdots \cap F_{s_n}$ where $F_{s_1}, \ldots, F_{s_n}$ are unique n facets of P. Let $m_v = \rho^{-1}(v)$ and $M_i = \rho^{-1}(F_{s_i})$ for $i = 1, \ldots, n$. Here the $\mathbb{Z}_2^n$-action on $M^n(P, \lambda)$ is locally standard. So, m_v is a fixed point, and $M_1, \ldots, M_n$ intersect to m_v transversely. Therefore, the Poincare dual of M_i represents a non-zero cohomology class in $H^1(M^n(P, \lambda); \mathbb{Z}_2)$. So by definition of cup-length, $n \leqslant \mathrm{cl}_{\mathbb{Z}_2}(M^n(P, \lambda))$. Therefore, by proposition 3.2, $n + 1 \leqslant \mathrm{cat}(M^n(P, \lambda))$. Also by proposition 3.3, we have $\mathrm{cat}(M^n(P, \lambda)) \leqslant \dim(M^n(P, \lambda)) + 1 = n + 1$. Hence, $\mathrm{cat}(M^n(P, \lambda)) = n + 1$. □

We remark that the LS-category of small covers has been studied in [**22**]. However, it is written in Chinese. So, we write a proof.

We recall a result from [**1**], which helps us to calculate the equivariant LS-category of a small cover over a simple polytope.

Proposition 3.5 [**1**, Theorem 3.3]. *Let Y be a G space and $\{[\mathcal{O}(y_i)]\}_{i \in \mathcal{A}}$ be the collection of all minimal orbit classes in Y. Let*

$$Y_i = \bigcup_{\mathcal{O}(y) \in [\mathcal{O}(y_i)]} \mathcal{O}(y).$$

Then

$$\#\mathcal{A} \leqslant \sum_{i \in \mathcal{A}} cat_G(Y_i) \leqslant cat_G(Y)$$

where $\#\mathcal{A}$ is the cardinality of $\mathcal{A}$.

Theorem 3.6. *Let $M^n(P, \lambda)$ be an n-dimensional small cover over a simple polytope P with k vertices. Then $cat_{\mathbb{Z}_2^n}(M^n(P, \lambda)) = k$.*

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Proof. Let $M := M^n(P, \lambda)$. We know that there is a bijection between the fixed point set $M^{\mathbb{Z}_2^n}$ and $V(P)$. Since the fixed points are isolated and minimal orbits, by proposition 3.5, we have $\mathrm{cat}_{\mathbb{Z}_2^n}(M) \geqslant |V(P)|$. So, it is enough to show that for any $v \in M^{\mathbb{Z}_2^n}$, there is a $\mathbb{Z}_2^n$-categorical subset X_v of M such that $M = \bigcup_{v \in M^{\mathbb{Z}_2^n}} X_v$. Let $\rho\colon M \to P$ be the orbit map. Now for $v \in M^{\mathbb{Z}_2^n}$, let

$$C_v = \bigcup_{\rho(v) \notin F} F, \; U_v = P - C_v, \; \text{and } X_v = \rho^{-1}(U_v),$$

where F is a face of P. Here X_v is $\mathbb{Z}_2^n$-invariant subset of M. Since U_v is a convex subset of P, it is contractible to v. So there exists a homotopy $h\colon U_v \times I \to P$ such that $h(x, 0) = x$ and $h(x, 1) = v$ for all $x \in U_v$ and preserves the face structure of $U_v \times I$. So, for any face F of U_v, we have $h(x, t) \in F$ for $x \in F$, $\; t \in I$. Thus, by proposition 1.8 of [8], we can say $X_v \cong (U_v \times \mathbb{Z}_2^n)/\sim$. Therefore, h induces a homotopy

$$h \times \mathrm{Id}\colon U_v \times I \times \mathbb{Z}_2^n \to P \times \mathbb{Z}_2^n$$

defined by $(x, (r', t)) \mapsto (h(x, r'), t)$. Since for each face F of U_v, we have

$$x \in F \Rightarrow h(x, r') \in F, \text{ for all } r' \in I,$$

$h \times \mathrm{Id}$ induces a homotopy $H\colon X_v \times I \to M$ with $([x, t], r') \mapsto [h(x, r'), t]$. Since

$$gH([x,t], r') = g[h(x, r'), t] = [h(x, r'), gt] = H([x, gt], r') = H(g[x,t], r'),$$

the map H is a $\mathbb{Z}_2^n$-homotopy. Also $H(x, 0) = x$, $H(x, 1) = \rho^{-1}(v) = v$, for all $x \in X_v$. Thus, X_v is $\mathbb{Z}_2^n$-categorical open invariant subset of M. Since $\{X_v \mid v \in V(P)\}$ covers M, $\mathrm{cat}_{\mathbb{Z}_2^n}(M) = |V(P)| = k$. □

Proposition 3.7. *Let P be an r-gon and $\mathbb{R}\mathcal{Z}_{K_P}$ be a moment angle manifold. Then $cat(\mathbb{R}\mathcal{Z}_{K_P}) = 3$.*

Proof. We know the cohomology ring $H^*(\mathbb{R}\mathcal{Z}_{K_P}; \mathbb{Z}_2)$ is generated by elements of degree only 0, 1 and 2. We can get two elements of degree 1 such that their cup product is non-zero in $H^*(\mathbb{R}\mathcal{Z}_{K_P}; \mathbb{Z}_2)$, see [4, Section 3]. Therefore, $2 \leqslant \mathrm{cl}_{\mathbb{Z}_2}(\mathbb{R}\mathcal{Z}_{K_P})$. Then, we have $3 \leqslant \mathrm{cat}(\mathbb{R}\mathcal{Z}_{K_P})$. Also $\dim(\mathbb{R}\mathcal{Z}_{K_P}) = \dim(P) = 2$. Therefore, $\mathrm{cat}(\mathbb{R}\mathcal{Z}_{K_P}) \leqslant 3$. Hence, $\mathrm{cat}(\mathbb{R}\mathcal{Z}_{K_P}) = 3$. □

Remark 3.8. Let K be a triangulated d-sphere for $d \leqslant 2$ or a connected sum of joins of such spheres. If K is k-Golod over $\mathbb{Z}_2$ (i.e. length $k+1$ cup products of positive degree elements in $H^*(\mathbb{R}\mathcal{Z}_K; \mathbb{Z}_2)$ vanish), then $k \leqslant \mathrm{cl}_{\mathbb{Z}_2}(\mathbb{R}\mathcal{Z}_K)$. Thus, $k + 1 \leqslant \mathrm{cat}(\mathbb{R}\mathcal{Z}_K)$, see [2, Theorem 4.2].

Theorem 3.9. *Let S be the set of all maximal simplices of a simplicial complex K on $[r]$. Then*

$$cat_{\mathbb{Z}_2^r}(\mathbb{R}\mathcal{Z}_K) = |S|.$$

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Proof. Note that if τ is a face of σ in K, then $(D^1, S^0)^\tau \subseteq (D^1, S^0)^\sigma$. So we have

$$\mathbb{R}\mathcal{Z}_K = \bigcup_{\sigma \in S} (D^1, S^0)^\sigma \subseteq (D^1)^r.$$

The topology on $\mathbb{R}\mathcal{Z}_K$ is the subspace topology of $(D^1)^r$. Also, any simplex of K is a face of a maximal simplex. So the set

$$\{(D^1, S^0)^\sigma \mid \sigma \in S\}$$

is an open covering for $\mathbb{R}\mathcal{Z}_K$. Moreover, $(D^1, S^0)^\sigma$ is a $\mathbb{Z}_2^r$-invariant subset which is equivariantly contractible to the orbit $(S^0)^\sigma$ in $\mathbb{R}\mathcal{Z}_K$ where

$$(S^0)^\sigma = \{(x_1, \ldots, x_r) \in \mathbb{R}\mathcal{Z}_K \mid x_i = 0 \text{ if } i \in \sigma \text{ and } |x_i| = 1 \text{ if } i \notin \sigma\}.$$

So we obtain that

$$\mathrm{cat}_{\mathbb{Z}_2^r}(\mathbb{R}\mathcal{Z}_K) \leqslant |S|.$$

Note that the set $\{(S^0)^\sigma \mid \sigma \in S\}$ is the set of all minimal orbits of $\mathbb{R}\mathcal{Z}_K$ with respect to $\mathbb{Z}_2^r$-action. So, by proposition 3.5, we have

$$\mathrm{cat}_{\mathbb{Z}_2^r}(\mathbb{R}\mathcal{Z}_K) \geqslant |S|.$$

□

4. Topological complexity of small covers

In this section, we recall the definition of topological complexity and zero-divisors-cup-length. Next, we try to give bounds for the topological complexity of a class of small covers over a product of simplices and real Bott manifolds.

