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On Riemannian and Ricci curvatures of homogeneous Finsler manifolds

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Abstract. The famous Cheng-Shen's conjecture in Riemann-Finsler geometry claims that every n-dimensional closed W-quadratic Randers manifold is a Berwald manifold. In this paper, first we study the Riemann and Ricci curvatures of homogeneous Finsler manifolds and obtain some rigidity theorems. Then, by using this investigation, we construct a family of W-quadratic Randers metrics which are not R-quadratic nor strongly Ricci-quadratic.

1 Introduction

In [5], Cheng and Shen studied the flag curvature of Finsler metrics and made the following conjecture:

Conjecture *Every W-quadratic Randers metric on a closed manifold is a Berwald metric.*

To answer Cheng-Shen's conjecture, one should find some hidden relations between the Riemann and Berwald curvatures of Finsler metrics. For this aim, we need to understand some important notions in Finsler geometry. In 1926, L. Berwald introduced a connection with two curvature tensors – namely, Riemann curvature $\mathbf{R}$ and Berwald curvature $\mathbf{B}$ [4]. For a Finsler manifold (M, F), the second variation of geodesics gives rise to a family of linear maps $\mathbf{R}_y : T_xM \to T_xM$, at any point $y \in T_xM$. The quantity $\mathbf{R}_y$ is called the Riemann curvature in the direction y. The Riemann curvature in Finsler geometry is not only a function of position but also depends on direction, while in Riemann geometry, it only depends on position. This situation complicates the understanding of Riemann curvature in Finsler geometry. A Finsler space is said to be R-quadratic if its Riemann curvature $\mathbf{R}_y$ is quadratic in $y \in T_xM$. It is well-known that every flat Finsler metric ($\mathbf{R} = 0$) and Berwald metric ($\mathbf{B} = 0$) are R-quadratic.

The notion of flag curvature is a natural extension of the Riemannian sectional curvature to Finsler manifolds. For a Finsler manifold (M, F), the flag curvature is a function $\mathbf{K} = \mathbf{K}(\Pi, y)$ of tangent planes $\Pi \subset T_xM$ and directions $y \in \Pi$. F is said to be of scalar flag curvature if the flag curvature $\mathbf{K}(\Pi, y) = \mathbf{K}(x, y)$ is independent of flags Π which are associated with any fixed flagpole y. Finsler metrics of scalar curvature

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are the natural extension of Riemannian metrics of isotropic sectional curvature. F is said to be of sectional flag curvature if its flag curvature depends only on the section. In this case, the flag curvature is independent of the choice of the flagpole $y \in \Pi$; that is, $\mathbf{K}(\Pi, y) = \mathbf{K}(\Pi)$.

Nowadays, the R-quadratic metrics problem is an important part of Finsler geometry, and there are many types of research about it [3][11][12]. This notion was introduced by Bácsó-Matsumoto in [3]. They studied R-quadratic Randers metrics. In [12], Mo proved that R-quadratic metrics have vanishing H-curvature. In [11], Li-Shen characterized R-quadratic Randers metrics and showed that these metrics must have constant S-curvature. However, up to now, very little attention has been paid to the subject of homogeneous R-quadratic metrics. Homogeneous Finsler manifolds are those Finsler manifolds (M, F) that the orbit of the natural action of $I(M, F)$ on M at any point of M is the whole M. Then, M is the quotient manifold $I(M, F)/H$, where H is the stabilizer subgroup at a point in M. In this paper, we study homogeneous R-quadratic Finsler metrics and prove the following.

Theorem 1.1 *Let (M, F) be a homogeneous Finsler manifold of scalar flag curvature or sectional flag curvature. Then, F is R-quadratic if and only if it is Riemannian or locally Minkowskian.*

The Ricci curvature is the trace of Riemann curvature $\mathbf{Ric}(y) := trace(\mathbf{R}_y)$. A Finsler space is said to be strongly Ricci-quadratic if its Ricci curvature $\mathbf{Ric}_y$ is quadratic in $y \in T_xM$ – namely, $\mathbf{Ric} = R_{i\ mj}^{\ m}(x)y^i y^j$. By definition, every R-quadratic metric and Berwald metric is strongly Ricci-quadratic. In general, it is quite challenging to characterize strongly Ricci-quadratic metrics [11]. In [9], Hu-Deng proved that a homogeneous Randers metric is Ricci-quadratic if and only if it is a Berwald metric. Here, we prove the following.

