COMPOSITIO MATHEMATICA

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Compositio Math. **146** (2010), 541–560.

doi:10.1112/S0010437X09004552

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Analytic continuation of overconvergent Hilbert eigenforms in the totally split case

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Abstract

We generalise results of Buzzard, Taylor and Kassaei on analytic continuation of p-adic overconvergent eigenforms over $\mathbb{Q}$ to the case of p-adic overconvergent Hilbert eigenforms over totally real fields F, under the assumption that p splits completely in F. This includes weight-one forms and has applications to generalisations of Buzzard and Taylor's main theorem. Next, we follow an idea of Kassaei's to generalise Coleman's well-known result that 'an overconvergent U_p-eigenform of small slope is classical' to the case of p-adic overconvergent Hilbert eigenforms of Iwahori level.

1. Introduction

Let $p \geqslant 5$ be a prime and let $\mathcal{O}$ be the ring of integers of a finite extension of $\mathbb{Q}_p$ with maximal ideal λ. Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O})$ be a continuous two-dimensional representation of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$. Suppose that the following conditions are satisfied.

(i) ρ ramifes at only finitely many primes.

(ii) ρ is unramified at p and $\rho(\mathrm{Frob}_p)$ has eigenvalues α and β in $\mathcal{O}$ which are distinct modulo λ.

(iii) ρ mod λ is absolutely irreducible and 'modular'.

Buzzard and Taylor proved in [BT99] that such a ρ arises from a holomorphic eigenform of weight one, in the sense of Deligne and Serre [DS74]. The arguments in [BT99] rely crucially on two key ingredients. First, one needs a 'companion forms theorem' due to Gross, Coleman and Voloch on congruences between ordinary forms of low weight; second, it is necessary to have results concerning 'analytic continuation of overconvergent eigenforms', extending an overconvergent eigenform to the non-ordinary locus of modular curves; this is crucial for 'gluing' weight-one liftings (overconvergent eigenforms of weight one and slope zero) of mod-p companion forms on their overlap.

Our motivation for this paper is to generalise the above result to the Hilbert case and follow Taylor's strategy (outlined in [Tay97]) to prove 'insoluble' cases of the strong Artin conjecture for totally odd, continuous representations $G_F \to \mathrm{GL}_2(\mathbb{C})$ of the absolute Galois group of totally real fields F. Since the Jacquet–Langlands correspondence does not transfer weight-one forms for GL_2 over F to forms on Shimura curves over F of the type considered by Carayol [Car86], one has to work directly with Hilbert modular varieties and develop a theory of p-adic (overconvergent) modular forms on Hilbert modular varieties.

Received 5 May 2008, accepted in final form 16 August 2009, published online 2 February 2010.
2000 Mathematics Subject Classification 11F33, 11F41 (primary), 11G18, 14G22, 14G35 (secondary).
Keywords: p-adic modular forms, Hilbert modular varieties.
This work was supported by a EPSRC project grant.

The companion-forms theorems of Gross, Coleman and Voloch have been generalised by Gee in [Gee08] to the setting of Hilbert modular forms over totally real fields F, under the assumption that p splits completely. However, proving a p-adic analytic continuation theorem for overconvergent Hilbert eigenforms, for example for weight-one specialisations of Hida families in the Hilbert case, has remained an open problem. We solve this problem in the present paper. In forthcoming work [Sas], the author will establish a modular lifting theorem for weight-one Hilbert modular forms that is analogous to the main theorem of Buzzard [Buz03]; this will help to prove many new cases of the strong Artin conjecture for totally odd, continuous representations $G_F \to \mathrm{GL}_2(\mathbb{C})$, under the local condition that *some* particular p splits completely in F.

The technique used to prove that certain overconvergent p-adic Hilbert eigenforms of level p are classical p-adic Hilbert eigenforms is analogous to that in [BT99, Buz03, Kas06]. In particular, Buzzard [Buz03] proved the optimal overconvergence results, and we shall follow his method. What is new here is that we write down more formally a function which controls the overconvergence on the generic fibre of the level-p HMV, and this generalises the *ad hoc* v' used in [Buz03].

Given an overconvergent eigenform of weight $k \geqslant 1$ and level $\Gamma_1(N) \cap \Gamma_0(p)$, Kassaei [Kas06] observed that even when its companion form is not assumed, one can explicitly write down (and then glue) another (over)convergent modular form if the valuation of the eigenvalue of U_p is strictly less than $k-1$. The key observation in [Kas06] is that, whilst an overconvergent eigenform f such that $f|U_p = \alpha f$ overconverges to 'the far end' of the supersingular annuli in $(X_{\Gamma_1(N)\cap\Gamma_0(p)})^{\mathrm{an}}$ but not any further, $f - (1/\alpha p) f|\mathrm{Frob}_p$ does extend (see [Gou88] for the definition of Frob_p) if we remove precisely that 'factor' which causes the problem. Based on this observation, Kassaei wrote down an infinite sum of overconvergent modular forms and endeavoured to glue it back onto f. In order to do this, a general 'gluing lemma' in p-adic 'integral' geometry was proved in [Kas06], and critical use was made of the overconvergence of the Frobenius operator, i.e. the fact that application of Frob_p makes supersingular elliptic curves 'more supersingular' (as opposed to U_p, which makes them 'less supersingular'). However, as noted by Gouvea (see the remark preceding [Gou88, Corollary II.2.5]), the Frobenius operator Frob_p is not 'integral' (but up to a power of the Hasse invariant), and precisely because of this, Kassaei needed to do a delicate calculation in [Kas06, Lemma 3.3] to ensure convergence of the infinite sum.

We generalise Kassaei's ideas to the setting of Hilbert modular forms and prove an analogue in the Hilbert case of Coleman's theorem [Col96] that an overconvergent p-adic U_p-eigenform of small slope is classical. More precisely, the result can be stated as follows.

Theorem. *Let F be a totally real field with $[F:\mathbb{Q}] = g > 1$, and let p be a prime which we assume splits completely in F. Let f be an overconvergent Hilbert modular form of weight $(k_1, \ldots, k_g)$ and level $\Gamma_1(N) \cap \Gamma_0(p)$ which is an eigenvector of U_{v_i} (where $v_i|p$) with non-zero eigenvalue a_i. Assume that $v_p(a_i) < k_i - 1$ for all $1 \leqslant i \leqslant g$. Then f is a classical Hilbert modular form.*

A constraint that has been necessary so far is that the prime p, which we fix, has to split completely in F. The author is currently trying to remove this assumption.

Following closely the original construction of Coleman and Mazur [CM98], Kisin and Lai constructed in [KL05] an 'eigenvariety' (a term coined by Buzzard in [Buz07]) for overconvergent Hilbert modular forms for GL_2 over a totally real field F. They substituted the Eisenstein series of weight $p-1$, whose p-adic variation property allowed Coleman to p-adically vary in [CM98] the weights of classical modular forms over $\mathbb{Q}$, for a classical Hilbert modular form lifting a full Hasse invariant of sufficiently large parallel weight. Since only full Hasse invariants of parallel weight lift to classical Hilbert modular forms in characteristic zero, their construction is one-dimensional; one would expect the weight space of the eigenvariety for Hilbert modular forms over F to be $[F:\mathbb{Q}]$. Buzzard, however, constructed an eigenvariety over the $[F:\mathbb{Q}]$-dimensional weight space by defining overconvergent Hilbert modular forms on a totally definite quaternion algebra D over F, i.e. on a zero-dimensional 'Hida variety'.

The missing ingredient in Kisin and Lai's construction [KL05] is that the locus of parallel-weight classical Hilbert modular forms is Zariski dense. For the Coleman–Mazur eigencurve, this follows from the main theorem of Coleman [Col96] (see [Che05], for example); our main theorem proves this. As a corollary, one can use the argument of [Che05] and p-adically vary the classical Jacquet–Langlands correspondence for GL_2 over F and D over F, for example. It should also be possible to apply our results to a conjecture of Fontaine and Mazur as given in the work of Kisin [Kis03]. Results of this kind will be proved elsewhere (see, e.g., [Sas]).

2. Hilbert modular varieties

Let F be a totally real field with $[F:\mathbb{Q}]=g>1$, and denote by $\mathfrak{d}$ its different. Let $\mathfrak{c}$ be a fractional ideal of F with a notion of positivity ('ordered' in [Tay01]), i.e. the choice for each embedding $\tau: F \hookrightarrow \mathbb{R}$ of an element of $\mathrm{Aut}(\mathfrak{c} \otimes_{\tau: F \hookrightarrow \mathbb{R}} \mathbb{R}) \simeq \mathbb{R}^\times$. The choice corresponds to the orientation of a one-dimensional vector space $\mathfrak{c} \otimes \mathbb{R}$. The isomorphism classes of such an object correspond precisely to the narrow ideal class group, the quotient of the group of fractional ideals in F by the principal ideals generated by totally positive elements. For fractional ideals $\mathfrak{a}$ and $\mathfrak{b}$, if $\mathfrak{a}\mathfrak{b}^{-1}$ is generated by a totally positive element in F, then we write $\mathfrak{a} \sim \mathfrak{b}$.

By an HBAV over a scheme S we shall mean an abelian variety A over S equipped with real multiplication $\mathcal{O}_F \hookrightarrow \mathrm{End}(A)$. Note that its dual $A^\vee$ is naturally also an HBAV.

We shall denote by $(A/S, \imath, \jmath)$ a triple consisting of:

(i) an HBAV A over S of relative dimension g;

(ii) a $\Gamma_1(N)$-level structure on the HBAV over S, that is, an embedding $\imath: (\mathcal{O}_F/N\mathcal{O}_F)(1) = \mathfrak{d}^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow A[N]$;

(iii) an $\mathcal{O}_F$-linear homomorphism $\jmath: \mathfrak{c} \to \mathcal{P}(A)$, where $\mathcal{P}(A)$ is the rank-one projective $\mathcal{O}_F$-module of $\mathcal{O}_F$-linear homomorphisms $f: A \to A^\vee$ which are symmetric (i.e. $f^\vee = f$) and such that:
   - the cone of totally positive elements $\mathfrak{c}^+$ in $\mathfrak{c}$ maps to the cone $\mathcal{P}(A)^+$ of polarisations in $\mathcal{P}(A)$;
   - the induced morphism of sheaves (on the large étale site of S) $A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A \otimes \mathcal{P}(A) \to A^\vee$, $a \otimes x \mapsto a \otimes \jmath(x) \mapsto \jmath(x)(a)$, is an isomorphism.

If $N \geqslant 4$, the functor that associates to a $\mathbb{Z}[1/N]$-scheme S the set of isomorphism classes of triples $(A, \imath, \jmath)$ is represented by a scheme over $\mathbb{Z}[1/N]$ (see [DT04]), which we shall henceforth denote by $Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}$.