Definition 4.1. *Let Y be a path-connected space. The topological complexity of the motion planning in Y is the least integer k such that $Y \times Y$ can be covered by k open subsets $U_1, \ldots, U_k$ on each of which there exists a section $s_i : U_i \to PY$ such that $\pi \circ s_i$ is homotopic to the inclusion id_{U_i}. If no such integer exists, then we set $\mathbf{TC}(Y) = \infty$.*

We note that in the above definition, we consider non-normalized topological complexity. The cup product map

$$\cup \colon H^*(Y; R) \otimes H^*(Y; R) \to H^*(Y; R) \tag{4.1}$$

is an algebra homomorphism whose kernel is called the ideal of zero-divisors of $H^*(Y; R)$. The multiplicative structure on the left in (4.1) is given by the formula $(\alpha \otimes \beta) \cdot (\gamma \otimes \delta) = (-1)^{|\beta| \cdot |\gamma|} \alpha\gamma \otimes \beta\delta$. Here $|\beta|$ and $|\gamma|$ denote the degrees of the cohomology classes β and γ, respectively.

Definition 4.2. *The zero-divisors-cup-length of $H^*(Y; R)$, denoted by $zcl_R(Y)$, is the length of the longest non-trivial product in the ideal of the zero-divisors of $H^*(Y; R)$.*

The following proposition gives a lower bound and an upper bound for $\mathbf{TC}(Y)$.

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PROPOSITION 4.3 [**11**, Theorem 4,5,7]. *If Y is a manifold, then we have*

$$max\{cat(Y), zcl_R(Y)+1\} \leqslant \mathbf{TC}(Y) \leqslant 2dim(Y)+1.$$

PROPOSITION 4.4 [**11**, Theorem 11]. *For any path-connected metric spaces $Y_1, \ldots, Y_m$, we have*

$$\mathbf{TC}(Y_1 \times \cdots \times Y_m) \leqslant \mathbf{TC}(Y_1) + \cdots + \mathbf{TC}(Y_m) - (m-1).$$

PROPOSITION 4.5. *Let $M^n(P, \lambda)$ be an n-dimensional small cover over a simple polytope P. Then*

$$n+1 \leqslant \mathbf{TC}(M^n(P, \lambda)) \leqslant 2n+1.$$

Proof. The proof follows from theorem 3.4 and proposition 4.3. □

Now we calculate the topological complexity of several small covers over a product of two simplices. Consider the set $\mathcal{S} := \{n \in \mathbb{N} \mid \binom{n}{i} \text{ is even for } 0 < i < n\}$. Let $n \leqslant 2^r - 1 < 2n$, $n_j \leqslant 2^{r_j} - 1 < 2n_j$ for $j = 1, 2$, and $n = n_1 + n_2$. Note that $\binom{2^s}{i}$ is even for $0 < i < 2^s$.

THEOREM 4.6. *Let $M^n(P, \lambda)$ be a small cover other than $\mathbb{RP}^{n_1} \times \mathbb{RP}^{n_2}$ over $P = \Delta^{n_1} \times \Delta^{n_2}$.*

(1) *Let $n_2 \in \mathcal{S}$ with $n_2 > n_1$. Then $2^{r_1} + 2^{r_2} - 1 \leqslant \mathbf{TC}(M^n(P, \lambda))$.*

(2) *Let $n_2 \in \mathcal{S}$ with n_2 divides n_1. Then $2^r \leqslant \mathbf{TC}(M^n(P, \lambda))$.*

(3) *Let $n_2 \in \mathcal{S} + 1$ with $n_2 > n_1 + 1$. Then, $2^r \leqslant \mathbf{TC}(M^n(P, \lambda))$.*

(4) *Let $n_2 \in \mathcal{S} + 2$ with $n_2 > n_1 + 2$. Then, $2^r \leqslant \mathbf{TC}(M^n(P, \lambda))$.*

In particular, if $n = 2^{s-1}$, then for the cases (2), (3) and (4), we have

$$2n \leqslant \mathbf{TC}(M^n(P, \lambda)) \leqslant 2n+1.$$

Proof. In the cohomology ring $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ described in proposition 2.2, the ideal $\tilde{I}$ is generated by $\alpha_1 = x_1 x_2 \cdots x_{n_1} y_1$ and $\alpha_2 = x_{n_1+1} x_{n_1+2} \cdots x_{n_1+n_2} y_2$. The ideal $\tilde{J}$ is generated by

$$x_1 = x_2 = \cdots = x_{n_1} = y_1, \text{ and } x_{n_1+1} = x_{n_1+2} = \cdots = x_{n_1+n_2} = y_1 + y_2.$$

Therefore, $\alpha_1 = y_1^{n_1+1} = 0$ and $y_1^{n_1} \neq 0$ in $H^*(M^n(P, \lambda); \mathbb{Z}_2)$. From (2.6), and the Poincare duality, we have $x_1 \cdots x_{n_1} x_{n_1+1} \cdots x_{n_1+n_2} \neq 0$. So, $y_1^{n_1}(y_1 + y_2)^{n_2} \neq 0$.

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Now,

$$\begin{aligned}
y_1^{n_1}(y_1+y_2)^{n_2} &= y_1^{n_1}\{y_2^{n_2}+y_1\cdot f(y_1,y_2)\}\\
&\qquad\text{(where } f(y_1,y_2) \text{ is a function of } y_1 \text{ and } y_2)\\
&= y_1^{n_1}y_2^{n_2}+y_1^{n_1+1}\cdot f(y_1,y_2)\\
&= y_1^{n_1}y_2^{n_2} \ (\text{as } y_1^{n_1+1}=0).
\end{aligned}$$

Therefore, we get the following:

$$y_1^{n_1}y_2^{n_2}\neq 0. \tag{4.2}$$

Let $\mathfrak{a}_j := 1\otimes y_j - y_j\otimes 1$ for $j=1,\,2$. Then $\mathfrak{a}_j$ is in the ideal of the zero-divisors of $H^*(M^n(P,\,\lambda);\mathbb{Z}_2)$. Let $c=2^r-1$, and $c_j=2^{r_j}-1$ for $j=1,\,2$.

(1) Let $n_2\in\mathcal{S}$ with $n_2>n_1$. Here,

$$\begin{aligned}
\alpha_2 &= (y_1+y_2)^{n_2}y_2\\
&= (y_1^{n_2}+y_2^{n_2})y_2 \ \left(\text{as } \binom{n_2}{i} \text{ is even for } 0<i<n\right)\\
&= y_2^{n_2+1} \ (\text{ as } y_1^{n_2}=0,\ \text{since } n_2\geqslant n_1+1 \text{ and } y_1^{n_1+1}=0).
\end{aligned}$$

Therefore, $y_2^{n_2+1}=0$. Now, for $j=1,\,2$,

$$\mathfrak{a}_j^{c_j} = (1\otimes y_j - y_j\otimes 1)^{c_j} = \sum_{k_j=0}^{c_j}(-1)^{c_j-k_j}\binom{c_j}{k_j}(y_j^{c_j-k_j}\otimes y_j^{k_j}).$$

Now, the binomial coefficient $\binom{2^{r_j}-1}{i_j}$ is odd for all $0\leqslant i_j\leqslant c_j$ for $j=1,\,2$. The binomial expansion of $\mathfrak{a}_j^{c_j}$ contains the term $(y_j^{c_j-n_j}\otimes y_j^{n_j})$ which is non-zero. Now, by (4.2), $y_1^{n_1}y_2^{n_2}\neq 0$. So, $\mathfrak{a}_1^{c_1}\mathfrak{a}_2^{c_2}$ contains the term $y_1^{n_1}y_2^{n_2}\otimes(y_1^{c_1-n_1}y_2^{c_2-n_2})$ which is non-zero and there is no other term of this form in the expression of $\mathfrak{a}_1^{c_1}\mathfrak{a}_2^{c_2}$.
Hence, zero-divisors-cup-length of $H^*(M^n(P,\,\lambda);\mathbb{Z}_2)$ is greater than or equal to c_1+c_2. Therefore, by proposition 4.3, we have, $2^{r_1}+2^{r_2}-1\leqslant \mathbf{TC}(M^n(P,\,\lambda))$.