Theorem 1.2 *Let (M, F) be a homogeneous Finsler manifold of scalar flag curvature. Then F is strongly Ricci-quadratic if and only if it is Riemannian or locally Minkowskian.*

Every two-dimensional Finsler manifold is of scalar flag curvature. Then, the special case $n = 2$ of Theorem 1.2 is an extension of Hu-Deng's theorem in [9], which has been proved for Randers metrics only. Our approach is completely different from theirs. Also, by Theorems 1.1 and 1.2, we conclude that a homogeneous Finsler metric of scalar flag curvature is R-quadratic if and only if it is strongly Ricci-quadratic. As an interesting application of Theorems 1.1 and 1.2, we construct a family of homogeneous W-quadratic Randers surfaces which are not R-quadratic nor strongly Ricci-quadratic (see Example 6).

A Finsler metric that is not Riemannian nor locally Minkowskian is called nontrivial. We study the Weyl curvature of homogeneous Finsler manifolds and prove the following rigidity result.

Theorem 1.3 *Every nontrivial homogeneous Randers surface is a W-quadratic metric which is not R-quadratic nor strongly Ricci-quadratic.*

Then, we study homogeneous Finsler metrics on closed connected surfaces and prove the following.

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Theorem 1.4 *Every homogeneous Finsler metric on a closed connected surface must be Riemannian or locally Minkowskian.*

In the final section, using Theorems 1.1–1.4 and the Zermelo navigation problem, we will construct three counterexamples for Cheng-Shen's conjecture. Example 4 is a four-dimensional Randers metric, and Examples 5 and 6 are two-dimensional W-quadratic Randers metrics which are not Berwaldian.

2 Preliminaries

Let M be an n-dimensional C^∞ connected manifold, $TM = \bigcup_{x\in M} T_xM$ the tangent bundle, and $TM_0 := TM - \{0\}$ the slit tangent bundle. Let (M, F) be a Finsler manifold and $\mathbf{G} = y^i\delta/\delta x^i$ be its induced spray on TM which in a standard coordinate (x^i, y^i) for TM_0 is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}, \quad G^i := \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

Then, for a vector $y \in T_xM_0$, the Riemann curvature is a family of linear transformation $\mathbf{R}_y : T_xM \to T_xM$ which is defined by $\mathbf{R}_y(u) := R^i_k(y)u^k\partial/\partial x^i$, where

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}. \tag{2.1}$$

The family $\mathbf{R} := \{\mathbf{R}_y\}_{y\in TM_0}$ is called the Riemann curvature. Let us put

$$R^i{}_{kl} := \frac{1}{3}\Big\{\frac{\partial R^i_k}{\partial y^l} - \frac{\partial R^i_l}{\partial y^k}\Big\}, \quad R^i_{j\ kl} := \frac{1}{3}\Big\{\frac{\partial^2 R^i_k}{\partial y^j \partial y^l} - \frac{\partial^2 R^i_l}{\partial y^j \partial y^k}\Big\}. \tag{2.2}$$

Then

$$R^i_k = R^i_{j\ kl}y^j y^l, \quad R^i{}_{kl} = R^i_{j\ kl}y^j, \quad R^i_{j\ kl} + R^i_{j\ lk} = 0. \tag{2.3}$$

Let (M, F) be an n-dimensional Finsler manifold. Put

$$\mathbf{Ric} := \sum_{i,j=1}^{n} g^{ij}\Big(\mathbf{R}_y(b_i), b_j\Big),$$

where $\{b_i\}$ is a basis for T_xM, $g_{ij} := g(b_i, b_j)$ and $(g^{ij}) := (g_{ij})^{-1}$. $\mathbf{Ric}$ is a well-defined scalar function on TM_0. We call $\mathbf{Ric}$ the Ricci curvature. In a local coordinate system,

$$\mathbf{Ric} = g^{ij}R_{ij} = R^m_m.$$

For a flag $\Pi := \mathrm{span}\{y, u\} \subset T_xM$ with flagpole y, the flag curvature $\mathbf{K} = \mathbf{K}(\Pi, y)$ is defined by

$$\mathbf{K}(x, y, \Pi) := \frac{\mathbf{g}_y\big(u, \mathbf{R}_y(u)\big)}{\mathbf{g}_y(y,y)\mathbf{g}_y(u,u) - \mathbf{g}_y(y,u)^2}. \tag{2.4}$$

The flag curvature $\mathbf{K}(x, y, \Pi)$ is a function of tangent planes $\Pi = \mathrm{span}\{y, v\} \subset T_xM$. This quantity tells us how curved the space is at a point. If F is a Riemannian metric,