By calculating its local model, Deligne and Pappas [DP94] showed that the fibre over a prime dividing the discriminant Δ is singular in a codimension-two closed subscheme, but, when Δ is invertible, Lie A is a locally free $\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_S$-module of rank one and thus $Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N\Delta]}$ coincides with Rapoport's smooth moduli space (see [DP94, Corollaire 2.9]).

We choose a set of representatives $\{\mathfrak{c}_1, \ldots, \mathfrak{c}_{h^+}\}$ for the narrow class group. We let

$$Y_{\Gamma_1(N),\mathbb{Z}[1/N]} \simeq \coprod_{\mathfrak{c}\,\in\{\mathfrak{c}_1,\ldots,\mathfrak{c}_{h^+}\}} Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}$$

and let

$$A_{\Gamma_1(N)} = \coprod_{\mathfrak{c}\,\in\{\mathfrak{c}_1,\ldots,\mathfrak{c}_{h^+}\}} A_{\Gamma_1(N;\mathfrak{c})},$$

with each $A_{\Gamma_1(N;\mathfrak{c})}$ being the universal HBAV over $Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}$, denote the 'universal' HBAV over $Y_{\Gamma_1(N),\mathbb{Z}[1/N]}$. The HBAV $A_{\Gamma_1(N)}$ comes equipped with the sheaf of relative differentials $\Omega_{A_{\Gamma_1(N)}/Y_{\Gamma_1(N),\mathbb{Z}[1/N]}}$ and we shall denote by $\omega_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}}$, or simply ω if there is no confusion anticipated, the pull-back of $\Omega_{A_{\Gamma_1(N)}/Y_{\Gamma_1(N),\mathbb{Z}[1/N]}}$ by the identity section. Following Rapoport and Deligne as attributed in [Rap78], we shall define in the next section a Hilbert modular form of weight $\vec{k} \in \mathbb{Z}_{\geqslant 0}^{\{\tau:F\hookrightarrow\mathbb{R}\}}$ and level $\Gamma \subset \mathrm{SL}_2(F)$ to be a section of a certain invertible sheaf $\omega^{\vec{k}}$ over the Hilbert modular variety Y_Γ. Note that 'congruence subgroups' are subgroups of $\mathrm{SL}_2(F)$ rather than $\mathrm{GL}_2(F)$ or its subgroup $\mathrm{GL}_2^+(F)$ of matrices in $\mathrm{GL}_2(F)$ with totally positive determinants, which one might expect from the classical theory of Hilbert modular forms. This is due to polarisations: the complex HBAVs, which canonically come equipped with 'polarisations faible' [Del71, 4.4], are parameterised by the quotient by $\mathrm{GL}_2^+(F)$ of $[F:\mathbb{Q}]$ copies of the complex upper half plane; since $\mathrm{GL}_2^+(F)$ acts on polarisations by determinant, the moduli space for complex HBAVs *with* a 'polarisation homogène' [Del71, 4.3] is parameterised by the quotient by the subgroup

$$\{M \in \mathrm{GL}_2^+(F) \mid \det M \in \mathbb{Q}\} = \mathrm{SL}_2(F).$$

3. Hilbert modular forms

Let $K \subset \mathbb{R}$ be the Galois closure of F over $\mathbb{Q}$ containing all the conjugates of F, and let $\mathcal{O}_K$ be its ring of integers. For brevity, we write ω for $\omega_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]}$. Since the discriminant Δ is invertible, this is a locally free $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]})$-module of rank one. We think of $\mathbb{Z}^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$ as the (algebraic) character group $\mathrm{Hom}_{\overline{\mathbb{Q}}}(\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1 \times \overline{\mathbb{Q}}, \mathrm{GL}_1 \times \overline{\mathbb{Q}})$ by identifying $\vec{k} = (k_1, \ldots, k_g) = \sum_{\tau\in\mathrm{Hom}(F,\overline{\mathbb{Q}})} k_\tau \tau$ with the character that sends $x \in F^\times$ to $\prod_{\tau\in\mathrm{Hom}(F,\overline{\mathbb{Q}})}(\tau(x))^{k_\tau}$ in $\overline{\mathbb{Q}}^\times$. The character corresponding to $\vec{k}$ gives rise to an invertible sheaf $\omega^{\vec{k}}$ on $Y_{\Gamma_1(N),\mathbb{Z}[1/N]} \times \mathcal{O}_K[1/(N\Delta)]$, which we define to be $\bigotimes_{\tau\in\mathrm{Hom}(F,\overline{\mathbb{Q}})}(\omega_\tau)^{\otimes k_\tau}$, where by ω_τ we mean the invertible sheaf of the $\mathcal{O}_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]}$-module obtained by tensoring ω with $\mathcal{O}_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]}$ as follows:

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]} \simeq \mathcal{O}^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]} \xrightarrow{\tau} \mathcal{O}_{Y_{\Gamma_1(N),\mathbb{Z}[1/N]}\times\mathcal{O}_K[1/(N\Delta)]}.$$

Definition. For an $\mathcal{O}_K[1/(\Delta N)]$-algebra R, an element of

$$H^0(Y_{\Gamma_1(N),\mathbb{Z}[1/N]} \times_{\mathcal{O}_K[1/(N\Delta)]} R, (\omega \times_{\mathcal{O}_K[1/(\Delta N)]} R)^{\vec{k}})$$

is called a Hilbert modular form defined over R of weight $\vec{k}$ and level $\Gamma_1(N)$.

Fix a prime p not dividing N, and assume that it splits completely in F. Let $v_1, \ldots, v_g$ denote the prime ideals in F above p, which we may think of as the (finite) places defining $F \hookrightarrow \overline{\mathbb{Q}}_p$. Fix an embedding $\imath : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ once and for all.

https://doi.org/10.1112/S0010437X09004552 Published online by Cambridge University Press

Let K_0 be the compositum of the images of F by all the field embeddings $F \hookrightarrow \overline{\mathbb{Q}} \overset{\imath}{\hookrightarrow} \overline{\mathbb{Q}}_p$. Since p splits completely in F, it is $\mathbb{Q}_p$. The composition with $\imath$ naturally defines a surjective map

$$\mathrm{Hom}(F, \overline{\mathbb{Q}}) = \{\tau\} \to \mathrm{Hom}(F, \overline{\mathbb{Q}}_p) = \{v|p\},$$

which is bijective since p splits completely in F. We assume that $\tau_i : F \hookrightarrow K \hookrightarrow \mathbb{R}$ corresponds to $v_i|p$ for every $1 \leqslant i \leqslant g$.

For $\vec{k} \in \mathbb{Z}^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$, define $\imath\vec{k} \in \mathbb{Z}^{\mathrm{Hom}(F,\overline{\mathbb{Q}}_p)}$ by $(\imath\vec{k})_v = k_{\imath^{-1}(v)}$, which corresponds to the character sending

$$x \in (F \otimes \mathbb{Q}_p)^\times = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_1(\mathbb{Q}_p) \quad \text{to} \quad \prod_{v|p} x_v^{(\imath\vec{k})_v}.$$

4. Hilbert modular varieties with Iwahori level structure

Let $Y_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p),\mathbb{Z}[1/N]}$ be the scheme over $\mathbb{Z}[1/N]$ in [Pap95], which represents the functor that sends a $\mathbb{Z}[1/N]$-scheme S to $\mathcal{O}_F$-linear isogenies $\alpha : (A, \imath, \jmath) \to (A', \imath', \jmath')$ of degree p^g such that:

(i) $(A, \imath, \jmath) \in Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}(S)$ and $(A', \imath', \jmath') \in Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}(S)$;

(ii) the kernel of $\alpha : A \to A'$ is annihilated by p;

(iii) for every $x \in \mathfrak{c}$, we have $\alpha^\vee \circ \jmath'(x) \circ \alpha = p\jmath(x)$;

(iv) the level structures are compatible: to be precise, $\imath' = \alpha \circ \imath$.

Remark. It does not seem possible to work with the '$\Gamma_0(p)$-level structure' as in [KM85, (3.4)]. The proof in [KM85] of its representability certainly does not work in the higher-dimensional case, so we work instead with a 'moduli space of isogenies' as in [KM85, (6.5)].

LEMMA 1. *Let $v_1, \ldots, v_g$ be the prime ideals in $\mathcal{O}_F$ lying above p. Giving a O_F-linear degree-p^g isogeny $\alpha : (A, \imath, \jmath) \to (A', \imath', \jmath')$ over S as above is equivalent to giving $(A, \imath, \jmath, C)$ where C is an $\mathcal{O}_F$-stable finite flat subgroup scheme of $A[p]$ over S of order p^g, which by the action of $\mathcal{O}_F$ decomposes as $C = \prod_{1 \leqslant i \leqslant g} C_i$ with each C_i being a finite flat subgroup of $A[v_i]$ of order equal to the cardinality of $\mathcal{O}_F/v_i$.*

Proof. This follows from [Pap95]. □

For each prime v_i above p, we choose a uniformiser ϖ_i of the integers of the completion of F_{v_i}. For $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant h^+$, we have $\varpi_i \mathfrak{c}_j \sim \mathfrak{c}_{j'}$ for some $j' \in [1, h^+]$ depending on ϖ_i and j, and we shall fix such a j'. With these choices made, we let

$$Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \simeq \coprod_{\mathfrak{c} \in \{\mathfrak{c}_1,\ldots,\mathfrak{c}_{h^+}\}} Y_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p),\mathbb{Z}[1/N]}.$$

Except when necessary, from now on we shall no longer mention $\mathfrak{c}$-polarisations.

There are canonically defined (representable) morphisms of schemes

$$\pi_1 : Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \to Y_{\Gamma_1(N),\mathbb{Z}[1/N]}$$

taking $(A, C, \imath)$ to $(A, \imath)$ and, for each

$$1 \leqslant i \leqslant g, \quad \pi_{2,i} : Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \to Y_{\Gamma_1(N),\mathbb{Z}[1/N]}$$

taking $(A, C, \imath)$ to $(A/C_i, \imath \bmod C_i)$. Here, by $(\imath \bmod C_i)$ we mean the composition of $\imath$ with the isogeny $A \to A/C_i$.

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For every $1 \leqslant i \leqslant g$, we have an automorphism w_i of $Y_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Z}_p$ which takes $(A, C = \prod_{1\leqslant j\leqslant g} C_j, \imath)$ to $(A/C_i, C', \bmod C_i)$, where C' is the $\mathcal{O}_F$-stable subgroup of $(A/C_i)[p]$ of order p^g defined by $C'_j = (C_j + C_i)/C_i$ for all $j \neq i$ and $C'_i = A[v_i]/C_i$. Note that $(C_i + C_i)/C_i$ is none other than the schematic closure in $A[p]/C_i$ of the image of C_j under the map $A[p] \to A[p]/C_j$. It is now clear that $\pi_{2,i} = \pi_1 \circ w_i$.