(2) Now, consider the case when $n_2\in\mathcal{S}$ with n_2 divides n_1. Let $n_1=\bar{n}n_2$ for some $\bar{n}\in\mathbb{Z}$. Since $\binom{n_2}{i}$ is even for $0<i<n_2$, so,

$$\alpha_2=(y_1+y_2)^{n_2}y_2=(y_1^{n_2}+y_2^{n_2})y_2=y_1^{n_2}y_2+y_2^{n_2+1}.$$

So, $y_2^{n_2+1}=y_1^{n_2}y_2$ in $H^*(M^n(P,\,\lambda);\mathbb{Z}_2)$. Thus,

$$y_1^{n_1}y_2^{n_2}=y_1^{\bar{n}n_2}y_2^{n_2}=y_1^{\bar{n}n_2}y_2^{\bar{n}}y_2^{n_2-\bar{n}}=y_2^{\bar{n}n_2+\bar{n}}y_2^{n_2-\bar{n}}=y_2^{\bar{n}n_2+n_2}=y_2^{n_1+n_2}=y_2^n.$$

Now, by (4.2), $y_1^{n_1}y_2^{n_2}\neq 0$. Thus, $y_2^n\neq 0$. Now,

$$\mathfrak{a}_2^{c}=(1\otimes y_2-y_2\otimes 1)^c=\sum_{k=0}^{c}(-1)^{c-k}\binom{c}{k}(y_2^{c-k}\otimes y_2^{k}).$$

Therefore, by similar arguments as in (1), we get $2^r\leqslant \mathbf{TC}(M^n(P,\,\lambda))$.

(3) Let $n_2 \in \mathcal{S}+1$ with $n_2 > n_1+1$. Now,

$$\begin{aligned}
\alpha_2 &= (y_1+y_2)^{n_2} y_2 \\
&= (y_1+y_2)^{n_2-1}(y_1+y_2)y_2 \\
&= (y_1^{n_2-1}+y_2^{n_2-1})(y_1+y_2)y_2 \ \left(\text{as } \binom{n_2-1}{i} \text{ is even for } 0<i<n_2-1\right) \\
&= (y_1+y_2)y_2^{n_2} \ \left(\text{ as } y_1^{n_2-1}=0, \ \text{since } n_2-1 \geqslant n_1+1 \text{ and } y_1^{n_1+1}=0\right) \\
&= y_1 y_2^{n_2} + y_2^{n_2+1}.
\end{aligned}$$

So, $y_2^{n_2+1} = y_1 y_2^{n_2}$ in $H^*(M^n(P,\lambda);\mathbb{Z}_2)$. Thus,

$$y_1^{n_1} y_2^{n_2} = y_1^{n_1-1}(y_1 y_2^{n_2}) = y_1^{n_1-1} y_2^{n_2+1} = \cdots = y_2^{n_1+n_2} = y_2^n.$$

Therefore, by similar arguments as in (2), we get $2^r \leqslant \mathbf{TC}(M^n(P,\lambda))$.

(4) Let $n_2 \in \mathcal{S}+2$ with $n_2 > n_1+2$. Now,

$$\begin{aligned}
\alpha_2 &= (y_1+y_2)^{n_2} y_2 \\
&= (y_1+y_2)^{n_2-2}(y_1+y_2)^2 y_2 \\
&= (y_1^{n_2-2}+y_2^{n_2-2})(y_1^2+y_2^2)y_2 \ \left(\text{as } \binom{n_2-2}{i} \text{ is even for } 0<i<n_2-2\right) \\
&= (y_1^2+y_2^2)y_2^{n_2-1} \ \left(\text{ as } y_1^{n_2-2}=0, \ \text{since } n_2-2 \geqslant n_1+1 \text{ and } y_1^{n_1+1}=0\right) \\
&= y_1^2 y_2^{n_2-1} + y_2^{n_2+1}.
\end{aligned}$$

So, $y_2^{n_2+1} = y_1^2 y_2^{n_2-1}$ in $H^*(M^n(P,\lambda);\mathbb{Z}_2)$. Let n_1 be even. Then,

$$y_1^{n_1} y_2^{n_2} = y_1^{n_1-2}(y_1^2 y_2^{n_2-1})y_2 = y_1^{n_1-2} y_2^{n_2+1} y_2 = \cdots = y_2^{n_1+n_2} = y_2^n.$$

Therefore, by similar arguments as in (2), we get $2^r \leqslant \mathbf{TC}(M^n(P,\lambda))$. Now, let n_1 be odd. Then,

$$y_1^{n_1} y_2^{n_2} = y_1^{n_1-2}(y_1^2 y_2^{n_2-1})y_2 = y_1^{n_1-2} y_2^{n_2+1} y_2 = \cdots = y_1 y_2^{n_1+n_2-1} = y_1 y_2^{n-1}.$$

Now, by (4.2), $y_1^{n_1} y_2^{n_2} \neq 0$. Therefore, $y_1 y_2^{n-1} \neq 0$. So, $y_2^{n-1} \neq 0$. We know $n \leqslant 2^r - 1 < 2n$, i.e. $2^r - 1 \leqslant 2n-1$. If $2^r - 1 = 2n-1$, then n is even. Since n_1 is odd, so n_2 is odd, which is not true. Therefore, $2^r - 1 \leqslant 2n-2$. Now,

$$\mathfrak{a}_2^c = (1 \otimes y_2 - y_2 \otimes 1)^c = \sum_{k=0}^{c} (-1)^{c-k} \binom{c}{k} (y_2^{c-k} \otimes y_2^k).$$

Here, the binomial expansion of $\mathfrak{a}_2^c$ contains the term $(y_2^{c-n+1} \otimes y_2^{n-1})$ which is non-zero, and there is no other same term in the expression of $\mathfrak{a}_2^c$. So $\mathfrak{a}_2^c$ is non-zero. Therefore, by similar arguments as in (2), we get $2^r \leqslant \mathbf{TC}(M^n(P,\lambda))$.

In particular, if $n = 2^{s-1}$ then $n \leqslant 2^s - 1 < 2n$. Therefore, by (2), (3) and (4) we have $2^s \leqslant \mathbf{TC}(M^n(P,\lambda))$. Thus, $2n \leqslant \mathbf{TC}(M^n(P,\lambda)) \leqslant 2n+1$ for any $s \geqslant 1$. □

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Proposition 4.7 [**16**, Corollary 8.1, 8.2]. *If n equals 1, 3 or 7, then* $\mathbf{TC}(\mathbb{RP}^n) = n+1$*, and if n is a power of 2, then* $\mathbf{TC}(\mathbb{RP}^n) = 2n$*.*

Proposition 4.8. *Let the small cover $M^n(P, \lambda)$ over the polytope $P = \prod_{j=1}^{m} \Delta^{n_j}$ be of the form $\mathbb{RP}^{n_1} \times \cdots \times \mathbb{RP}^{n_m}$.*

(1) *If $n_j = 2^{s_j - 1}$ for all $j \in \{1, 2, \ldots, m\}$, then $\mathbf{TC}(M^n(P, \lambda)) = 2^{s_1} + \cdots + 2^{s_m} - (m-1)$.*

(2) *If $n_j = 1, 3$ or 7 for all $j \in \{1, 2, \ldots, m\}$, then $\mathbf{TC}(M^n(P, \lambda)) = n+1$.*

Proof. Here $M^n(P, \lambda) = \mathbb{RP}^{n_1} \times \cdots \times \mathbb{RP}^{n_m}$. In the cohomology ring $H^*(M^n(P, \lambda); \mathbb{Z}_2)$, the ideal $\tilde{I}$ is generated by $\alpha_j = x_{\mathcal{N}_{j-1}+1} x_{\mathcal{N}_{j-1}+2} \cdots x_{\mathcal{N}_j} y_j$ for $j = 1, \ldots, m$, and the ideal $\tilde{J}$ is generated by

$$x_{\mathcal{N}_{j-1}+1} = x_{\mathcal{N}_{j-1}+2} = \cdots = x_{\mathcal{N}_j} = y_j. \tag{4.3}$$