$\mathbf{K}(x, y, \Pi) = \mathbf{K}(x, \Pi)$ is independent of $y \in \Pi \backslash \{0\}$. A Finsler manifold (M, F) is said to have scalar flag curvature if $\mathbf{K}(\Pi_x, y) = \mathbf{K}(x, y)$ is only a function of $(x, y) \in TM_0$. As a special case, (M, F) is called of weakly isotropic flag curvature if

$$\mathbf{K} = \frac{3\theta}{F} + \sigma,$$

where $\theta = \theta_i(x)y^i$ is a 1-form and $\sigma = \sigma(x)$ is a scalar function on M. Also, it is called of isotropic flag curvature if $\mathbf{K}(\Pi_x, y) = \mathbf{K}(x)$ – namely, the flag curvature is only a function of $x \in M$. (M, F) is said to have constant flag curvature if $\mathbf{K}(\Pi_x, y) = constant$ everywhere.

Let (M, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xM \times T_xM \to \mathbb{R}$ is called fundamental tensor:

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, one can define $\mathbf{C}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM_0}$ is called the Cartan torsion.

Let $c = c(t)$ be a C^∞ curve and $U(t) = U^i(t)\partial/\partial x^i|_{c(t)}$ be a vector field along c. Define the covariant derivative of $U(t)$ along c by

$$D_{\dot{c}}U(t) := \left\{\frac{dU^i}{dt}(t) + U^j(t)\frac{\partial G^i}{\partial y^j}\big(c(t), \dot{c}(t)\big)\right\}\frac{\partial}{\partial x^i}\Big|_{c(t)}.$$

$U(t)$ is said to be linearly parallel if $D_{\dot{c}}U(t) = 0$.

For a vector $y \in T_xM$, define

$$\mathbf{L}_y(u, v, w) := \frac{d}{dt}\Big[\mathbf{C}_{\dot{\sigma}(t)}\big(U(t), V(t), W(t)\big)\Big]|_{t=0},$$

where $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$, $\dot{\sigma}(0) = y$, and $U(t), V(t), W(t)$ are linearly parallel vector fields along σ with $U(0) = u$, $V(0) = v$, $W(0) = w$. We call $\mathbf{L}_y$ the Landsberg curvature. The Landsberg curvature measures the rate of change of the Cartan torsion along geodesics.

Define $\mathbf{B}_y : T_xM \times T_xM \times T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^jv^kw^l\partial/\partial x^i|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

The quantity $\mathbf{B}$ is called the Berwald curvature. F is called a Berwald metric if $\mathbf{B} = 0$. In this case, G^i are quadratic in $y \in T_xM$ for all $x \in M$; that is, there exists $\Gamma^i{}_{jk} = \Gamma^i{}_{jk}(x)$ such that

$$G^i = \Gamma^i{}_{jk}y^jy^k.$$

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Taking a trace of Berwald curvature **B** give us the mean of Berwald curvature **E** which is defined by $\mathbf{E}_y : T_xM \times T_xM \to \mathbb{R}$, where

$$\mathbf{E}_y(u,v) := \frac{1}{2}\sum_{i=1}^{n} g^{ij}(y) g_y\Big(\mathbf{B}_y(u,v,e_i), e_j\Big). \tag{2.5}$$

The family $\mathbf{E} = \{\mathbf{E}_y\}_{y\in TM\setminus\{0\}}$ is called the mean Berwald curvature. In local coordinates, $\mathbf{E}_y(u,v) := E_{ij}(y)u^i v^j$, where

$$E_{ij} := \frac{1}{2} B^m{}_{mij}.$$

Taking a horizontal derivation of the mean of Berwald curvature **E** gives us the H-curvature **H** which is defined by $\mathbf{H}_y = H_{ij} dx^i \otimes dx^j$, where

$$H_{ij} := E_{ij|m} y^m.$$

Here, "|" denotes the horizontal covariant differentiation with respect to the Berwald connection.

3 Proof of Theorem 1.1

In this section, we are going to prove Theorem 1.1. For this aim, we will need some useful geometrical and topological facts about the homogeneous Finsler manifolds. In [17], by using the property of exponential map, the following is proved.

Lemma 3.1 [17] *Every homogeneous Finsler manifold is complete.*

Every two points of a homogeneous Finsler manifold (M,F) map to each other under an isometry. This causes the norm of an invariant tensor under the isometries of a homogeneous Finsler manifold to be a constant function on the manifold M, and consequently, it has a bounded norm. Then, we have the following.

Lemma 3.2 [16] *Let (M,F) be a homogeneous Finsler manifold. Then, every invariant tensor under the isometries of F has a bounded norm with respect to it.*

Now, we consider the flag curvature of homogeneous R-quadratic Finsler metrics and prove the following.