For each prime v above p, we let $Y_{\Gamma_1(N)\cap\Gamma_0(p)\cap\Gamma^0(v),\mathbb{Z}[1/(Np)]}$ denote the Hilbert modular variety parameterising $A, C, \imath$ and $\jmath$ as above, equipped with a finite flat subgroup $D \subset A[v]$ of order p which, locally f.p.p.f, admits a $\mathcal{O}_F/v$-generator in the sense of [KM85] and has trivial intersection with C. One can show that this is relatively representable to $Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times \mathbb{Z}[1/(Np)]$, as in [KM85, proof of Theorem 3.7.1], and therefore defines a scheme in characteristic prime to Np. There is a natural map

$$\pi_1 : Y_{\Gamma_1(N)\cap\Gamma_0(p)\cap\Gamma^0(v)\mathbb{Z}[1/(Np)]} \to Y_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Z}[1/(Np)]$$

which forgets D, and also a natural map

$$\pi_{2,v} : Y_{\Gamma_1(N)\cap\Gamma_0(p)\cap\Gamma^0(v),\mathbb{Z}[1/(Np)]} \to Y_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Z}[1/(Np)]$$

which quotients out by D. With these maps, we define a Hecke operator U_v on $Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times \mathbb{Z}[1/(Np)]$ as in [KL05, (1.11)] or [Dim05, 2.4].

For $(\vec{k}, \vec{w}) \in \mathbb{Z}^{\mathrm{Hom}(F,\overline{\mathbb{Q}})} \times \mathbb{Z}^{\mathrm{Hom}(F,\overline{\mathbb{Q}})}$ such that $k_\tau + 2w_\tau$ is independent of τ and $w \overset{\mathrm{def}}{=} k_\tau + 2w_\tau \geqslant 0$, we define an invertible sheaf of the $(\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{Y_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathcal{O}_K[1/N]})$-module $\omega^{(\vec{k},\vec{w})}$ to be

$$\Omega_{A_{\Gamma_1(N)\cap\Gamma_0(p)}/Y_{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathcal{O}_K[1/N]} \otimes\biggl(\bigotimes_\tau \bigwedge^2 (\mathbb{R}^1\pi_*\Omega^\bullet_{A_{\Gamma_1(N)\cap\Gamma_0(p)}/Y_{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathcal{O}_K[1/N]})_\tau^{\otimes w_\tau} \otimes \omega_\tau^{\vec{k}_\tau - 2}\biggr),$$

where π denotes

$$A_{\Gamma_1(N)\cap\Gamma_0(p)} \to Y_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathcal{O}_K[1/N]$$

and a subscript τ means tensoring with $\mathcal{O}_{Y_{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathcal{O}_K[1/N]}$ by

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \mathcal{O}_{Y_{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathcal{O}_K[1/N]} \xrightarrow{\tau} \mathcal{O}_{Y_{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathcal{O}_K[1/N]}.$$

Note that what Hida denotes by w (respectively, $n + 2v$) in [Hid88] is our $-\vec{w}$ (respectively, $-w$). Following [KL05, (1.11)], we use the degeneracy maps π_1 and $\pi_{2,v}$ above to define a Hecke operator U_v on

$$H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times \mathcal{O}_K[1/(Np)], \omega^{(\vec{k},\vec{w})})$$

which is compatible, via the projection map that 'averages the action of the totally positive units' as in [KL05, (1.11.8)], with the classical U_v operator on the Hilbert modular forms on $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_2$. Upon changing base from $\mathcal{O}_K[1/(Np)]$ to K and choosing a (canonical) trivialisation of $\mathbb{R}^1\pi_*\Omega^\bullet_{A_{\Gamma_1(N)\cap\Gamma_0(p)}/Y_{\Gamma_1(N)\cap\Gamma_0(p)}}$, we have an isomorphism

$$H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times K, \omega^{(\vec{k},\vec{w})}) \simeq H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times K, \omega^{\vec{k}}),$$

and we define U_v on $H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times K, \omega^{\vec{k}})$ by composing the operator with this isomorphism.

https://doi.org/10.1112/S0010437X09004552 Published online by Cambridge University Press

5. Generic fibres

As in [DR80, RT83], an (unramified) $\Gamma_1(N)$-cusp C of $Y_{\Gamma_1(N;\mathfrak{c})}$ over a ring R consists of the following data, up to isomorphisms:

(i) projective rank-one $\mathcal{O}_F$-modules $\mathfrak{a}$ and $\mathfrak{b}$ such that $\mathfrak{b}^{-1}\mathfrak{a} \simeq \mathfrak{c}$;

(ii) an $\mathcal{O}_F$-linear isomorphism $N^{-1}\mathcal{O}_F/\mathcal{O}_F \simeq N^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$;

(iii) an $(\mathcal{O}_F \otimes R)$-linear isomorphism $\mathfrak{a}^{-1} \otimes R \simeq \mathcal{O}_F \otimes R$.

Fix an (unramified) $\Gamma_1(N)$-cusp C. Let $U_N \subset \mathcal{O}_F^\times$ denote the group of units in $\mathcal{O}_F$ which are congruent to 1 mod N. Let $X = \mathfrak{a}\mathfrak{b}$, let X^* denote its dual $\mathrm{Hom}_{\mathcal{O}_F}(X, \mathfrak{d}^{-1}) \overset{\mathrm{tr}_{F/\mathbb{Q}}}{=} \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, and let $X^{*,+}$ denote the subset of X^* consisting of totally positive elements. One can choose a '$\Gamma_1(N)$-admissible' smooth polyhedral cone decomposition $\{\sigma\}$ of $X^{*,+} \otimes_{\mathbb{Z}} \mathbb{R} \cup \{0\}$, depending on C, such that $\{\sigma\}$ is invariant under the action of U_N^2 and $\{\sigma\}/U_N^2$ is finite.

Let $X_N \overset{\mathrm{def}}{=} N^{-1}X$. Then let $S_N^0 = \mathrm{Spec}\,\mathbb{Z}[X_N] \hookrightarrow S_N(\{\sigma\})$ be the torus embedding of a split torus over $\mathbb{Z}$ corresponding to $\{\sigma\}$ (see [CF90, ch. IV, Theorem 2.5]), and take $\widehat{S_N(\{\sigma\})}$ to be the formal completion of $S_N(\{\sigma\})$ along $S_N(\{\sigma\}) \backslash S_N^0$. Note that $\widehat{S_N(\{\sigma\})}$ has a covering by open formal subschemes of the form $\widehat{S_N(\sigma)} \overset{\mathrm{def}}{=} \mathrm{Spf}\, R_\sigma$, where R_σ is the completion of $\mathbb{Z}[q^\xi]_{\xi \in X_N \cap \check{\sigma}}$. Here, by $\check{\sigma}$ we mean the dual cone of a cone σ along the ideal $\bigcap_{\xi \in X_N \cap \check{\sigma}} (q^\xi)$. We set $R_\sigma^0 = R_\sigma[1/q^\xi]_{\xi \in \check{\sigma} \cap X_N}$.

By the 'Mumford construction', there exists a semi-abelian scheme with the action of $\mathcal{O}_F$ over $\mathrm{Spec}\, R_\sigma$ extending an HBAV over R_σ^0. The main theorem of [Rap78] for the full level-N structure (and of [Dim04] for the level-$\Gamma_1(N)$ structure, following [Rap78]) says that one can construct an algebraic space by 'gluing' a disjoint union of finitely many 'good algebraic models' (see [CF90, ch. IV, Definition 4.5]) $\coprod_C \coprod_\sigma \mathrm{Spec}\, R_\sigma$ over $\mathbb{Z}[1/N, \zeta_N]$ to $Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]} \times \mathbb{Z}[1/N, \zeta_N]$ and get a proper scheme $X_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N,\zeta_N]}$ over $\mathbb{Z}[1/N, \zeta_N]$ (here ζ_N means the group scheme of Nth roots of unity). In the Hilbert case, the $\mathbb{Q}$-rank is zero; in other words, we deal only with 'totally degenerate cusps', and the subtle analysis from [CF90, pp. 104–106 in ch. IV, § 3] is not necessary. In particular, this scheme is smooth over the base. Also, from [KL05, (1.6.5)] we know that it descends to $X_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}$ over $\mathbb{Z}[1/N]$. We remark that since the torus embeddings are fibrewise open dense by definition, $Y_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}$ is fibrewise open dense in $X_{\Gamma_1(N;\mathfrak{c}),\mathbb{Z}[1/N]}$.

Let $X_{\Gamma_1(N)} \times R$ denote a smooth toroidal compactification $\coprod_{\mathfrak{c} \in \{\mathfrak{c}_1,\ldots,\mathfrak{c}_{h^+}\}} X_{\Gamma_1(N;\mathfrak{c})} \times_{\mathbb{Z}[1/N]} R$ of $Y_{\Gamma_1(N),\mathbb{Z}[1/N]} \times R$ for any $\mathbb{Z}[1/N]$-algebra R as defined in [Rap78] or, in a more precise way, in [Dim04]. It depends on our choice of polyhedral cone decompositions, but we omit this dependence from the notation.

Let $X_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p),\mathbb{Z}[1/N]}$ be a 'partial' toroidal compactification of $Y_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p)} \times \mathbb{Z}[1/N]$ as in [Cha90, 4.5.2]. The partial compactification $X_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p),\mathbb{Z}[1/N,\zeta_N]}$ is obtained from $Y_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p),\mathbb{Z}[1/N,\zeta_N]}$ by compactifying at the (isomorphism classes of) *unramified* $\Gamma_1(N)$ cusps of $Y_{\Gamma_1(N),\mathbb{Z}[1/N,\zeta_N]}$, and it descends to a proper scheme over $\mathbb{Z}[1/N]$ as argued in [KL05, 1.6.5]. We remark that [KL05] uses the partial compactification of level-$\Gamma_1(Np^r)$ Hilbert modular varieties to construct an eigenvariety for Hilbert modular forms. We then let $X_{\Gamma_1(N)\cap\Gamma_0(p)} \times R$ denote the toroidal partial compactification $\coprod_{\mathfrak{c} \in \{\mathfrak{c}_1,\ldots,\mathfrak{c}_{h^+}\}} X_{\Gamma_1(N;\mathfrak{c})\cap\Gamma_0(p)} \times_{\mathbb{Z}[1/N]} R$ of $Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/N]} \times R$, for any $\mathbb{Z}[1/N]$-algebra R.