Therefore, $\alpha_j = y_j^{n_j+1} = 0$ in $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ for $j = 1, \ldots, m$. Let

$$\mathfrak{a}_j := 1 \otimes y_j - y_j \otimes 1$$

for $j = 1, \ldots, m$. Then $\mathfrak{a}_j$ belongs to the ideal of the zero-divisors of $H^*(M^n(P, \lambda); \mathbb{Z}_2)$. Let $c_j = 2^{s_j} - 1$ for $j = 1, \ldots, m$. Now, by lemma 2.6, $y_j^{n_j} \neq 0$. From (2.6), and the Poincare duality, we have

$$x_1 \cdots x_{\mathcal{N}_1} x_{\mathcal{N}_1+1} \cdots x_{\mathcal{N}_2} x_{\mathcal{N}_2+1} \cdots x_{\mathcal{N}_{m-1}+1} \cdots x_{\mathcal{N}_m} \neq 0.$$

Using (4.3), we have, $y_1^{n_1} \cdots y_m^{n_m} \neq 0$. Therefore, by similar arguments as in the proof of theorem 4.6 (1), we have $\mathfrak{a}_1^{c_1} \mathfrak{a}_2^{c_2} \cdots \mathfrak{a}_m^{c_m} \neq 0$. Thus, $c_1 + \cdots + c_m + 1 \leqslant \mathbf{TC}(M^n(P, \lambda))$.
That is

$$2^{s_1} + \cdots + 2^{s_m} - (m-1) \leqslant \mathbf{TC}(M^n(P, \lambda)).$$

Also, from proposition 4.4, we have

$$\mathbf{TC}(M^n(P, \lambda)) \leqslant \mathbf{TC}(\mathbb{RP}^{n_1}) + \cdots + \mathbf{TC}(\mathbb{RP}^{n_m}) - (m-1). \tag{4.4}$$

(1) If $n_j = 2^{s_j - 1}$ then $n_j \leqslant 2^{s_j} - 1 < 2n_j$. Now by proposition 4.7, we have $\mathbf{TC}(\mathbb{RP}^{n_j}) = 2^{s_j}$. So, the right inequality can be obtained using (4.4). Hence,

$$\mathbf{TC}(M^n(P, \lambda)) = 2^{s_1} + \cdots + 2^{s_m} - (m-1).$$

(2) If $n_j = 1, 3$ or 7, then there exists some s_j which satisfies $n_j \leqslant 2^{s_j} - 1 < 2n_j$ and $n_j + 1 = 2^{s_j}$. So, $2^{s_1} + \cdots + 2^{s_m} = n_1 + \cdots + n_m + m = n + m$. Thus, we have $n + m - (m-1) = n + 1 \leqslant \mathbf{TC}(M^n(P, \lambda))$. By proposition 4.7, we have $\mathbf{TC}(\mathbb{RP}^{n_j}) = n_j + 1$. So, the right inequality can be obtained using (4.4). Hence,

$$\mathbf{TC}(M^n(P, \lambda)) = n + 1.$$

□

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We remark that if $M^n(P, \lambda)$ is not $\mathbb{RP}^{n_1} \times \cdots \times \mathbb{RP}^{n_m}$ then computation of $\mathbf{TC}(M^n(P, \lambda))$ is a challenging problem.

We recall that the n-th stage real Bott manifold is a small cover $M^n(P, \lambda)$ over the polytope $P = (\Delta^1)^n$ (an n-dimensional cube) and λ be as in (2.7). In this case, the elements of the $(n \times n)$ matrix coming from (2.15) are scalars. Note that the diagonal elements of this matrix are 1 that follows from the definition of λ. Since the Bott matrix is unique up to conjugation, different $\beta_l^{m\prime}s$ give different real Bott manifolds up to equivariant diffeomorphism. Now we calculate some lower bounds (possibly tight) of the topological complexity of the real Bott manifolds.

THEOREM 4.9. *For $n \geqslant 3$, let the elements β_k^{k+1} in the Bott matrix (2.15) be 1 for $k = 1, \ldots, n-1$, and the remaining elements β_l^m be zero for $l = 1, \ldots, n-2$, and $m = 3, \ldots, n$. If $n \leqslant 2^r - 1 < 2n$, then the topological complexity of the real Bott manifold $M^n(P, \lambda)$ is greater than or equal to 2^r. In particular, if $n = 2^{s-1}$, then $2n \leqslant \mathbf{TC}(M^n(P, \lambda)) \leqslant 2n+1$.*

Proof. In the cohomology ring $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ as in proposition 2.2, the generators of the ideal $\tilde{I}$ are $\alpha_j = x_j y_j$ for $j = 1, \ldots, n$, and the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$ and $x_j + y_j + y_{j-1}$ for $j = 2, \ldots, n$. Now for $j \in \{2, \ldots, n\}$,

$$\alpha_j = x_j y_j = (y_{j-1} + y_j) y_j = y_{j-1} y_j + y_j^2.$$

Our claim is that $y_j^2 \neq 0$ for $j = 2, \ldots, n$. For this, it is enough to show that $y_{j-1} y_j \neq 0$ for $j = 2, \ldots, n$ as $\alpha_j = 0$ in $H^*(M^n(P, \lambda); \mathbb{Z}_2)$. Note that in this case, $P = \prod_1^n \Delta^1$, an n-cube. So, the facets corresponding to the indeterminates x_j and y_j don't intersect. But the facets corresponding to the indeterminates y_{j-1} and y_j intersect to an $(n-2)$-dimensional face. So $y_{j-1} y_j$ is non-zero in $H^*(M^n(P, \lambda); \mathbb{Z}_2)$. Therefore, $y_j^2 \neq 0$ for $j = 2, \ldots, n$.

Since P is an n-dimensional simple polytope, there is a vertex where the facets corresponding to the indeterminates $y_1, y_2, \ldots, y_n$ intersect. In other words, $y_1 y_2 \cdots y_n \neq 0$, by Poincare duality. From the relation $y_{j-1} y_j = y_j^2$ for $j = 2, \ldots, n$, we have $y_1 y_2 \cdots y_n = y_n^n$. Therefore, $y_n^n \neq 0$.

Let $\mathfrak{a}_n := 1 \otimes y_n - y_n \otimes 1$. Then $\mathfrak{a}_n$ is in the ideal of the zero-divisors of $H^*(M^n(P, \lambda); \mathbb{Z}_2)$. Then,

$$\begin{aligned}
\mathfrak{a}_n^{2^r-1} &= (1 \otimes y_n - y_n \otimes 1)^{2^r-1} \\
&= \sum_{k=0}^{2^r-1} (-1)^{2^r-1-k} \binom{2^r-1}{k} (1 \otimes y_n)^k (y_n \otimes 1)^{2^r-1-k} \\
&= \sum_{k=0}^{2^r-1} (-1)^{2^r-1-k} \binom{2^r-1}{k} (y_n^{2^r-1-k} \otimes y_n^k).
\end{aligned}$$

The binomial coefficients $\binom{2^r-1}{i}$ are odd for all $0 \leqslant i \leqslant 2^r - 1$. The binomial expansion of $\mathfrak{a}_n^{2^r-1}$ contains the term $(y_n^{2^r-1-n} \otimes y_n^n)$ which is non-zero and there is no other term of this form in the expression of $\mathfrak{a}_n^{2^r-1}$. So $\mathfrak{a}_n^{2^r-1}$ is non-zero. Therefore, zero-divisors-cup-length of $H^*(M^n(P, \lambda); \mathbb{Z}_2)$ is greater than or equal to $2^r - 1$. Hence, by proposition 4.3, we have $2^r \leqslant \mathbf{TC}(M^n(P, \lambda))$.

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If $n = 2^{s-1}$ then r satisfies $n \leqslant 2^s - 1 < 2n$. Thus, $2^s = 2n \leqslant \mathbf{TC}(M^n(P, \lambda)) \leqslant 2n + 1$. □

We recall that for $n = 3$, the Bott matrix is given by $\begin{pmatrix} 1 & 0 & 0 \\ \beta_1^2 & 1 & 0 \\ \beta_1^3 & \beta_2^3 & 1 \end{pmatrix}$. We denote the corresponding real Bott manifold $M^3(P, \lambda)$ by $M^3(\beta_1^2, \beta_1^3, \beta_2^3)$.