Lemma 3.3 *Every homogeneous R-quadratic Finsler metric of scalar flag curvature has constant flag curvature.*

Proof In [12], Mo proved that R-quadratic Finsler metrics satisfy $\mathbf{H} = 0$. By Akbar-Zadeh's theorem, every Finsler metric of scalar flag curvature $\mathbf{K} = \mathbf{K}(x,y)$ has isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{H} = 0$. However, every scalar function on M – namely, $\mathbf{K} = \mathbf{K}(x)$ – which is invariant under isometries of (M,F) is a constant function. Thus, the homogeneity of (M,F) and invariancy of the flag curvature under isometries of F imply that $\mathbf{K} = constant$. ∎

Here, we give a relation between homogeneous R-quadratic metrics and Landsberg metrics.

Lemma 3.4 *Every homogeneous R-quadratic Finsler metric is a Landsberg metric.*

Proof The following Bianchi identity for the Berwald connection of F holds:

$$B^h{}_{mjk|i} - B^h{}_{mik|j} = R^h{}_{mij,k}. \tag{3.1}$$

For more details, see page 136 in [14]. The R-quadratic Finsler metric is characterized by $R^h{}_{mij,k} = 0$. Then (3.1) reduces to

$$B^h{}_{mjk|i} = B^h{}_{mik|j}. \tag{3.2}$$

Contacting (3.2) with $y_h y^i$ yields

$$L_{mjk|i} y^i = 0. \tag{3.3}$$

For any geodesic $c = c(t)$ and any parallel vector field $U = U(t)$ along c, we define

$$\mathbf{C}(t) := \mathbf{C}_{\dot{c}}\big(U(t), U(t), U(t)\big), \quad \mathbf{L}(t) := \mathbf{L}_{\dot{c}}\big(U(t), U(t), U(t)\big). \tag{3.4}$$

By (3.4) and the definition of $\mathbf{L}_y$, we get

$$\mathbf{L}(t) = \mathbf{C}^{'}(t). \tag{3.5}$$

By (3.3), we obtain

$$\mathbf{L}^{'}(t) = 0. \tag{3.6}$$

The equation (3.6) implies that

$$\mathbf{L}(t) = \mathbf{L}(0). \tag{3.7}$$

Considering (3.5) and taking an integral of (3.7) yields

$$\mathbf{C}(t) = \mathbf{L}(0)t + \mathbf{C}(0). \tag{3.8}$$

By using Lemma 3.1, one can put $t \to \infty$ in (3.8) and then Lemma 3.2 implies $\mathbf{L}(0) = 0$. Therefore, F reduces to a Landsberg metric. ∎

Here, we consider homogeneous R-quadratic Randers metrics of nonzero flag curvature and prove the following rigidity result.

Corollary 3.1 *Let (M, F) be a homogeneous Randers space of R-quadratic type. If the flag curvature of F is everywhere nonzero, then it is a Riemannian metric.*

Proof By Lemma 3.4, F is a Landsberg metric. Every Randers metric with vanishing Landsberg curvature is a Berwald metric [15]. In [8], Deng-Hu proved that a homogeneous Randers space of Berwald type with nonzero flag curvature is a Riemannian metric. This completes the proof. ∎

By using the celebrated Akbar-Zadeh theorem for Finsler metrics of vanishing flag curvature, we prove the following result which plays a key lemma in our investigation.

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Lemma 3.5 *Every homogeneous flat Finsler metric must be locally Minkowskian.*

Proof Let (M, F) be a homogeneous Finsler manifold with vanishing flag curvature. The well-known Akbar-Zadeh theorem stated that every positively complete Finsler manifold with vanishing flag curvature must be locally Minkowskian if the Cartan torsion and its vertical covariant derivative are bounded [1]. By Lemma 3.1, (M, F) is a complete manifold. By Lemma 3.2, for homogeneous Finsler metrics, the Cartan torsion and its vertical covariant derivative are bounded. Then, by Akbar-Zadeh's theorem, F is locally Minkowskian. ∎

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 The proof is divided into two main cases as follows:

Case (I): Let F be a homogeneous R-quadratic Finsler metric of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. This case divided to two cases:

Case (a): $n = 2$. By Lemma 3.4, homogeneous R-quadratic metrics are Landsbergian. In [18], it is proved that every homogeneous Landsberg surface is Riemannian or locally Minkowskian.

Case (b): $n \geq 3$. By Lemma 3.3, $\mathbf{K} = constant$. This case is divided into two subcases as follows:

Case (b1): Let $\mathbf{K} \neq 0$. According to the Numata theorem, every Landsberg metric of nonzero scalar flag curvature is a Riemannian metric of constant sectional curvature (see page 158 of [14]). By Lemma 3.4, we get the proof for this case.