The morphisms π_1 , $\pi_{2,i}$ $(1 \leqslant i \leqslant g)$ and w_i $(1 \leqslant i \leqslant g)$ as well as the Hecke operators U_{v_i} $(1 \leqslant i \leqslant g)$ on $Y_{\Gamma_1(N)\cap\Gamma_0(p),\mathbb{Z}[1/(Np)]} \times R$ naturally extend to the cusps. We think of these as correspondences and take, for example, their schematic closures.

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For Γ being $\Gamma_1(N)$ or $\Gamma_1(N)\cap\Gamma_0(p)$, let $(\widehat{Y_\Gamma\times\mathbb{Z}_p})^{\mathrm{rig}}$ (respectively, $(\widehat{X_\Gamma\times\mathbb{Z}_p})^{\mathrm{rig}}$) be the generic fibre, in the sense of Raynaud [Ber], of the formal completion $\widehat{Y_\Gamma\times\mathbb{Z}_p}$ (respectively, $\widehat{X_\Gamma\times\mathbb{Z}_p}$) of $Y_\Gamma\times\mathbb{Z}_p$ (respectively, $X_\Gamma\times\mathbb{Z}_p$) along its special fibre; moreover, let $(Y_\Gamma\times\mathbb{Q}_p)^{\mathrm{an}}$ (respectively, $(X_\Gamma\times\mathbb{Q}_p)^{\mathrm{an}}$) be the rigid space, in the sense of [BGR84], associated to the generic fibre $Y_\Gamma\times\mathbb{Q}_p$ (respectively, $X_\Gamma\times\mathbb{Q}_p$). One may think of $(Y_\Gamma\times\mathbb{Q}_p)^{\mathrm{an}}$ as an admissible open subset of $(\widehat{X_\Gamma\times\mathbb{Z}_p})^{\mathrm{rig}}$ containing $(\widehat{Y_\Gamma\times\mathbb{Z}_p})^{\mathrm{rig}}$ via the canonical isomorphism

$$(\widehat{X_{\Gamma_1(N)}\times\mathbb{Z}_p})^{\mathrm{rig}}\simeq(X_{\Gamma_1(N)}\times\mathbb{Q}_p)^{\mathrm{an}}\hookleftarrow(Y_{\Gamma_1(N)}\times\mathbb{Q}_p)^{\mathrm{an}};$$

see [Ber, Proposition 0.3.5].

For simplicity, we shall use the same notation π_1 to represent $(\pi_1)^{\mathrm{an}}$. Similarly, we write $(\pi_{2,i})^{\mathrm{an}}$ as $\pi_{2,i}$ and $(w_i)^{\mathrm{an}}$ as w_i.

Definition. Taking Γ to be $\Gamma_1(N)$ or $\Gamma_1(N)\cap\Gamma_0(p)$, the sheaf ω on $Y_\Gamma\times\mathbb{Q}_p$ is a locally free $(\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{Y_\Gamma\times\mathbb{Q}_p})$-module of rank one and, for $\vec{k}\in\mathbb{Z}^{\mathrm{Hom}(F,\mathbb{R})}$, we define $\omega^{\vec{k}}$ to be the invertible sheaf corresponding to $\imath\vec{k}$, i.e. $\bigotimes_{v|p}\omega_v^{\otimes(\imath\vec{k})_v}$, where by ω_v we mean the invertible sheaf of the $\mathcal{O}_{Y_\Gamma\times\mathbb{Q}_p}$-module obtained by tensoring ω with $\mathcal{O}_{Y_\Gamma\times\mathbb{Q}_p}$ via

$$\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{Y_\Gamma\times\mathbb{Q}_p}\simeq\mathcal{O}_{Y_\Gamma\times\mathbb{Q}_p}^{\mathrm{Hom}(F,\mathbb{Q}_p)}\xrightarrow{\ v\ }\mathcal{O}_{Y_\Gamma\times\mathbb{Q}_p}.$$

Because of Koecher's principle, we may call an element of $H^0((Y_\Gamma\times\mathbb{Q}_p)^{\mathrm{an}},\omega^{\vec{k}})$ a classical (p-adic) Hilbert modular form of weight $\vec{k}$ and level Γ.

6. Overconvergent Hilbert modular forms of level $\Gamma_1(N)$

We shall write down a function v on $(X_{\Gamma_1(N)}\times\mathbb{Q}_p)^{\mathrm{an}}$ and define overconvergence in terms of v. Let $x\in(X_{\Gamma_1(N)}\times\mathbb{Q}_p)^{\mathrm{an}}$ be a point. If it is not a cusp, then it corresponds to a closed point of $Y_{\Gamma_1(N)}\times\mathbb{Q}_p$, and there is a finite extension L of $\mathbb{Q}_p$ such that x corresponds to $(A,\imath,\jmath)$ over L. Consider a model over the ring of integers R of L, equipped with a norm which we normalise so that $|p|=1/p$. Suppose that it has good reduction. Then the Serre–Tate theorem gives that its formal completion $\widehat{A}$ along the identity section on the closed fibre is smooth over R and is equivalent to the connected component of the associated p-divisible group; in particular, it comes with the action of $\mathcal{O}_F$. The underlying ring is a formal group in g parameters and decomposes as the product of g one-parameter formal groups $R[[Y_i]]$, $1\leqslant i\leqslant g$. If we take $X_i=\sum_{\zeta\in\mu}[\zeta]Y_i/\zeta$, where μ is the group of $(p-1)$th roots of unity and $[\cdot]$ denotes the action of $\mathcal{O}_{F_v}\simeq\mathbb{Z}_p$, then $[\zeta]X_i=\zeta X_i$ and we have isomorphisms $R[[X_i]]\simeq R[[Y_i]]$ for all i. We then define $v(x)$ to be the g-tuple $(v_i(x))_{1\leqslant i\leqslant g}$ where each $v_i(x)$ is the minimum of 1 and the (normalised) p-adic variation of the coefficient of X_i^p in $[p]X_i$. Note that although the coefficient itself depends on a choice of parameters, its p-adic valuation does not when less than one and depending only on the HBAV.

If x does not have good reduction, define $v_i(x)$ to be zero for all $1\leqslant i\leqslant g$.

This definition works in families. Let A be the universal HBAV over $Y_{\Gamma_1(N)}\times\mathbb{Z}_p$, and let $\widehat{A}\to\widehat{Y_{\Gamma_1(N)}}$ be the induced map of formal completions along special fibres. Since $X_{\Gamma_1(N)}\times\mathbf{F}_p$ is of finite type, one may choose a finite affine covering for it, $\{U=\mathrm{Spec}\,R\}$. For each U, we let $\widehat{U}$ be an open formal affine subscheme of $\widehat{X_{\Gamma_1(N)}\times\mathbb{Z}_p}$ satisfying $U=\widehat{U}\cap(X_{\Gamma_1(N)}\times\mathbf{F}_p)$. It is then clear that $\{\widehat{U}=\mathrm{Spf}\,R\}$ is a finite formal affine covering of $\widehat{X_{\Gamma_1(N)}\times\mathbb{Z}_p}$, and if we write $\mathrm{sp}^{-1}(\widehat{U})=\mathrm{Sp}(R\otimes_{\mathbb{Z}}\mathbb{Q}_p)$ simply as U^{rig}, it follows from [Ber, Proposition 1.1.14] that $\{U^{\mathrm{rig}}\}$ is an admissible covering of $(X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}$. In fact, $]U[= (X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}} \cap U^{\mathrm{rig}}$ upon identifying $(X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}$ with the Raynuad generic fibre of the formal completion $\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p}$ along its special fibre (see [Ber, Proposition 0.3.5]).

Let $\widehat{U}' = \widehat{U} \times_{\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p}} \widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p}$. By shrinking $\widehat{U}$ if necessary, we can assume that $\{\widehat{U}' = \operatorname{Spf} R'\}$ is an affine formal covering of $\widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p}$ and that $\{U'^{\mathrm{rig}} = \mathrm{sp}^{-1}(\widehat{U}')\}$ is an admissible covering of the Raynaud generic fibre $(\widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p})^{\mathrm{rig}}$ (see [Ber, Proposition 0.2.3(iii)]). By shrinking the covering if necessary, we may assume that $\mathrm{Lie}(\widehat{A})$ is trivialised on each $\widehat{U}'$. Then the formal group associated to $\widehat{A}/\widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p}$, when restricted to $\widehat{U}'$, gives a formal group in g variables with coefficients in R'. Considering the action of $\mathcal{O}_F$, it is isomorphic to the product of g one-parameter formal groups $R'[[X_i]]$, where X_i is normalised as above. The coefficient in R' of X_i^p in $[p]X_i = pX_i + \cdots$ can be thought of as a function on $\widehat{U}'$; we call it $h_i \in \mathcal{O}_{\widehat{U}'}(\widehat{U}')$. One may think of this as a *lift* of the partial Hasse invariant. It follows from Koecher's principle that h_i extends to the cusps and gives an element of $\mathcal{O}_{\widehat{U}}(\widehat{U})$, which we shall again denote by h_i.

A point $x \in (Y_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}$ corresponds to a map

$$x : \operatorname{Sp} L \to (Y_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}} \hookrightarrow (\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{\mathrm{rig}}$$

for a finite extension L over $\mathbb{Q}_p$, and factors through $U^{\mathrm{rig}} \hookrightarrow (\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{\mathrm{rig}}$ for some U^{rig} in the covering. We then alternatively define $v(x)$ to be the g-tuple $(v_i(x))_{1 \leqslant i \leqslant g}$ with $v_i(x) = \min\{1, v_p(x^* h_i^{\mathrm{rig}})\}$ where h_i^{rig} is the rigid analytic function on U^{rig} defined as the image of $h_i \in \mathcal{O}_{\widehat{U}}(\widehat{U})$ under the map

$$\Gamma(\widehat{U}, \mathcal{O}_{\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \to \Gamma(U^{\mathrm{rig}}, (X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}});$$

see [Ber, (0.2.3.4)]. One can check that this does not depend on the choice of covering $\{\widehat{U}\}$.