Theorem 4.10. *$5 \leqslant \mathbf{TC}(M^3(1, 0, 0)), \mathbf{TC}(M^3(0, 1, 0)), \mathbf{TC}(M^3(0, 0, 1)), \mathbf{TC}(M^3(0, 1, 1)) \leqslant 7$.*

Proof. The generators of the ideal $\tilde{I}$ in proposition 2.2 are $\alpha_j = x_j y_j$ where $j = 1, 2, 3$. Let

$$\mathfrak{a}_j := 1 \otimes y_j - y_j \otimes 1$$

for $j = 1, 2, 3$. Then $\mathfrak{a}_j$ is in the ideal of the zero-divisors of $H^*(M^3(\beta_1^2, \beta_1^3, \beta_2^3); \mathbb{Z}_2)$. Note that $\mathbf{TC}(M^3(\beta_1^2, \beta_1^3, \beta_2^3)) \leqslant 7$ by proposition 4.5. The manifolds $M^3(1, 0, 0)$, $M^3(0, 1, 0)$, $M^3(0, 0, 1)$, and $M^3(0, 1, 1)$ are diffeomorphic to each other by [**23**, Theorem 4].

Consider the real Bott manifold $M^3(1, 0, 0)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$, and $x_3 + y_3$. So, $x_1 = y_1$, $x_2 = y_1 + y_2$ and $x_3 = y_3$ in $H^*(M^3(1, 0, 0); \mathbb{Z}_2)$. Therefore, we have $y_1^2 = y_3^2 = 0$, and $y_2^2 = y_1 y_2$. Now,

$$\begin{aligned} \mathfrak{a}_2^3 \mathfrak{a}_3 &= (1 \otimes y_2^3 - y_2 \otimes y_2^2 + y_2^2 \otimes y_2 - y_2^3 \otimes 1)(1 \otimes y_3 - y_3 \otimes 1) \\ &= y_1 y_2 \otimes y_2 y_3 + y_2 y_3 \otimes y_1 y_2 - y_1 y_2 y_3 \otimes y_2 - y_2 \otimes y_1 y_2 y_3. \end{aligned}$$

So, the product $\mathfrak{a}_2^3 \mathfrak{a}_3$ contains an element $y_1 y_2 \otimes y_2 y_3$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(\mathbf{TC}(M^3(1, 0, 0)); \mathbb{Z}_2)$ is greater than or equal to 4. Hence, by proposition 4.3, we have $5 \leqslant \mathbf{TC}(M^3(1, 0, 0))$. □

We remark that $M^3(1, 1, 0)$ is the 3-dimensional Klein Bottle, and [**7**, Theorem 3.1] gives $\mathbf{TC}(M^3(1, 1, 0)) = 6$.

Theorem 4.11. *$6 \leqslant \mathbf{TC}(M^3(1, 0, 1)), \mathbf{TC}(M^3(1, 1, 1)) \leqslant 7$.*

Proof. The generators of the ideal $\tilde{I}$ in proposition 2.2 are $\alpha_j = x_j y_j$ where $j = 1, 2, 3$. Let $\mathfrak{a}_j := 1 \otimes y_j - y_j \otimes 1$ for $j = 1, 2, 3$. Then $\mathfrak{a}_j$ is in the ideal of the zero-divisors of $H^*(M^3(P, \lambda); \mathbb{Z}_2)$. Note that $\mathbf{TC}(M^3(\beta_1^2, \beta_1^3, \beta_2^3)) \leqslant 7$ by proposition 4.5. The manifolds $M^3(1, 0, 1)$ and $M^3(1, 1, 1)$ are diffeomorphic by [**23**, Theorem 4].

Consider the real Bott manifold $M^3(1, 0, 1)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$ and $x_3 + y_2 + y_3$. So, $x_1 = y_1$, $x_2 = y_1 + y_2$ and $x_3 = y_2 + y_3$ in $H^*(M^3(1, 0, 1); \mathbb{Z}_2)$. Now,

$$\begin{aligned} \mathfrak{a}_2^2 \mathfrak{a}_3^3 &= (1 \otimes y_2^2 + y_2 \otimes 1)(1 \otimes y_3^3 - y_3 \otimes y_3^2 + y_3^2 \otimes y_3 - y_3^3 \otimes 1) \\ &= (y_1 y_2 + y_2 y_3) \otimes y_1 y_2 y_3 - y_1 y_2 y_3 \otimes (y_1 y_2 + y_2 y_3). \end{aligned}$$

So, the product $\mathfrak{a}_2^2 \mathfrak{a}_3^3$ contains an element $y_1 y_2 \otimes y_1 y_2 y_3$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(M^3(1, 0, 1); \mathbb{Z}_2)$ is greater than or equal to 5. Hence, by proposition 4.3, we have $6 \leqslant \mathbf{TC}(M^3(1, 0, 1))$. □

Now, for $n = 4$, the Bott matrix is given by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_1^2 & 1 & 0 & 0 \\ \beta_1^3 & \beta_2^3 & 1 & 0 \\ \beta_1^4 & \beta_2^4 & \beta_3^4 & 1 \end{pmatrix}$. In this case, we denote $M^4(P, \lambda)$ by $M^4(\beta_1^2, \beta_1^3, \beta_2^3, \beta_1^4, \beta_2^4, \beta_3^4)$.

THEOREM 4.12. *Let $\beta_1^2 = 1$. If at least one of $\{\beta_1^3, \beta_2^3\}$ is 1, and at least two of $\{\beta_1^4, \beta_2^4, \beta_3^4\}$ are 1, then $8 \leqslant \mathbf{TC}(M^4(1, \beta_1^3, \beta_2^3, \beta_1^4, \beta_2^4, \beta_3^4)) \leqslant 9$.*

Proof. The generators of the ideal $\tilde{I}$ in the cohomology ring $H^*(M^4(1, \beta_1^3, \beta_2^3, \beta_1^4, \beta_2^4, \beta_3^4); \mathbb{Z}_2)$ are $\alpha_j = x_j y_j$ where $j = 1, 2, 3, 4$. Let $\mathfrak{a}_j := 1 \otimes y_j - y_j \otimes 1$ for $j = 1, 2, 3, 4$. Then $\mathfrak{a}_j$ is in the ideal of the zero-divisors of $H^*(M^4(1, \beta_1^3, \beta_2^3, \beta_1^4, \beta_2^4, \beta_3^4); \mathbb{Z}_2)$. Note that $\mathbf{TC}(M^4(1, \beta_1^3, \beta_2^3, \beta_1^4, \beta_2^4, \beta_3^4)) \leqslant 9$ by proposition 4.5. By [**23**, Theorem 5], it is enough to consider the following manifolds to prove the claim; $M^4(1, 1, 0, 1, 1, 0)$, $M^4(1, 0, 1, 1, 1, 0)$, $M^4(1, 0, 1, 0, 1, 1)$, $M^4(1, 0, 1, 1, 0, 1)$, and $M^4(1, 1, 1, 1, 1, 0)$

(1) Consider the real Bott manifold $M^4(1, 1, 0, 1, 1, 0)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$, $x_3 + y_1 + y_3$ and $x_4 + y_1 + y_2 + y_4$. So $x_1 = y_1$, $x_2 = y_1 + y_2$, $x_3 = y_1 + y_3$, and $x_4 = y_1 + y_2 + y_4$ in $H^*(M^4(1, 1, 0, 1, 1, 0); \mathbb{Z}_2)$. Therefore, we have $y_1^2 = 0$, $y_2^2 = y_1 y_2$, $y_3^2 = y_1 y_3$, $y_4^2 = y_1 y_4 + y_2 y_4$. Now,

$$\begin{aligned} \mathfrak{a}_2 \mathfrak{a}_3^3 \mathfrak{a}_4^3 &= (1 \otimes y_2 - y_2 \otimes 1)(1 \otimes y_3^3 - y_3 \otimes y_3^2 \\ &\quad + y_3^2 \otimes y_3 - y_3^3 \otimes 1)(1 \otimes y_4^3 - y_4 \otimes y_4^2 + y_4^2 \otimes y_4 - y_4^3 \otimes 1) \\ &= y_1 y_2 y_3 y_4 \otimes y_1 y_2 y_3 - y_1 y_2 y_3 \otimes y_1 y_2 y_3 y_4 + y_1 y_2 y_3 y_4 \otimes y_1 y_3 y_4 \\ &\quad - y_1 y_3 y_4 \otimes y_1 y_2 y_3 y_4. \end{aligned}$$

So, the product $\mathfrak{a}_2 \mathfrak{a}_3^3 \mathfrak{a}_4^3$ contains an element $y_1 y_2 y_3 \otimes y_1 y_2 y_3 y_4$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(M^4(1, 1, 0, 1, 1, 0); \mathbb{Z}_2)$ is greater than or equal to 7. Hence by proposition 4.3, we have $8 \leqslant \mathbf{TC}(M^4(1, 1, 0, 1, 1, 0))$.