Case (b2): Let $\mathbf{K} = 0$. By Lemma 3.5, F is a locally Minkowskian metric.

Case (II): Now, suppose that F is a homogeneous R-quadratic Finsler metric of sectional flag curvature $\mathbf{K} = \mathbf{K}(\Pi)$. In [19], Wu proved that every Landsberg metric of nonzero sectional flag curvature must be Riemannian. Then, by the same method used for the case of Finsler metrics of scalar flag curvature – namely, Case (I) – we get the proof. ∎

The condition of scalar flag curvature in Theorem 1.1 cannot be ignored. For example, see the following.

Example 1 Let G/K be an irreducible symmetric space of compact type (i.e., it has compact G and K). Any G-invariant metric on G/K is Berwaldian, so it is R-quadratic. Suppose G/K is not of rank 1. Since the rank of G/K is bigger than 1, it admits G-invariant Finsler metrics which are not Riemannian or locally Minkowski. Those metrics are not of scalar flag curvature.

Also, in Theorem 1.1, the condition R-quadratic is necessary. Here, we give an example that certifies our claim.

Example 2 The Bao-Shen's Randers metrics on $\mathbb{S}^3$ are of constant flag curvature (see page 31 in [14]) which are not R-quadratic. These metrics are not Riemannian nor locally Minkowskian.

We must mention that Theorem 1.1 does not hold for non-homogeneous Finsler metrics. See the following example.

Example 3 Let us consider the following Randers metric defined nearby the origin in $\mathbb{R}^n$:

$$F := \frac{\sqrt{|y|^2(1-|xA|^2)+\langle y, xA\rangle^2}}{1-|xA|^2} - \frac{\langle y, xA\rangle}{1-|xA|^2},$$

where $|.|$ and $\langle , \rangle$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively, and $A := (a^i_j)$ is a nonzero and anti-symmetric matrix. F is a R-quadratic and is of scalar flag curvature [11]. However, it is not Riemannian nor locally Minkowskian.

Here, we give another alternative proof for a special case of Theorem 1.1 which is independent of Numata's Theorem.

Proposition 3.1 *Let (M, F) be a homogeneous Finsler manifold of scalar flag curvature. Suppose that the flag curvature of F is everywhere nonzero. Then F is R-quadratic if and only if it is a Riemannian metric.*

Proof Suppose that the flag curvature of F is nonzero everywhere. We just need to prove $F = F(x, \cdot)$ is Euclidean when restricted to each tangent plane P in T_xM. We restrict the Riemann curvature to P (i.e., $(\mathbf{R}_y)|_P : P \to P$) for $y \in T_xM_0$. For $y \in P - \{0\}$ with $F(y) = 1$, y is an eigenvector of $(\mathbf{R}_y)|_P$ for the 0 eigenvalue; another eigenvalue of $(\mathbf{R}_y)|_P$ is nonzero. The corresponding eigenvector is tangent to the indicatrix at y. Using the quadratic property of $\mathbf{R}$, one can determine the ODE for $F = 1$ in P, which integral curve must be an ellipsoid centered at 0. To summarize, $F|_P$ is Euclidean for each P, so the Cartan tensor $\mathbf{C}_y(v, v, v) = 0$ for all y and all v $\mathbf{g}_y$-orthogonal to y. Thus, $\mathbf{C} = 0$ (i.e., F is Riemannian). The converse is trivial. ■

Randers metrics are special (α, β)-metrics [5]. As an interesting result, we study homogeneous R-quadratic (α, β)-metrics and prove the following.

Corollary 3.2 *Every homogeneous (α, β)-metric on a manifold of dimension $n \geq 3$ is R-quadratic if and only if it is a Berwald metric.*

Proof According to Lemma 3.4, F is a Landsberg metric. In [15], it is proved that every regular (α, β)-metric on a manifold of dimension $n \geq 3$ with vanishing Landsberg curvature is a Berwald metric. This gives us the proof. ■

Let (M, F) be an n-dimensional Finsler manifold, TM its tangent bundle, and (x^i, y^i) the coordinates in a local chart on TM. Let F be a scalar function on TM defined by $F = \sqrt[m]{A}$, where A is given by $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$, with $a_{i_1 \dots i_m}$ symmetric in all its indices. F is called an m-th root Finsler metric.

An m-th root Finsler metric $F = \sqrt[m]{a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}}$ is regarded as a direct generalization of Riemannian metric in the sense that the second root metric is a Riemannian metric $F = \sqrt{a_{ij}(x) y^i y^j}$. The fourth root metrics $F = \sqrt[4]{a_{ijkl}(x) y^i y^j y^k y^l}$

are called quartic metric. The special 4-th root metric – namely, Berwald-Moór metric – plays an important role in the theory of space-time structure, gravitation, and general relativity.