For a vector $([0, r_i])_{1 \leqslant i \leqslant g}$ of g intervals where $r_i \in p^{\mathbb{Q}}$ and $r_i \in [0, 1)$ for all i, we define for each U^{rig} a rational subdomain $U^{\mathrm{rig}}([0, r_1], [0, r_2], \ldots, [0, r_g])$ in the aforementioned covering of $(X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}$ to be the set of $x \in U^{\mathrm{rig}}$ such that $v(h_i^{\mathrm{rig}}(x)) \in [0, r_i]$ for every $1 \leqslant i \leqslant g$. We then define $X_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g])$ by gluing $U^{\mathrm{rig}}([0, r_1], \ldots, [0, r_g])$. By construction, this is clearly an admissible subset of $(X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}$. Note that, as remarked in [Buz03], this construction does not depend on the choices of parameters for the formal groups. We define $Y_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g])$ to be

$$X_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g]) \times_{(X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}} (Y_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}.$$

Definition. Let $\vec{r} = (r_i)_{1 \leqslant i \leqslant g}$. We say that an element of

$$H^0(Y_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g]), \omega^{\vec{k}})$$

is an $\vec{r}$-overconvergent (p-adic) Hilbert modular form of weight $\vec{k}$ and level $\Gamma_1(N)$. We shall denote the sections $H^0(Y_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g]), \omega^{\vec{k}})$ by $\mathbf{M}_{\vec{k}}(\Gamma_1(N))([0, r_1], \ldots [0, r_g])$. By Koecher's principle [KL05, Lemma 4.1.4], the sections of $\omega^{\vec{k}}$ extend to sections over the quasi-compact $X_{\Gamma_1(N)}([0, r_1], \ldots, [0, r_g])$, and this naturally gives a Banach space. See the remark in [KL05, (2.4)]. Alternatively, for $1 \leqslant i \leqslant g$, the collection of sections $H^0(Y_{\Gamma_1(N)}([0, r_1], \ldots, [0, r_g]), \omega_i^{k_i})$, which we think of as

$$H^0(X_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g]), \omega_i^{k_i}),$$

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has a natural norm (in the natural equivalence class) that makes it a Banach space. More precisely, for $f \in \mathbf{M}_{k_i}(\Gamma_1(N))([0, r_1], \ldots, [0, r_g])$, set $|f|_{\vec{r}} = \sup_{x \in X_{\Gamma_1(N)}([0,r_1],\ldots,[0,r_g])} |f|_x$ with $|f|_x$ being the p-adic norm of $(f/h_i)(x)$, where h_i is as in the definition of overconvergent modular forms above; we are deriving a Banach space structure by mapping the space into the Banach space of overconvergent functions on $X_{\Gamma_1(N)}([0, r_1], \ldots, [0, r_g])$. Finally, since all the $H^0(X_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g]), \omega_i^{k_i})$ are complete,

$$\mathbf{M}_{\vec{k}}(\Gamma_1(N))([0, r_1], \ldots, [0, r_g]) = \bigotimes H^0(X_{\Gamma_1(N)}([0, r_1], [0, r_2], \ldots, [0, r_g]), \omega_i^{k_i})$$

rather than their completed tensor product, which makes it a Banach space.

The union of all $\vec{r}$-overconvergent modular forms of weight $\vec{k}$ and level N over all $\vec{r}$ satisfying $r_i \in (0, 1)$ for all i is called the space of overconvergent (p-adic) Hilbert modular forms of weight $\vec{k}$ and level $\Gamma_1(N)$.

7. Overconvergent Hilbert modular forms of level $\Gamma_1(N) \cap \Gamma_0(p)$

In this section, we define functions

$$v' = (v'_i)_{1 \leqslant i \leqslant g} : (X_{\Gamma_1(N) \cap \Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}} \to [0, 1]^g$$

and overconvergent modular forms of level $\Gamma_1(N) \cap \Gamma_0(p)$ in terms of v'.

If a point $x \in (X_{\Gamma_1(N) \cap \Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}}$ is not a cusp, it corresponds to a closed point of $Y_{\Gamma_1(N) \cap \Gamma_0(p)} \times \mathbb{Q}_p$ and hence there is a finite extension L of $\mathbb{Q}_p$ equipped with the valuation, normalised so that $|p| = 1/p$, such that x corresponds to a L-valued point of $Y_{\Gamma_1(N) \cap \Gamma_0(p)} \times \mathbb{Q}_p$. Suppose that it has good reduction. Let A be the model over the ring of integers R of L of the corresponding HBAV over L, which comes equipped with the action of $\mathcal{O}_F$ lifting the action on the generic fibre. By taking the schematic closure in $A[p]$, A has an $\mathcal{O}_F$-stable finite flat group scheme C over R. For brevity, we shall denote the quotient A/C by B. The isogeny $A \to B$ induces a map of locally free sheaves $\mathrm{Lie}^\vee B \to \mathrm{Lie}^\vee A$. It is $\mathcal{O}_F$-linear and decomposes as the sum of morphisms $(\mathrm{Lie}^\vee B)_i \to (\mathrm{Lie}^\vee A)_i$, where $(\mathrm{Lie}^\vee A)_i$ and $(\mathrm{Lie}^\vee B)_i$ are both locally free $\mathcal{O}_{\mathrm{Spec}\, R}$-modules of rank one and $\mathcal{O}_F$ acts by τ_i. We may assume that $(\mathrm{Lie}^\vee A)_i$ and $(\mathrm{Lie}^\vee B)_i$ are simultaneously trivialised on $\mathrm{Spec}\, R$, in which case, for each $1 \leqslant i \leqslant g$, the Fitting ideal [MW84] of its cokernel is generated by one element, $g_i \in R$, say. We then define $v'(x)$ to be the g-tuple $(1 - v(g_i))_{1 \leqslant i \leqslant g}$.

The isogeny $A \to B$ induces a homomorphism of the formal groups over R; it is $\mathcal{O}_F$-linear and decomposes as the product of g homomorphisms $R[[X_i]] \to R[[Y_i]]$ of one-parameter formal groups with the normalised parameters X_i and Y_i, as before. In fact, it sends X_i to $Y_i = \prod_x G(X_i, x)$, where x is a point in C_i and G denotes the group law on $R[[X_i]]$ (see [Lub67]). The annihilator of the cokernel of the induced map of the R-modules of the invariant differential forms on the formal groups is the first derivative at $X_i = 0$ of the homomorphism with respect to X_i, and its p-adic valuation is therefore $(p-1)v(c)$ with $c \in \{c_0 = 0, c_1, \ldots, c_{p-1}\} = C_i(\overline{L})$. As a result, we have $v'_i(x) = 1 - (p-1)v(c)$.

A more conceptual way of thinking about v' is as follows. One can check that locally on $U = \mathrm{Spec}\, \mathcal{O}_U$, the Fitting ideal of the cokernel of $(\mathrm{Lie}^\vee B)_i \to (\mathrm{Lie}^\vee A)_i$ is isomorphic to the Fitting ideal of $(\mathrm{Lie}^\vee C)_i$, which equals the Fitting ideal of R/δ_i if C_i, over U, is of the form $\mathrm{Spec}\, \mathcal{O}_U[T]/(T^p - \delta_i T)$, i.e. if δ_i is the Oort–Tate or, more generally, the Raynaud parameter of C_i. Note that $(\mathrm{Lie}^\vee C)_i \simeq \mathrm{Lie}^\vee C_i$.

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If the model A over R does not have good reduction, i.e. if x is a cusp, we define $(v_i'(x))_{1\leqslant i\leqslant g}$ to be that of any ordinary point in the same component.

More generally, and possibly more amenable to generalisation, one may paraphrase the above in terms of cotangent complexes. Let A and $B=A/C$ be HBAVs over $\operatorname{Spec} R$ as above; in particular, they are smooth and locally complete intersections over R. Let $L_{C/\operatorname{Spec} R}$ denote the cotangent complex of C over $\operatorname{Spec} R$ in the derived category $\mathbf{D}(\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\operatorname{Spec} R})$ (see [Ill71, Ill72]) of complexes of $(\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\operatorname{Spec} R})$-modules. If we let e be the unit section $\operatorname{Spec} R\to C$, the complex $l_{C/\operatorname{Spec} R}:=\mathbf{L}e^*L_{C/\operatorname{Spec} R}$ of $(\mathcal{O}_F\otimes_{\mathbb{Z}}\mathcal{O}_{\operatorname{Spec} R})$-modules that are locally free as $\mathcal{O}_{\operatorname{Spec} R}$-modules is perfect and concentrated in degrees $[-1,0]$. It turns out to be isomorphic to the two-term complex $0\to\operatorname{Lie}^\vee B\to\operatorname{Lie}^\vee A\to 0$ and, by taking 'determinant divisors', one can deduce the same result.

For (A,C) as above, we fix $i\in[0,g]$, take $A'=A/C_i$ and let $C'\subset A'$ be the $\mathcal{O}_F$-stable subgroup of $A[p]$ of order p^g defined by $C_j'=(C_j+C_i)/C_i$ for all $j\neq i$ and $C_i'=A[v_i]/C_i$. Note that $\operatorname{Fit}_R(\operatorname{Lie}^\vee C_i)\cdot\operatorname{Fit}_R(\operatorname{Lie}^\vee C_i')\subset\operatorname{Fit}_R((\operatorname{Lie}^\vee A[p])_i)$ (cf. [MW84, Appendix]), and since $\operatorname{Lie}^\vee(A[p])$ is a locally free $((\mathcal{O}_F/p)\otimes_{\mathbb{Z}}\mathcal{O}_{\operatorname{Spec} R})$-module of rank one, the right-hand side is the R-module generated by p and we have $v_i'(A,C)+v_i'(A',C')=1$.

For a vector $([0,r_i])_{1\leqslant i\leqslant g}$ of g intervals where $r_i\in p^{\mathbb{Q}}$ and $r_i\in[0,1)$ for all i, we define an admissible subset $Y_{\Gamma_1(N)\cap\Gamma_0(p)}([0,r_1],\dots,[0,r_g])$ (respectively, $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0,r_1],\dots,[0,r_g])$) of $(Y_{\Gamma_1(N)\cap\Gamma_0(p)})^{\mathrm{an}}$ (respectively, $(X_{\Gamma_1(N)\cap\Gamma_0(p)})^{\mathrm{an}}$) to be the component containing the points whose v_i' are all zero of the inverse image under π_1 of $Y_{\Gamma_1(N)}([0,r_1],\dots,[0,r_g])$ (respectively, $X_{\Gamma_1(N)}([0,r_1],\dots,[0,r_g])$). One can check that this coincides with the set of points $x\in(Y_{\Gamma_1(N)\cap\Gamma_0(p)})^{\mathrm{an}}$ such that $v_i'(x)\in[0,r_i]$ for all $1\leqslant i\leqslant g$.