(2) Consider the real Bott manifold $M^4(1, 0, 1, 1, 1, 0)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$, $x_3 + y_2 + y_3$ and $x_4 + y_1 + y_2 + y_4$. So, $x_1 = y_1$, $x_2 = y_1 + y_2$, $x_3 = y_2 + y_3$ and $x_4 = y_1 + y_2 + y_4$ in $H^*(M^4(1, 0, 1, 1, 1, 0); \mathbb{Z}_2)$. Now,

$$\begin{aligned} \mathfrak{a}_2 \mathfrak{a}_3^3 \mathfrak{a}_4^3 &= (1 \otimes y_2 - y_2 \otimes 1)(1 \otimes y_3^3 - y_3 \otimes y_3^2 + y_3^2 \otimes y_3 - y_3^3 \otimes 1)(1 \otimes y_4^3 \\ &\quad - y_4 \otimes y_4^2 + y_4^2 \otimes y_4 - y_4^3 \otimes 1) \\ &= y_1 y_2 y_3 y_4 \otimes (y_1 y_2 y_3 + y_2 y_3 y_4 + y_1 y_3 y_4) \\ &\quad - (y_1 y_2 y_3 + y_2 y_3 y_4 + y_1 y_3 y_4) \otimes y_1 y_2 y_3 y_4. \end{aligned}$$

So, the product $\mathfrak{a}_2 \mathfrak{a}_3^3 \mathfrak{a}_4^3$ contains an element $y_1 y_2 y_3 \otimes y_1 y_2 y_3 y_4$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(M^4(1, 0, 1, 1, 1, 0); \mathbb{Z}_2)$ is greater than or equal to 7. Hence, by proposition 4.3, we have $8 \leqslant \mathbf{TC}(M^4(1, 0, 1, 1, 1, 0))$.

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(3) Consider the real Bott manifold $M^4(1, 0, 1, 0, 1, 1)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$, $x_3 + y_2 + y_3$ and $x_4 + y_2 + y_3 + y_4$. So, $x_1 = y_1$, $x_2 = y_1 + y_2$, $x_3 = y_2 + y_3$ and $x_4 = y_2 + y_3 + y_4$ in $H^*(M^4(1, 0, 1, 0, 1, 1); \mathbb{Z}_2)$. Now,

$$\begin{aligned}\mathfrak{a}_2\mathfrak{a}_3^3\mathfrak{a}_4^3 &= (1\otimes y_2 - y_2\otimes 1)(1\otimes y_3^3 - y_3\otimes y_3^2 + y_3^2\otimes y_3 - y_3^3\otimes 1)(1\otimes y_4^3\\ &\quad - y_4\otimes y_4^2 + y_4^2\otimes y_4 - y_4^3\otimes 1)\\ &= y_1y_2y_3y_4\otimes(y_1y_2y_3 + y_2y_3y_4 + y_1y_2y_4) - (y_1y_2y_3 + y_2y_3y_4 +\\ &\quad y_1y_2y_4)\otimes y_1y_2y_3y_4.\end{aligned}$$

So, the product $\mathfrak{a}_2\mathfrak{a}_3^3\mathfrak{a}_4^3$ contains an element $y_1y_2y_3\otimes y_1y_2y_3y_4$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(M^4(1, 0, 1, 0, 1, 1); \mathbb{Z}_2)$ is greater than or equal to 7. Hence, by proposition 4.3, we have $8 \leqslant \mathbf{TC}(M^4(1, 0, 1, 0, 1, 1))$.

(4) Consider the real Bott manifold $M^4(1, 0, 1, 1, 0, 1)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$, $x_3 + y_2 + y_3$ and $x_4 + y_1 + y_3 + y_4$. So $x_1 = y_1$, $x_2 = y_1 + y_2$, $x_3 = y_2 + y_3$ and $x_4 = y_1 + y_3 + y_4$ in $H^*(M^4(1, 0, 1, 1, 0, 1); \mathbb{Z}_2)$. Now,

$$\begin{aligned}\mathfrak{a}_2\mathfrak{a}_3^3\mathfrak{a}_4^3 &= (1\otimes y_2 - y_2\otimes 1)(1\otimes y_3^3 - y_3\otimes y_3^2 + y_3^2\otimes y_3\\ &\quad - y_3^3\otimes 1)(1\otimes y_4^3 - y_4\otimes y_4^2 + y_4^2\otimes y_4 - y_4^3\otimes 1)\\ &= y_1y_2y_3y_4\otimes y_1y_2y_4 - y_1y_2y_4\otimes y_1y_2y_3y_4 + y_1y_2y_3y_4\otimes y_1y_3y_4\\ &\quad - y_1y_3y_4\otimes y_1y_2y_3y_4.\end{aligned}$$

So, the product $\mathfrak{a}_2\mathfrak{a}_3^3\mathfrak{a}_4^3$ contains an element $y_1y_2y_4\otimes y_1y_2y_3y_4$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(M^4(1, 0, 1, 1, 0, 1); \mathbb{Z}_2)$ is greater than or equal to 7. Hence, by proposition 4.3, we have $8 \leqslant \mathbf{TC}(M^4(1, 0, 1, 1, 0, 1))$.

(5) Consider the real Bott manifold $M^4(1, 1, 1, 1, 1, 0)$. Then from proposition 2.2, the ideal $\tilde{J}$ is generated by the elements $x_1 + y_1$, $x_2 + y_1 + y_2$, $x_3 + y_1 + y_2 + y_3$ and $x_4 + y_1 + y_2 + y_4$. So, $x_1 = y_1$, $x_2 = y_1 + y_2$, $x_3 = y_1 + y_2 + y_3$ and $x_4 = y_1 + y_2 + y_4$ in $H^*(M^4(1, 1, 1, 1, 1, 0); \mathbb{Z}_2)$. Now,

$$\begin{aligned}\mathfrak{a}_2\mathfrak{a}_3^3\mathfrak{a}_4^3 &= (1\otimes y_2 - y_2\otimes 1)(1\otimes y_3^3 - y_3\otimes y_3^2 + y_3^2\otimes y_3 - y_3^3\otimes 1)(1\otimes y_4^3\\ &\quad - y_4\otimes y_4^2 + y_4^2\otimes y_4 - y_4^3\otimes 1)\\ &= y_1y_2y_3y_4\otimes y_2y_3y_4 - y_2y_3y_4\otimes y_1y_2y_3y_4.\end{aligned}$$

So, the product $\mathfrak{a}_2\mathfrak{a}_3^3\mathfrak{a}_4^3$ contains an element $y_2y_3y_4\otimes y_1y_2y_3y_4$ which is non-zero. Therefore, the zero-divisors-cup-length of $H^*(M^4(1, 1, 1, 1, 1, 0); \mathbb{Z}_2)$ is greater than or equal to 7. Hence, by proposition 4.3, we have $8 \leqslant \mathbf{TC}(M^4(1, 1, 1, 1, 1, 0))$.

□

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5. Symmetric topological complexity of small covers

In this section, we recall the definition of symmetric topological complexity. Then we compute this invariant for a class of small covers.

Let Y be a path-connected space. The path fibration $\pi\colon PY \to Y \times Y$ restricts to a fibration

$$\pi'\colon P'Y \to F(Y;2), \tag{5.1}$$

where $F(Y;2) = \{(x,\, y) \in Y \times Y \mid x \neq y\}$ is the space of ordered pairs of distinct points in Y, and $P'Y$ is the subspace $\{\gamma\colon I \to Y \mid \gamma(0) \neq \gamma(1)\} \subseteq PY$ consisting of paths with distinct endpoints.

The group $\mathbb{Z}_2$ acts on $F(Y;2)$ by permutation of factors, and acts on $P'Y$ by sending a path γ to its inverse $\bar{\gamma}$ given by $\bar{\gamma}(t) = \gamma(1-t)$. So, the group $\mathbb{Z}_2$ acting on the spaces $P'Y$ and $F(Y;2)$ freely. Observe that $\pi'\colon P'Y \to F(Y;2)$ is an equivariant map of free $\mathbb{Z}_2$-spaces. So, it induces a map

$$\pi''\colon P'Y/\mathbb{Z}_2 \to B(Y;2), \tag{5.2}$$

where $B(Y;2)$ denotes the orbit space $F(Y;2)/\mathbb{Z}_2$ of unordered pairs of distinct points in Y. This map is also a fibration.