Corollary 3.3 *Every homogeneous fourth root metric has weakly isotropic flag curvature if and only if it is locally Minkowskian.*

Proof For a fourth root Finsler metric $F = \sqrt[4]{A}$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$, let us put $A_{ij} = [A]_{y^i y^j}$. Suppose that the matrix (A_{ij}) defines a positive definite tensor and (A^{ij}) denotes its inverse. Then the spray coefficients G^i of F are given by following:

$$G^k = \frac{1}{2}(A_{0j} - A_{x^j})A^{kj}, \tag{3.9}$$

where $A_{x^i} = [A]_{x^i}$ and $A_{0l} := A_{x^m y^l} y^m$. By (3.9), it follows that the spray coefficients of fourth root Finsler metrics are rational functions in y. Therefore, the Riemannian curvature $\mathbf{R}$ and Ricci curvature $\mathbf{Ric}$ are rational functions in y. By assumption, F has weakly isotropic flag curvature. Thus, we have

$$R^i_j = \left(\frac{3\theta}{F} + \sigma\right) F^2 h^i_j, \tag{3.10}$$

where $\theta = \theta_i(x)y^i$ is a 1-form and $\sigma = \sigma(x)$ and is a scalar function on M. Taking a trace of (3.10) yields

$$\sigma F^2 + 3\theta F - \frac{1}{n-1}\mathbf{Ric} = 0. \tag{3.11}$$

We have two main cases as follows:

Case (i): Let at a point $x_0 \in M$, we have $\sigma(x_0) = 0$. In this case, (3.11) reduces to

$$\mathbf{Ric}(x_0, y) = 3(n-1)\theta(x_0, y)F^2(x_0, y), \quad \forall y \in T_{x_0}M. \tag{3.12}$$

The left side of (3.12) is a rational function in y, while the right side is an irrational function in y. Thus, $\theta(x_0, y) = 0$ and, by considering (3.10), $F|_{x_0}$ reduces to a R-flat metric.

Case (ii): Let at a point $x_0 \in M$, $\sigma(x_0) \neq 0$ holds. In this case, (3.11) is written as follows:

$$\sigma(x_0)F^2(x_0, y) + 3\theta(x_0, y)F(x_0, y) - \frac{1}{n-1}\mathbf{Ric}(x_0, y) = 0. \tag{3.13}$$

For a fourth root metric F, we have $F(x_0, -y) = F(x_0, y)$. By assumption, F is R-quadratic and then $R^i{}_j(x, -y) = R^i{}_j(x, y)$. It follows that $\mathbf{Ric}(x_0, -y) = \mathbf{Ric}(x_0, y)$. Thus, $y \to -y$ in (3.13) gives us

$$\sigma(x_0)F^2(x_0, y) - 3\theta(x_0, y)F(x_0, y) - \frac{1}{n-1}\mathbf{Ric}(x_0, y) = 0. \tag{3.14}$$

By (3.13) and (3.14), we get

$$F^2(x_0, y) = \frac{1}{(n-1)\sigma(x_0)}\mathbf{Ric}(x_0, y). \tag{3.15}$$

According to (3.15), we find that $F^2|_{x_0}$ is rational in y which is a contradiction. Thus, only the case (i) holds, and by Lemma 3.5, F is locally Minkowskian. The converse is trivial. ∎

4 Proof of Theorems 1.2, 1.3, and 1.4

In this section, we are going to prove Theorems 1.2, 1.3, and 1.4. To prove Theorem 1.2, we need to extend Mo's result in [12] to strongly Ricci-quadratic metrics. More precisely, we show that every strongly Ricci-quadratic Finsler metric satisfies $\mathbf{H} = 0$. Our approach is completely different from Mo. For this aim, we need the structure equations of the Berwald connection.