Definition. Let $\vec{r}=(r_i)_{1\leqslant i\leqslant g}$. We define an element of

$$H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p)}([0,r_1],[0,r_2],\dots,[0,r_g]),\omega^{\vec{k}})$$

to be a $\vec{r}$-overconvergent (p-adic) Hilbert modular form of weight $\vec{k}$ and level $\Gamma_1(N)\cap\Gamma_0(p)$. We shall denote the sections

$$H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p)}([0,r_1],[0,r_2],\dots,[0,r_g]),\omega^{\vec{k}})$$

by

$$\mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))([0,r_1],\dots[0,r_g]).$$

This is a Banach space. The union of all $\vec{r}$-overconvergent modular forms of weight $\vec{k}$ and level N over all $\vec{r}$ satisfying $r_i\in(0,1)$ for all i is called the space of overconvergent (p-adic) Hilbert modular forms of weight $\vec{k}$ and level $\Gamma_1(N)\cap\Gamma_0(p)$.

We remark that if we let $(Y_{\widehat{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathbb{Z}_p})^{\mathrm{rig}}([0,r_1],\dots,[0,r_g])$ be the admissible open subset of the points $x\in(Y_{\widehat{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathbb{Z}_p})^{\mathrm{rig}}$ such that $v_i'(x)\in[0,r_i]$ for all $1\leqslant i\leqslant g$, then we have natural morphisms

$$\begin{aligned}H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([0,r_1],\dots,[0,r_g]),\omega^{\vec{k}})&\to H^0(Y_{\Gamma_1(N)\cap\Gamma_0(p)}([0,r_1],\dots,[0,r_g]),\omega^{\vec{k}})\\&\to H^0((Y_{\widehat{\Gamma_1(N)\cap\Gamma_0(p)}\times\mathbb{Z}_p})^{\mathrm{rig}}([0,r_1],\dots,[0,r_g]),\omega^{\vec{k}}),\end{aligned}$$

which are isomorphisms. To check this, follow [KL05, proof of Lemma 4.1.4]. Henceforth, we tacitly extend overconvergent modular forms to the cusps.

https://doi.org/10.1112/S0010437X09004552 Published online by Cambridge University Press

8. Canonical subgroups of Hilbert–Blumenthal abelian varieties

In this section, we extend the notion of canonical subgroups of elliptic curves over p-adically complete rings, as in the work of Katz [Kat73], to the case of abelian varieties with the action of real multiplication $\mathcal{O}_F$.

Definition. Let A be an HBAV over a p-adically complete $\mathbb{Z}_p$-algebra R, and assume that $0 < v_i(A) < p/(p+1)$ for $1 \leqslant i \leqslant g$. Then the identity component of the p-divisible group $A[v_i^\infty]$ is formally smooth of dimension one, and we let $R[[X_i]]$ denote the underlying formal group with one variable X_i, normalised as before. We define the canonical subgroup H_i of A to be the finite flat group scheme of order p in the p-torsion subgroup of $R[[X_i]]$, defined by the equation $X_i^p - t_{\mathrm{can}} X_i$ as in Katz's paper [Kat73]. If $v_i(A) = 0$, then $h_i(A)$ is invertible and the ith component of the kernel of Frobenius in the special fibre of A gives a finite flat group scheme of order p. Since its dual is étale, one can lift the dual to R by Hensel's lemma, and the dual of such an object is the canonical subgroup H_i in this case.

The following lemma will be needed. For an HBAV A over R as above such that $v_i(A) \in [0, p/(p+1))$, we follow [Kat73] and write $p/h_i(A)$ as r_1. Then $r_1 \in R$ and $v(r_1) > 1/(p+1)$, and we have the following result.

Lemma 2. *Modulo r_1, the canonical subgroup H_i of A is the kernel of Frobenius.*

Proof. Note that, modulo r_1, the canonical subgroup is defined by X_i^p in $\ker[p]$. In the notation of [Kat73], $r_1 = 0$ in R/r_R implies that $t_0 = 0$, and thus $t_{\mathrm{can}} = 0 \bmod r_1$. □

Definition. Let L be a finite extension of $\mathbb{Q}_p$ and let A be an HBAV over L. We say that A over L is not too v_i-non-ordinary if, after changing base to a finite extension L' of L as necessary, $v_i(A \times_{\mathrm{Spec}\, L} \mathrm{Spec}\, L') < p/(p+1)$.

One can readily generalise the argument of Katz in [Kat73, Theorem 3.10.7] to associate a canonical subgroup H_i to a not too v_i-non-ordinary HBAV A, since this essentially involves calculations with formal groups. It is then easy to deduce the following.

Lemma 3. *Let A be an HBAV over a finite extension L of $\mathbb{Q}_p$.*

(i) *If $v_i(A) = 0$, then the canonical subgroup H_i of A is the finite étale subgroup of $A[v_i]$.*

(ii) *If $v_i(A) < 1/(p+1)$, then A/H_i is not too v_i-non-ordinary and $v_i(A/H_i) = pv_i(A)$ while $v_j(A/H_i) = v_j(A)$ for all j with $1 \leqslant j \leqslant g, j \neq i$.*

(iii) *If $v_i(A) = 1/(p+1)$, then A/H_i is too v_i-non-ordinary.*

(iv) *If $1/(p+1) < v_i(A) < p/(p+1)$, then A/H_i is not too v_i-non-ordinary and $v_i(A/H_i) = 1 - v_i(A)$ while $v_j(A/H_i) = v_j(A)$ for all j with $1 \leqslant j \leqslant g, j \neq i$. Furthermore, the canonical subgroup of A/H_i is $A[v_i]/H_i$.*

(v) *If $v_i(A) < p/(p+1)$ and $C_i \subset A[v_i]$ is a finite subgroup not equal to H_i and of order p, then $v_i(A/C_i) = v_i(A)/p$ while $v_j(A/C_i) = v_j(A)$ for all j with $1 \leqslant j \leqslant g, j \neq i$. The canonical subgroup of A/C_i is $A[v_i]/C_i$.*

(vi) *If $p/(p+1) \leqslant v_i(A)$ and $C_i \subset A[v_i]$ is a finite flat subgroup of order p, then $v_i(A/C_i) = 1/(p+1)$ and the canonical subgroup of A/C_i is $A[v_i]/C_i$.*

Remark. Let $(A, C, \imath)$ over R correspond to a point of $(X_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}}$. One can check that $v_i'(A, C) = v_i(A)$ if A has the canonical subgroup and $C_i = H_i$. In fact, this follows from [Kat73].

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Note that it is impossible that $C_i \subset A[v_i]$ and $A[v_i]/C_i \subset A[p]/C_i$ are *not* canonical at the same time; therefore, if $v_i'(A, C) < 1/(p+1)$, A has to have the canonical subgroup and $C_i = H_i$; on the other hand, if $v_i'(A, C) > p/(p+1)$, then C_i cannot be the canonical subgroup.

9. Analytic continuation of overconvergent eigenforms

For $\vec{I} = (I_i)_{1 \leqslant i \leqslant g}$ where each $I_i \subseteq [0, 1]$ is an interval of the form $[0, r_i)$ or $[0, r_i]$ for $0 < r_i \leqslant 1$, we let $X_{\Gamma_1(N)\cap\Gamma_0(p)}(\vec{I})$ denote the set of points $x \in (X_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}}$ such that $v_i(x) \in I_i$ for all $1 \leqslant i \leqslant g$ and let $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))(\vec{I})$ denote the space $H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}(\vec{I}), \omega^{\vec{k}})$ of overconvergent modular forms of weight $\vec{k}$.

Let f be an overconvergent Hilbert modular form over $\mathbb{Q}_p$ of weight $\vec{k}$ and level $\Gamma_1(N) \cap \Gamma_0(p)$. Then there exists a g-tuple $\vec{r} = (r_i)_{1 \leqslant i \leqslant g}$ such that $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, r_1], \ldots, [0, r_g])$.

9.1 Analytic continuation of overconvergent U_v-eigenforms, I

Fix $1 \leqslant i \leqslant g$. We shall take $X_{\Gamma_1(N)\cap\Gamma_0(p)}(I_i)$ to mean $X_{\Gamma_1(N)\cap\Gamma_0(p)}(\vec{I})$ and $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))(I_i)$ to mean $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))(\vec{I})$. Similarly for $X_{\Gamma_1(N)}$.

Proposition 4. *The $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n-1}(p+1))])$, $n \in \mathbb{Z}_{\geqslant 0}$, form an admissible covering of $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1))$.*

Proof. It essentially follows from the maximum modulus principle [BGR84] that these sets form an admissible covering of the admissible open set $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1))$. See, for example, [BGR84, 9.1.4, Proposition 5]. □

For π_1 and

$$\pi_{2,v} : (X_{\Gamma_1(N)\cap\Gamma_0(p)\cap\Gamma^0(v)} \times \mathbb{Q}_p)^{\mathrm{an}} \to (X_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}}$$

as above, if U and V are admissible open subsets of $(X_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}}$ such that $(\pi_1)^{-1}(U) \subseteq (\pi_{2,v})^{-1}(V)$, one can define the 'trace map' $\mathrm{tr}_v : H^0(V, \omega^{\vec{k}}) \to H^0(U, \omega^{\vec{k}})$ as in [Buz03, § 5]; if, furthermore, U and V satisfy $V \subseteq U$, we define U_v to be $p^{-1}\mathrm{tr}_v$ followed by the restriction map $H^0(U, \omega^{\vec{k}}) \to H^0(V, \omega^{\vec{k}})$.

Proposition 5. *If $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, r])$ is an eigenform for U_v with non-zero eigenvalue a, then it extends to an eigenform in $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, 1))$.*

In order to prove this, we need two lemmas.