Definition 5.1. *The symmetric topological complexity of Y, denoted by $\mathbf{TC}^S(Y)$, is defined to be one plus the sectional category of the fibration π''. In other words, $\boldsymbol{TC}^S(Y) = 1 + secat(\pi'')$.*

We adopt the convention that the sectional category of $p\colon E \to B$ vanishes if and only if $E = B = \emptyset$. The space $B(Y;2)$ is empty if and only if Y is a single point, and so in this case, $\mathbf{TC}^S(Y) = 1$. If Y contains more than one point then $\mathrm{secat}(\pi'') \geqslant 1$, and therefore $\mathbf{TC}^S(Y) \geqslant 2$.

Example 5.2. Let Y be a contractible space. Then there exists a continuous map $y \mapsto \gamma_y \in PY$ such that $\gamma_y(0) = y$ and $\gamma_y(1) = y_0$. Then setting $s(a,\, b)$ to be equal to the concatenation of γ_a and the inverse path to γ_b gives a symmetric equivariant section of (5.1). Therefore, for any contractible space Y with more than one point, we have $\mathbf{TC}^S(Y) = 2$. We note that if Y is a path-connected space with $\mathbf{TC}^S(Y) = 2$, then Y is contractible.

Let N_Y be the sub-ring of $H^*(Y) \otimes H^*(Y)$ spanned by the norm elements (i.e. the elements of the form $x \otimes y + y \otimes x$ with $x \neq y$). The following result follows from corollary 9, proposition 10 and theorem 17 in [**13**].

Proposition 5.3. *Let Y be a closed smooth manifold. Then*

$$max\{\mathbf{TC}(Y), cl(N_Y) + 2\} \leqslant \mathbf{TC}^S(Y) \leqslant 2dimY + 1.$$

Next, we calculate the symmetric topological complexity of the circle.

Corollary 5.4. *If P is a 1-simplex, then $M^1(P,\, \lambda) = \mathbb{RP}^1$ and $\mathbf{TC}^S(\mathbb{RP}^1) = 3$.*

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Proof. Note that the non-zero element $1 \otimes y_1 + y_1 \otimes 1$ is the norm element of $H^*(M^1(P, \lambda); \mathbb{Z}_2) \otimes H^*(M^1(P, \lambda); \mathbb{Z}_2)$. So, by proposition 5.3, we get $3 \leqslant \mathbf{TC}^S(M^1(P, \lambda))$. The small cover over a 1-simplex is $\mathbb{RP}^1 = \mathbb{S}^1$. Therefore, by proposition 5.3, we get

$$\mathbf{TC}^S(M^1(P, \lambda)) \leqslant 2\mathrm{dim}(M^1(P, \lambda)) + 1 = 3.$$

□

We note that the conclusion of corollary 5.4 can be obtained from [**13**, Corollary 18].

REMARK 5.5. The element $1 \otimes y_j + y_j \otimes 1$ is same as $1 \otimes y_j - y_j \otimes 1$ in $N_{M^n(P,\lambda)}$. Thus, the zero-divisors-cup-length of $M^n(P, \lambda)$ is the same as the cup-length of $N_{M^n(P,\lambda)}$.

THEOREM 5.6. *Let $M^n(P, \lambda)$ be a small cover other than $\mathbb{RP}^{n_1} \times \mathbb{RP}^{n_2}$ over $P = \Delta^{n_1} \times \Delta^{n_2}$.*

(1) *Let $n_2 \in \mathcal{S}$ with $n_2 > n_1$. Then $2^{r_1} + 2^{r_2} \leqslant \mathbf{TC}^S(M^n(P, \lambda))$.*

(2) *Let $n_2 \in \mathcal{S}$ with n_2 divides n_1. Then $2^r + 1 \leqslant \mathbf{TC}^S(M^n(P, \lambda))$.*

(3) *Let $n_2 \in \mathcal{S} + 1$ with $n_2 > n_1 + 1$. Then, $2^r + 1 \leqslant \mathbf{TC}^S(M^n(P, \lambda))$.*

(4) *Let $n_2 \in \mathcal{S} + 2$ with $n_2 > n_1 + 2$. Then, $2^r + 1 \leqslant \mathbf{TC}^S(M^n(P, \lambda))$.*

In particular, if $n = 2^{s-1}$, then for the cases (2), (3) and (4), we have $\mathbf{TC}^S(M^n(P, \lambda)) = 2n + 1$.

Proof. Let $N_{M^n(P,\lambda)}$ denote the sub-ring of $H^*(M^n(P, \lambda); \mathbb{Z}_2) \otimes H^*(M^n(P, \lambda); \mathbb{Z}_2)$ spanned by the norm elements. Consider the norm elements $\mathfrak{a}_j := 1 \otimes y_j + y_j \otimes 1$ in $N_{M^n(P,\lambda)}$ for $j = 1, 2$. Then the proof follows from theorem 4.6, proposition 5.3 and remark 5.5. □

THEOREM 5.7. *For $n \geqslant 3$, let the elements β_k^{k+1} in the Bott matrix (2.15) be 1 for $k = 1, \ldots, n-1$, and the remaining elements β_l^m be zero for $l = 1, \ldots, n-2$ and $m = 3, \ldots, n$. If $n \leqslant 2^r - 1 < 2n$, then the symmetric topological complexity of the real Bott manifold $M^n(P, \lambda)$ is greater than or equal to $2^r + 1$. In particular, if $n = 2^{s-1}$, then $\mathbf{TC}^S(M^n(P, \lambda)) = 2n + 1$.*

Proof. The proof follows from theorem 4.9, proposition 5.3 and remark 5.5. □

REMARK 5.8.

(1) From theorem 4.11, proposition 5.3 and remark 5.5, we get,

$$\mathbf{TC}^S(M^3(1,1,0)) = \mathbf{TC}^S(M^3(1,0,1)) = \mathbf{TC}^S(M^3(1,1,1)) = 7.$$

(2) Let $\beta_1^2 = 1$. If at least one of $\{\beta_1^3, \beta_2^3\}$ is 1, and at least two of $\{\beta_1^4, \beta_2^4, \beta_3^4\}$ are 1. Then from theorem 4.12, proposition 5.3 and remark 5.5, we get,

$$\mathbf{TC}^S(M^4(1, \beta_1^3, \beta_2^3, \beta_1^4, \beta_2^4, \beta_3^4)) = 9.$$

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6. $\mathcal{D}$-topological complexity of small covers

In this section, we recall the $\mathcal{D}$-topological complexity and the LS one-category of a space. We compute LS one-category for generalized real Bott manifolds and for a class of small covers. Then, we give some bounds for the $\mathcal{D}$-topological complexity of small covers over finite product of simplices.

DEFINITION 6.1. *Let Y be a path-connected space with the fundamental group $G = \pi_1(Y, y_0)$. The $\mathcal{D}$-topological complexity, denoted by $\mathbf{TC}^{\mathcal{D}}(Y)$, is defined as the minimal number k such that $Y \times Y$ can be covered by k open subsets $U_1, \ldots, U_k$ with the property that for each $i \in \{1, ..., k\}$ and for every choice of the base point $u_i \in U_i$, the homomorphism $\pi_1(U_i, u_i) \to \pi_1(Y \times Y, u_i)$ induced by the inclusion $U_i \to Y \times Y$ takes values in a subgroup conjugate to the diagonal $\Delta \subseteq G \times G$.*

Note that there is an isomorphism $\pi_1(Y \times Y, u_i) \to \pi_1(Y \times Y, (y_0, y_0)) \cong G \times G$ determined uniquely up to conjugation, and the diagonal inclusion $Y \to Y \times Y$ induces the inclusion $G \to G \times G$ onto the diagonal Δ.

We recall the Lusternik–Schnirelmann one-category (in short LS one-category) of a space which is denoted by $\mathrm{cat}_1(Y)$ for a space Y.

DEFINITION 6.2. *Let Y be a connected, locally path-connected, and semi-locally simply connected space with the universal cover $p \colon \widetilde{Y} \to Y$. Then the LS one-category is the sectional category of the map p. That is, $cat_1(Y) = secat(p)$.*

Similar to $\mathrm{cat}(Y)$ and $\mathbf{TC}(Y)$ there is a relation between $\mathrm{cat}_1(Y)$ and $\mathbf{TC}^{\mathcal{D}}(Y)$.