Throughout this paper, we use the Berwald connection on Finsler manifolds. Let $\{e_j\}$ be a local frame for $\pi^* TM$, $\{\omega^i, \omega^{n+i}\}$ be the corresponding local coframe for $T^*(TM_0)$, and $\{\omega^i_j\}$ be the set of local Berwald connection forms with respect to $\{e_j\}$. Then the connection forms of the Berwald connection are characterized by the following structure equations:

- Torsion freeness:

$$d\omega^i = \omega^j \wedge \omega^i{}_j. \tag{4.1}$$

- Almost metric compatibility:

$$dg_{ij} - g_{jk}\Omega^k{}_i - g_{ik}\Omega^k{}_j = -2L_{ijk}\omega^k + 2C_{ijk}\omega^{n+k}, \tag{4.2}$$

  where $\omega^i := dx^i$, $\omega^{n+k} := dy^k + y^j\omega^k{}_j$, and $\Omega^i{}_j$ are the curvature forms of the Berwald connection defined by following:

$$\Omega^i{}_j = d\omega^i{}_j - \omega^k{}_j \wedge \omega^i{}_k := \frac{1}{2}R^i{}_{jkl}\omega^k \wedge \omega^l - B^i{}_{jkl}\omega^k \wedge \omega^{n+l}. \tag{4.3}$$

The horizontal and vertical covariant derivations with respect to the Berwald connection are denoted by "|" and " , " respectively. Thus,

$$g_{ij|k} = -2L_{ijk}, \quad g_{ij,k} = 2C_{ijk}.$$

For more details, one can see [14].

Lemma 4.1 *Every strongly Ricci-quadratic Finsler metric satisfies* $\mathbf{H} = 0$.

Proof Differentiating (4.2) yields the following Ricci identity:

$$\begin{aligned} g_{pj}\Omega^p{}_i - g_{pi}\Omega^p{}_j = -2L_{ijk|l}\omega^k \wedge \omega^l - 2L_{ijk,l}\omega^k \wedge \omega^{n+l} - 2C_{ijl|k}\omega^k \wedge \omega^{n+l} \\ -2C_{ijl,k}\omega^{n+k} \wedge \omega^{n+l} - 2C_{ijp}\Omega^p{}_l y^l. \end{aligned} \tag{4.4}$$

It follows from (4.4) that

$$C_{ijl|k} + L_{ijk,l} = \frac{1}{2}g_{pj}B^p{}_{ikl} + \frac{1}{2}g_{ip}B^p{}_{jkl}. \tag{4.5}$$

Contracting (4.5) with y^j yields

$$L_{jkl} = -\frac{1}{2} y^m g_{im} B^i{}_{jkl}. \tag{4.6}$$

Differentiating of (4.3) implies that

$$d\Omega_i^{\ j} - \omega_i^{\ k} \wedge \Omega_k^{\ j} + \omega_k^{\ j} \wedge \Omega_i^{\ k} = 0. \tag{4.7}$$

Define $B^i{}_{jkl|m}$ and $B^i{}_{jkl,m}$ by

$$dB^i{}_{jkl} - B^i{}_{mkl}\omega^m_i - B^i{}_{jml}\omega^m_k - B^i{}_{jkm}\omega^m_l + B^i{}_{jkl}\omega^i{}_m := B^i{}_{jkl|m}\omega^m + B^i{}_{jkl,m}\omega^{n+m}. \tag{4.8}$$

Similarly, one can define $R^i{}_{jkl|m}$ and $R^i{}_{jkl,m}$ by following

$$dR^i{}_{jkl} - R^i{}_{mkl}\omega^m_i - B^i{}_{jml}\omega^m_k - R^i{}_{jkm}\omega^m_l + R^i{}_{jkl}\omega^i{}_m := R^i{}_{jkl|m}\omega^m + R^i{}_{jkl,m}\omega^{n+m}. \tag{4.9}$$

By (4.7), (4.8), and (4.9), one obtains the following Bianchi identities:

$$R^h{}_{mij|k} + R^h{}_{mjk|i} + R^h{}_{mki|j} = -B^h{}_{mir}R^r{}_{jk} - B^h{}_{mjr}R^r{}_{ki} - B^h{}_{mkr}R^r{}_{ij}, \tag{4.10}$$

$$B^i{}_{jkl|m} - B^i{}_{jkm|l} = R^i{}_{jkl,m}, \tag{4.11}$$

$$B^i{}_{jkl,m} = B^i{}_{jkm,l}. \tag{4.12}$$

Letting $i = k$ in (4.11) yields

$$B^k{}_{jkl|m} - B^k{}_{jkm|l} = R^k{}_{jkl,m}. \tag{4.13}$$

F is a strongly Ricci-quadratic metric; then from (4.13), we get

$$B^k{}_{jkl|m} = B^k{}_{jkm|l}. \tag{4.14}$$

Multiplying (4.14) with y^m implies that

$$H_{jk} = E_{jk|m}y^m = 0.$$

This completes the proof. ∎

Generally, two-dimensional Finsler metrics have some different special Riemannian and non-Riemannian curvature properties from the higher dimensions. In [20], Yan-Deng studied homogeneous Einstein (α, β)-metrics and proved that any homogeneous Ricci-flat (α, β)-metric with vanishing S-curvature must be a Minkowski space. Here, we prove the following.