Lemma 6. *Let $(A, C, \imath)$ correspond to a point x in $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^n(p+1))])$. Let D_i be a subgroup of $A[v_i]$ of order p which does not meet C non-trivially. If we let y denote the point corresponding to $(A/D_i, (C + D_i)/D_i, \imath \bmod D_i)$, then $y \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n-1}(p+1))])$.*

Proof. This follows from case-by-case calculations as in [Buz03, Lemma 4.2]. □

Lemma 7. *If $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, r_i])$ is an eigenform for U_{v_i} with non-zero eigenvalue a_i, then it extends to an eigenform in $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, 1 - 1/(p^{-1}(p+1))])$.*

Proof. If $0 < v_i'(A, C) < p/(p+1)$, then

$$v_i'(A, C) > v_i'(A/D_i, (C + D_i)/D_i) = v_i'(A, C)/p$$

for a subgroup D_i of $A[v_i]$ of order p such that $D_i \neq C_i$. To prove this, observe that since $C_i = H_i$, we have $v_i'(A, C) = v_i(A)$ (and also $v_i'(A/D_i, (C+D_i)/D_i) = v_i'(A/D_i)$); it then follows from Lemma 3(v) that $v_i(A) > v_i(A/D_i) = v_i(A)/p$. If $r_i \geqslant 1/(p+1)$, there is nothing to prove, so we assume that $r_i < 1/(p+1)$. Suppose that $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, t_i])$ for some $r_i \leqslant t_i < 1/(p+1)$ and that it is an eigenform for U_{v_i} with eigenvalue a_i. Then

$$(f|p^{-1}\mathrm{tr}_{v_i})/a_i \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, pt_i]),$$

since for $(A, C, \imath) \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, pt_i])$ we have by definition that

$$((f|p^{-1}\mathrm{tr}_{v_i})/a_i)(A, C, \imath) = 1/(a_i p) \sum_{D_i} (\mathrm{pr}_i)^* f(A/D_i, (C+D_i)/D_i, \imath \bmod D_i),$$

where pr_i denotes $A \to A/D_i$. Since f is an eigenform for U_{v_i}, the restriction to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, t_i])$ of $f|p^{-1}\mathrm{tr}_{v_i}$ is $a_i f$; in other words, the restriction of $(f|p^{-1}\mathrm{tr}_{v_i})/a_i$ to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, t_i])$ is f, and therefore $f|p^{-1}\mathrm{tr}_{v_i}$ extends f. Since the trace map and the restriction map 'commute', as observed in [Kas09, Lemma 2.18(1)], one can check that $f|p^{-1}\mathrm{tr}_{v_i}$ is also an eigenform for U_{v_i} with eigenvalue a_i. Repeating the argument gives the result. □

Remark. In [Buz03], the analytic continuation of overconvergent eigenforms is typically proved by first showing that admissible open subsets, over which overconvergent modular forms are defined, are connected, and then using the q-expansion principle at cusps to ensure that $f|p^{-1}\mathrm{tr}$ does in fact extend f because they have the same q-expansions. In calculating q-expansions, $p^{-1}\mathrm{tr}$ is not any different from U_p. Kassaei, however, observes in [Kas09, § 3] that establishing that admissible subsets are connected is not actually necessary; this substantially simplifies the argument in [Buz03]: in fact, one does not need to compute q-expansions! Although almost all the admissible subsets that we consider in this paper are undoubtedly connected, we shall follow Kassaei's approach.

Proof of Proposition 5. It follows from the preceding lemma that the Hecke operator $p^{-1}\mathrm{tr}_{v_i}$ defines a map of sections

$$\begin{aligned} p^{-1}\mathrm{tr}_{v_i} : H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n-1}(p+1))]), \omega^{\vec{k}}) \\ \to H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n}(p+1))]), \omega^{\vec{k}}). \end{aligned}$$

More explicitly, for an element

$$\begin{aligned} f_n &\in H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n-1}(p+1))]), \omega^{\vec{k}}) \\ &= \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, 1 - 1/(p^{n-1}(p+1))]) \end{aligned}$$

and for $(A, C, \imath) \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n}(p+1))])$,

$$f_n|p^{-1}\mathrm{tr}_{v_i} \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, 1 - 1/(p^{n}(p+1))])$$

is defined by

$$(f_n|p^{-1}\mathrm{tr}_{v_i})(A, C, \imath) = (1/p) \sum_{D_i \neq C_i} (\mathrm{pr}_i)^* f_n(A/D_i, (C+D_i)/D_i, \imath \bmod D_i).$$

By the preceding lemma, $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, r_i])$ extends to a U_{v_i}-eigenform $f_0 \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, 1/(p+1)])$. Inductively, we define $f_{n+1} \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0, 1 - 1/(p^{n}(p+1))])$ by $(f_n|p^{-1}\mathrm{tr}_{v_i})/a_i$ for all $n \in \mathbb{Z}_{\geqslant 0}$, which is an U_{v_i}-eigenform with eigenvalue a_i when restricted to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1 - 1/(p^{n-1}(p+1))])$. Since f_n is an eigenform for U_{v_i},

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the restriction of f_{n+1} to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1-1/(p^{n-1}(p+1))])$ is therefore f_n, and hence f_{n+1} extends f_n. Since the

$$\{X_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1-1/(p^{n-1}(p+1))]\}_{n\in\mathbb{Z}_{\geqslant 0}}$$

form an admissible covering of $X_{\Gamma_1(N)\cap\Gamma_0(p)}([0,1))$, this inductive construction gives an element of $\mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))([0,1))$ that extends f; it is still an eigenform for U_{v_i} with eigenvalue a_i. □

Corollary 8. *If f is an overconvergent Hilbert modular form of weight $\vec{k}$ which is an eigenform for U_v with non-zero eigenvalue for all $v|p$, then it extends to an eigenform in $\mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))([0,1),\ldots,[0,1))$.*

9.2 Analytic continuation of overconvergent U_v-eigenforms, II

Fix any $0\leqslant t_i < 1/(p+1)$. For convenience, we use$Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1]])$ to represent

$$Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1)) \quad \text{or } Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1]),$$

since the following construction applies to both of these. Similarly for $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1]])$.

Define

$$s_i : Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_1/p, 1]]) \to Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_1, 1]])$$

as taking $(A, C, \imath)$ to $(A/H_i, (C+H_i)/H_i, \imath \bmod H_i)$ where $H_i \subset A[v_i]$ is the canonical subgroup of A. It is important to have $C_i \neq H_i$; see the end of the last section. Note that s_i extends to

$$X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p)]) \to X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i)]).$$

At a cusp with a choice of polyhedral cone decomposition $\{\sigma\}$, for an $\mathcal{O}_F$-stable semi-abelian scheme G_σ over $\mathrm{Spec}(R_\sigma\otimes\mathbb{Z}_p)$ extending a $\mathfrak{c}$-polarised HBAV A_σ over $\mathrm{Spec}(R_\sigma^0\otimes\mathbb{Z}_p)$, s_i corresponds to the isogeny $A_\sigma \to A_\sigma/H_{\sigma,i}$ where $H_{\sigma,i}$ is the kernel of the morphism of p-divisible groups $A_\sigma[v_i^\infty]\to A_\sigma[v_i^\infty]$ corresponding to $q\mapsto q^p$ in the base. Since $X_N\cap\check{\sigma}\ni\xi\mapsto\xi p$ leaves $X_N\cap\check{\sigma}$ stable, the isogeny extends to G_σ, and consequently s_i extends to

$$X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p)]) \to X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i)]),$$

which we shall again denote by s_i.

Define

$$\mathrm{id} : X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1]]) \to X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1]])$$

as taking $(A, C, \imath)$ to itself.

We shall define the operator

$$V_i : \mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))[1-t_i, 1)] \to \mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))([1-t_i/p, 1)])$$

as follows. Let p_i^* denote the natural morphism of sheaves $s_i^*\omega\to(\mathrm{id})^*\omega$ on $Y_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1-t_i/p])$ which takes the universal HBAV over $Y_{\Gamma_1(N)\cap\Gamma_0(p)}([0, 1-t_i/p])$ to its quotient by the canonical subgroup in the v_i-torsion points. It clearly extends to the cusps. We shall use the same notation for the variant $s_i^*\omega^{\vec{k}}\to(\mathrm{id})^*\omega^{\vec{k}}$. Now define V_i to be the composite

$$\begin{aligned} V_i : \mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))[1-t_i, 1)] &\xrightarrow{s_i^*} H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)]), s_i^*\omega^{\vec{k}}) \\ &\xrightarrow{p^{-k_i}p_i^*} H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)]), (\mathrm{id}^*)\omega^{\vec{k}}) \\ &\longrightarrow \mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))[1-t_i/p, 1)]. \end{aligned}$$

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Let $x \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)])$ be a point corresponding to $\mathrm{Sp}\, L \to X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)])$ for some finite extension L of $\mathbb{Q}_p$. Denote by x_1 (respectively, x_2) the composite

$$\mathrm{Sp}\, L \xrightarrow{x} X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)])$$
$$\xrightarrow{\text{id (respectively, } s_i)} X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1)])(\hookrightarrow (X_{\Gamma_1(N)\cap\Gamma_0(p)} \times \mathbb{Q}_p)^{\mathrm{an}}).$$

Upon specialisation to x, the map p_i^* induces a map of the sections,

$$p_i^* : H^0(\mathrm{Sp}\, L, x_2^*\omega^{\vec{k}}) \to H^0(\mathrm{Sp}\, L, x_1^*\omega^{\vec{k}}),$$

and one can check that for $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i, 1)])$,

$$f|V_i \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i/p, 1)])$$

satisfies $(f|V_i)(x_1) = p^{-k_i}p_i^*(x_2^*f) = p^{-k_i}p_i^*f(x_2)$. In other words, if x corresponds to a point $(A, C, \imath)$, then

$$(f|V_i)(A, C, \imath) = p^{-k_i}\mathrm{pr}_i^* f(A/H_i, (C+H_i)/H_i, \imath \bmod H_i),$$

where by pr_i^* we mean the pull-back by the isogeny $\mathrm{pr}_i : A \to A/H_i$.

Lemma 9. *Let $x \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)])$ be a point corresponding to*

$$\mathrm{Sp}\, L \to X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)])$$

for some finite extension L over $\mathbb{Q}_p$. Let h_i be an element of L such that $|h_i| = |h_i(\pi_1(x))|$, and let $x_1 \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1)])$ (respectively, $x_2 \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1)])$) be the image of x under id *(respectively, s_i). Then, for $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i, 1)])$, we have $|(f|V_i)(x_i)| \leqslant |f(x_2)||h_i|^{-k_i}$ with respect to the normalised norm on L.*