PROPOSITION 6.3 [**15**, Proposition 2.4, Proposition 2.11]. *If Y is a connected, locally path-connected, and semi-locally simply connected topological space, then*

$$cat_1(Y) \leqslant \mathbf{TC}^{\mathcal{D}}(Y) \leqslant min\{\mathbf{TC}(Y), cat_1(Y \times Y)\}.$$

We recall a result that gives a lower bound for the sectional category of fibrations.

PROPOSITION 6.4 [**6**, Proposition 9.14]. *Let $F \to E \xrightarrow{p} B$ be a fibration. If there exists $y_1, \ldots, y_k \in H^*(B; R)$ with $p^*(y_1) = \cdots = p^*(y_k) = 0$ and $y_1 \cup \cdots \cup y_k \neq 0$, then $secat(p) \geqslant k + 1$.*

The following result gives the computation of LS one-category of infinitely many small covers.

THEOREM 6.5. *Let $M^n(P, \lambda)$ be a small cover over a simple polytope P such that $\mathbb{R}\mathcal{Z}_{K_P}$ is simply connected. Then $cat_1(M^n(P, \lambda)) = n + 1$.*

Proof. Consider the principal $\mathbb{Z}_2^m$-bundle map $p \colon \mathbb{R}\mathcal{Z}_{K_P} \to M^n(P; \lambda)$ given by proposition 2.7. So we get the induced graded ring homomorphism

$$p^* \colon H^*(M^n(P, \lambda); \mathbb{Z}_2) \to H^*(\mathbb{R}\mathcal{Z}_{K_P}; \mathbb{Z}_2).$$

Note that p^* carries $H^j(M^n(P, \lambda); \mathbb{Z}_2)$ to $H^j(\mathbb{R}\mathcal{Z}_{K_P}; \mathbb{Z}_2)$. Now, for $j = 1$, $H^1(\mathbb{R}\mathcal{Z}_{K_P}; \mathbb{Z}_2) = 0$ as $\mathbb{R}\mathcal{Z}_{K_P}$ is simply connected. Therefore, each v in

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$H^1(M^n(P,\lambda);\mathbb{Z}_2)$ maps to 0 in $H^*(\mathbb{R}\mathcal{Z}_{K_P};\mathbb{Z}_2)$. Hence, $p^*(v)=0$ for $v\in H^1(M^n(P,\lambda);\mathbb{Z}_2)$. Since P is a simple polytope, at each vertex, exactly n many facets intersect. So the cup product of corresponding n indeterminates is non-zero. Therefore, by proposition 6.4, $\mathrm{secat}(p)\geqslant n+1$. Since p is the universal cover, so by definition of LS one-category, $\mathrm{secat}(p)=\mathrm{cat}_1(M^n(P,\lambda))$. Therefore, $n+1\leqslant \mathrm{cat}_1(M^n(P,\lambda))$.

On the other hand, we know that $\mathrm{cat}_1(M^n(P,\lambda))\leqslant \mathrm{cat}(M^n(P,\lambda))$ as discussed in [**15**]. Since $\mathrm{cat}(M^n(P,\lambda))=n+1$ (by theorem 3.4), so $\mathrm{cat}_1(M^n(P,\lambda))\leqslant n+1$. Hence, we get the result. □

Corollary 6.6. *Let $M^n(P,\lambda)$ be a small cover over $P=\prod_{j=1}^m\Delta^{n_j}$ such that $n_j\geqslant 2$ for $j=1,\ldots,m$. Then $cat_1(M^n(P,\lambda))=n+1$.*

Proof. The moment angle manifold $\mathbb{R}\mathcal{Z}_{K_P}$ for the polytope $P=\prod_{j=1}^m\Delta^{n_j}$ is $\mathbb{S}^{n_1}\times\cdots\times\mathbb{S}^{n_m}$. Thus, $\mathbb{R}\mathcal{Z}_{K_P}$ is simply connected and orientable for $n_j\geqslant 2$ for $j=1,\ldots,m$. Therefore, by theorem 6.5, $\mathrm{cat}_1(M^n(P,\lambda))=n+1$. □

Theorem 6.7. *Let $M^n(P,\lambda)$ be a small cover over $P=\prod_{j=1}^m\Delta^{n_j}$. Then $cat_1(M^n(P,\lambda))=n+1$.*

Proof. Corollary 6.6 gives the proof for all $n_j\geqslant 2$.

Now consider the small cover $M^n(P,\lambda)$ over $P=\prod_{j=1}^m\Delta^{n_j}$ where some $n_j=1$. Without loss of generality, we assume that $n_1=n_2=\cdots=n_{s-1}=1$ and the remaining n_j's are greater than or equal to 2. Then the map $\bar{p}\colon (\prod_1^{s-1}\mathbb{R}\times\prod_{j=s}^m\mathbb{S}^{n_j})\to(\prod_1^{s-1}\mathbb{S}^1\times\prod_{j=s}^m\mathbb{S}^{n_j})/\mathbb{Z}_2^m=M^n(P,\lambda)$ is the universal cover where $\mathbb{R}\to\mathbb{S}^1$ is given by exponential map. This induces a ring homomorphism

$$\bar{p}^*\colon H^*(M^n(P,\lambda);\mathbb{Z}_2)\to H^*\left(\prod_1^{s-1}\mathbb{R}\times\prod_{j=s}^m\mathbb{S}^{n_j};\mathbb{Z}_2\right).$$

We know that the cohomology ring of $M^n(P,\lambda)$ is generated by $y_1,\ldots,y_m$ and $y_j^{n_j}\neq 0$ for $j=1,2,\ldots,m$. Hence,

$$p^*(y_1)=\cdots=p^*(y_m)=0.$$

Since all $y_j^{n_j}\neq 0$ for $j=1,\ldots,m$, so

$$y_1\cup\cdots\cup y_{s-1}\cup\underbrace{y_s\cup\cdots\cup y_s}_{n_s\text{ times}}\cup\cdots\cup\underbrace{y_m\cup\cdots\cup y_m}_{n_m\text{ times}}\neq 0.$$

Therefore, by proposition 6.4, $\mathrm{secat}(\bar{p})\geqslant n_1+\cdots+n_m+1=n+1$. By the definition of LS one-category, $\mathrm{secat}(p)=\mathrm{cat}_1(M^n(P,\lambda))$. Therefore, $n+1\leqslant \mathrm{cat}_1(M^n(P,\lambda))$. Using theorem 3.4, $\mathrm{cat}(M^n(P,\lambda))=n+1$. Thus, $\mathrm{cat}_1(M^n(P,\lambda))\leqslant n+1$. Hence, $\mathrm{cat}_1(M^n(P,\lambda))=n+1$. □

We give some bounds on $\mathbf{TC}^{\mathcal{D}}(M^n(P,\lambda))$ in the following.

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THEOREM 6.8. *Let $M^n(P, \lambda)$ be a small cover over a product of simplices P. Then*

$$n+1 \leqslant \mathbf{TC}^{\mathcal{D}}(M^n(P,\lambda)) \leqslant 2n+1.$$

In particular, if $M^n(P, \lambda) = \mathbb{RP}^{n_1} \times \cdots \times \mathbb{RP}^{n_m}$ with $n_j \in \{1, 3, 7\}$, then $\mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda)) = n+1$.

Proof. By proposition 6.3, we have $\mathrm{cat}_1(M^n(P, \lambda)) \leqslant \mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda))$. Therefore, by theorem 6.7, we have $n+1 \leqslant \mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda))$. By proposition 6.3, $\mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda)) \leqslant \mathbf{TC}(M^n(P, \lambda))$. So, the upper bound of $\mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda))$ is $2n+1$, i.e. $\mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda)) \leqslant 2n+1$.

The second part follows from corollary 4.8 and $\mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda)) \leqslant \mathbf{TC}(M^n(P, \lambda))$. In this case $\mathbf{TC}^{\mathcal{D}}(M^n(P, \lambda)) \leqslant n+1$. □

Acknowledgements

The authors thank the anonymous referee for several helpful comments and suggestions. The authors thank Navnath for the helpful discussion. The first author would like to thank the Indian Institute of Technology Madras for the Ph.D. fellowship. The second author would like to thank UGC India for his Ph.D. fellowship. The last author would like to thank ISI Kolkata for its financial support.

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