Proof. If x is a point in $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1])$, its image in $(X_{\Gamma_1(N)} \times \mathbb{Q}_p)^{\mathrm{an}}$ under $\pi_{2,i}$ has $v_i(x)$ satisfying $0 \leqslant v_i(x) \leqslant t_i/p < p/(p+1)$. Let x correspond to an HBAV over a finite extension L of $\mathbb{Q}_p$, with residue field k, which has a model A over the ring of integers $\mathcal{O}_L$. Assume first that A has good reduction. Then A has the canonical subgroup $H_i \subset A[v_i]$ and we let $B_i = A/H_i$. Observe that $p_i^* : s_i^*\omega^{\vec{k}} \to (\mathrm{id})^*\omega^{\vec{k}}$ induces the map of sections $p_i^* : H^0(B_i, \Omega_{B_i}^{\vec{k}}) \to H^0(A, \Omega_A^{\vec{k}})$ of sheaves of relative differentials over $\mathrm{Spec}\, \mathcal{O}_L$. We claim that the 'unit ball' sections

$$H^0(B_i[v_i^\infty], \Omega^{k_i}_{B_i[v_i^\infty], v_i}) \subset H^0(B_i[p^\infty], \Omega^{k_i}_{B_i[p^\infty], v_i}) \subset H^0(B_i, \Omega^{\vec{k}}_{B_i})$$

of the sheaf of relative differentials of the p-divisible group $B_i[v_i^\infty]$ over $\mathrm{Spf}\, \mathcal{O}_L$ on which $\mathcal{O}_F$ acts by $\mathcal{O}_F \hookrightarrow \mathcal{O}_{F_{v_i}} \to \mathcal{O}_L$ map under p_i^* to $(p/h_i)^{k_i} H^0(A[v_i^\infty], \Omega^{k_i}_{A[v_i^\infty], v_i}) \subset H^0(A, \Omega^{k_i}_{A/S})$. This follows from observing that the map $\Omega_{B_i[v_i^\infty], v_i} \to \Omega_{A[v_i^\infty], v_i}$ reduces modulo p/h_i (see Lemma 2) to $\varphi^*\Omega_{(A[v_i^\infty]\times k)^{(p)}, v_i} \to \Omega^{k_i}_{A[v_i^\infty]\times k, v_i}$ where φ is the relative Frobenius, which is zero. For

$$f = \bigotimes_{j=1}^{g} f_j \in H^0\left(Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1]), \bigotimes_{j=1}^{g} \omega_{v_j}^{k_j}\right),$$

if $|f_i(x_2)| \leqslant 1$, then $f_i(x_2) \in H^0(B_1[v_1^\infty], \Omega^{k_1}_{B[v_i^\infty], v_i})$ and therefore

$$|(f_i|V_i)(x_1)| = |p^{-k_i}p_i^* f_i(x_1)| = |p^{-k_i}(p/h_i)^{k_i} f_i(x_1)| \leqslant |h_i|^{-k_i},$$

since $f_i(x_1) \in H^0(A[v_i^\infty], \Omega^{k_i}_{A[v_i^\infty], v_i})$ and so $|f_i(x_1)| \leqslant 1$, which, as in [Kat73], suffices.

Secondly, if A does not have good reduction, then by definition we have $v_i'(x) = 1$. We denote by $(Y_{\Gamma_1(N)} \times \mathbf{F}_p)^{v_i\, \mathrm{ord}}$ (respectively, $(X_{\Gamma_1(N)} \times \mathbf{F}_p)^{v_i\, \mathrm{ord}}$) the open subscheme of $Y_{\Gamma_1(N)} \times \mathbf{F}_p$

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(respectively, $X_{\Gamma_1(N)} \times \mathbf{F}_p$) where the ith partial Hasse invariant (respectively, the extension by Koecher's principle of the ith partial Hasse invariant) does *not* vanish. We also let $(\widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}}$ (respectively, $(\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}}$) denote the formal completion of $Y_{\Gamma_1(N)} \times \mathbb{Z}_p$ (respectively, $X_{\Gamma_1(N)} \times \mathbb{Z}_p$) along $(Y_{\Gamma_1(N)} \times \mathbf{F}_p)^{v_i \,\mathrm{ord}}$ (respectively, $(X_{\Gamma_1(N)} \times \mathbf{F}_p)^{v_i \,\mathrm{ord}}$).

Let x_1 (respectively, x_2) again denote the composition

$$\begin{aligned}\mathrm{Sp}\, L \xrightarrow{x} Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1]) &\xrightarrow{\text{id (respectively } s_i)} Y_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1]) \\ &\xrightarrow{\pi_1} X_{\Gamma_1(N)}([0,0]) \simeq ((\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}})^{\mathrm{rig}},\end{aligned}$$

let $\widehat{x_1}$ (respectively, $\widehat{x_2}$) denote its formal model $\mathrm{Spf}\,\mathcal{O}_L \to (\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{\mathrm{ord}}$, and let $\overline{x}_1$ (respectively, $\overline{x}_2$) denote its specialisation in $(X_{\Gamma_1(N)} \times \mathbf{F}_p)^{\mathrm{ord}}$. Note that $\overline{x}_1$ and $\overline{x}_2$ lie in the complement of $Y_{\Gamma_1(N)} \times \mathbf{F}_p$ in $X_{\Gamma_1(N)} \times \mathbf{F}_p$ or, in other words, the cusps of $Y_{\Gamma_1(N)} \times \mathbf{F}_p$.

Now, p_i^* induces $x_2^*\omega \to x_1^*\omega$ upon specialising to x, and it suffices to establish that

$$H^0(\mathrm{Spf}\,\mathcal{O}_L, (\widehat{x_2})^*\omega) \subset H^0(\mathrm{Spf}\,\mathcal{O}_L, (\widehat{x_2})^*\omega) \otimes L \simeq H^0((\mathrm{Spf}\,\mathcal{O}_L)^{\mathrm{rig}}, x_2^*\omega) \simeq H^0(\mathrm{Sp}\,L, x_2^*\omega)$$

maps under p_i^* to $pH^0(\mathrm{Spf}\,\mathcal{O}_L, (\widehat{x_1})^*\omega) \subset H^0(\mathrm{Sp}\,L, x_1^*\omega)$. This follows from observing that p_i^* induces the zero morphism on $(\widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}} \times \mathbf{F}_p$ by the preceding argument and that, since $(\widehat{Y_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}} \times \mathbf{F}_p$ is open dense in $(\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}} \times \mathbf{F}_p$, p_i^* has to be zero on $(\widehat{X_{\Gamma_1(N)} \times \mathbb{Z}_p})^{v_i \,\mathrm{ord}} \times \mathbf{F}_p$. The rest of the argument is analogous to that in the first case. □

We shall prove that if $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0,1))$ is an eigenvactor for U_{v_i} with non-zero eigenvalue a_i such that $v(a_i) < k_i - 1$, then f extends to $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0,1])$. Fix $t_i \in (0, p/(p+1))$. For simplicity, we shall typically use f to mean the restriction of $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([0,1))$ to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i, 1)])$. Write $b_i = p^{k_i-1}/a_i$ for brevity. We then have the following lemma.

Lemma 10. *$f - b_i(f|V_i) \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i/p, 1))$ extends to $\mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i/p, 1])$.*

Proof. On the non-cuspidal points, simply define $f' \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i/p, 1])$ to be

$$f'((A, C, \imath)) = 1/(a_i p) \sum_{D_i \neq H_i, D_i \neq C_i} p_i^* f((A/D_i, (C+D_i)/D_i, \imath \bmod D_i))$$

and follow the remark at the end of § 7 to extend to the cusps. Then, for

$$(A, C, \imath) \in X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1)),$$

we have

$$\begin{aligned}f'(A, C, \imath) &= (1/a_i)(f|U_{v_i})(A, C, \imath) - (1/(a_i p)) f(A/H_i, (C+H_i)/H_i, (\imath \bmod H_i)) \\ &= (f - (p^{k_i-1}/a_i) f|V)(A, C, \imath).\end{aligned}$$

□

For $n \in \mathbb{Z}_{\geqslant 1}$, define $g_n \in \mathbf{M}_{\vec{k}}(\Gamma_1(N) \cap \Gamma_0(p))([1-t_i/p^n, 1])$ to be $\sum_{j=0}^{n-1} b_i^j (f'|V_i^j)$, where by $f'|V_i^j$ we mean the iterated expression $(\cdots((f'|V_i)|V_i)\cdots)|V_i$. One can check that, since the restriction of f' to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p, 1))$ is $f - b_i(f|V_i)$ by definition, the restriction of g_n to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p^n, 1))$ is $f|_{X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p^n, 1))} - b_i^n(f|V_i^n)$. Since $v(b_i) = v(p^{k_i-1}/a_i) = (k_i - 1) - v(a_i) > 0$ and because it follows from Lemma 9 that for all j we have $|f'|V_i^j| \leqslant |f'| < \infty$ on $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])$, g_n converges to give a section of $\omega^{\vec{k}}$ over $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])$, which we shall denote by g. This is the (over)convergent form that we will glue to f. Unravelling the definition, one can verify that $g - b_i^n(g|V_i^n)$ is in fact equal to $\sum_{j=0}^{n-1} b_i^j(f|V_i^j)$ for any $n \in \mathbb{Z}_{\geqslant 1}$. However, observe that the former is defined over $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])$, and therefore the latter is none other than the restriction of g_n to $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])$.

In order to apply Kassaei's gluing lemma, we need the following result.

Lemma 11. *The $g|V_i^n$ on $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])$ and the $f|V_i^n$ on $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_1/p^n,1))$ are uniformly bounded.*

Proof. For any

$$n \in \mathbb{Z}_{\geqslant 1}, \quad |(g|V_i^n)|_{X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])} \leqslant |g|_{X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])} < \infty.$$

This follows from Lemma 9 and the quasi-compactness of $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])$. See [Kas06, Lemmas 4.4 and 4.5] for the second assertion; note that [Kas06, Lemma 3.3] is critical to the proof of [Kas06, Lemma 4.5], but we have its generalisation in Lemma 9 and thus can argue similarly to prove the assertion. □

To summarise, for $f \in H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i,1)),\omega^{\vec{k}})$, we have

$$g_n \in H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p^n,1]),\omega^{\vec{k}}) \quad \text{and} \quad g \in H^0(X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1]),\omega^{\vec{k}}),$$

which satisfy

$$|f-g_n|_{X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i/p^n,1))} \to 0 \quad \text{and} \quad |g_n - g|_{X_{\Gamma_1(N)\cap\Gamma_0(p)}([1,1])} \to 0$$

as $n \to \infty$. As a result, we have the following proposition.

Proposition 12. *Let $f \in \mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))([0,1))$. If f is eigenform for U_{v_i} with non-zero eigenvalue a_i such that $v(a_i) < k_i - 1$, then it extends to $\mathbf{M}_{\vec{k}}(\Gamma_1(N)\cap\Gamma_0(p))([0,1])$.*

Proof. This follows from [Kas06, Lemma 3.2]: apply the lemma to affinoid coverings of $X_{\Gamma_1(N)\cap\Gamma_0(p)}([1-t_i,1])$. □

We can repeat the argument, essentially, for the remaining indices.

Theorem 13. *Let f be an overconvergent Hilbert modular form of weight $\vec{k} = (k_i)_{1\leqslant i\leqslant g}$ and level $\Gamma_1(N)\cap\Gamma_0(p)$. If f is an eigenform for U_{v_i} with non-zero eigenvalue a_i satisfying $v(a_i) < k_i - 1$ for all $1 \leqslant i \leqslant g$, then it is a classical Hilbert eigenform.*

Acknowledgements

The author would like to thank his PhD supervisor Kevin Buzzard for useful conversations on numerous occasions. He is also grateful to Gaetan Chenevier, Payman Kassaei, Jacques Tilouine and Alain Genestier for inspirational discussions and Teruyoshi Yoshida for moral support. He regrets the delay in writing up this work and thanks Gaetan Chenevier and Payman Kassaei for unflagging encouragement over the years, without which the writing process would have taken even longer.

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