Lemma 4.2 *A homogeneous Finsler surface is Ricci-flat if and only if it is locally Minkowskian.*

Proof Every Finsler surface has scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ – namely, F satisfies

$$R^i_j = \mathbf{K}F^2 h^i_j. \tag{4.15}$$

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Taking a trace of (4.15) implies that $\mathbf{Ric} = \mathbf{K}F^2$. From the assumption, we find that $\mathbf{K} = 0$ and then F is a R-flat Finsler metric. By Lemma 3.5, we get the proof. ∎

Lemma 4.3 *Every homogeneous strongly Ricci-quadratic metric of isotropic flag curvature is a Riemannian metric or locally Minkowskian.*

Proof Let F be a homogeneous Finsler metric of isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$. Thus, we have $R^i{}_j = \mathbf{K}F^2 h^i_j$ and taking a trace of it yields

$$\mathbf{Ric} = (n-1)\mathbf{K}F^2. \tag{4.16}$$

By assumption, F is strongly Ricci-quadratic

$$\mathbf{Ric} = R^m_m = R_j{}^m{}_{ml}(x)y^j y^l. \tag{4.17}$$

Comparing (4.16) and (4.17) implies that

$$\mathbf{K}F^2 = \frac{1}{n-1} R_j{}^m{}_{ml}(x)y^j y^l. \tag{4.18}$$

Let at a point $x_0 \in M$, $\mathbf{K}(x_0) = 0$ holds. In this case, the homogeneousness of (M, F) implies that $\mathbf{K}(x) = 0$, $\forall x \in M$. By (4.16), F is Ricci-flat. By assumption, we get $R^i{}_j = 0$. Then, Lemma 3.5 implies that F is locally Minkowskian.

Now, let $\mathbf{K}(x) \neq 0$, $\forall x \in M$. Then (4.18) yields

$$F = \sqrt{\frac{1}{(n-1)\mathbf{K}} R_i{}^m{}_{mj} y^i y^j}.$$

In this case, F is Riemannian. ∎

Proof of Theorem 1.2 Let (M, F) be an n-dimensional strongly Ricci-quadratic Finsler manifold of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$. By Lemma 4.1 and Akbar-Zadeh's theorem, we find that F has isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$. By Lemma 4.3, we get the proof. ∎

A Finsler metric F on an n-dimensional manifold M is called an Einstein metric if its Ricci curvature satisfies $\mathbf{Ric} = (n-1)\mu F^2$, where $\mu = \mu(x)$ is a scalar function on M. In [7], Deng-Hou proved that a homogeneous Einstein-Randers space with negative Ricci curvature is Riemannian. Here, an extension of their result to two-dimensional homogeneous metrics is presented.

Corollary 4.1 *Every homogeneous negatively curved Einstein Finsler surface is a Riemannian metric of constant sectional curvature or locally Minkowskian.*

Proof Every Finsler surface of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ satisfies $R^i_j = \mathbf{K}F^2 h^i_j$. Taking a trace of it yields

$$\mathbf{Ric} = \mathbf{K}F^2. \tag{4.19}$$

By assumption, F is an Einstein Finsler metric

$$\mathbf{Ric} = \mu F^2, \tag{4.20}$$

where $\mu = \mu(x)$ is a scalar function on M. By considering (4.19) and (4.20), we get $\mathbf{K} = \mu(x)$ (i.e., F is of isotropic flag curvature). Every scalar function on M which is invariant under isometries of (M, F) is a constant function. Thus, the homogeneity of (M, F) and invariancy of the flag curvature under isometries of F imply that $\mathbf{K} = constant$. If $\mathbf{K} = 0$, then by Lemma 3.5, F is locally Minkowskian. If $\mathbf{K} < 0$, then by Lemmas 3.1 and 3.2, and Akbar-Zadeh's theorem in [1], F reduces to a Riemannian metric. This completes the proof. ∎

In [20], Deng-Yan proved that a homogeneous (α, β)-space with vanishing S-curvature and negative Ricci curvature must be Riemannian. As a rigidity result, they showed that a homogeneous Ricci-flat (α, β)-space with vanishing S-curvature must be locally Minkowskian. Here, we extend their theorem for the Finsler metric satisfying $\mathbf{H} = 0$ and prove the following.

Corollary 4.2 *A homogeneous Finsler surface of non-positive Ricci curvature has vanishing H-curvature if and only if it is Riemannian or locally Minkowskian.*

Proof By Akbar-Zadeh's theorem, every Finsler metric of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ has isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$ if and only if $\mathbf{H} = 0$. By the same argument used in Corollary 4.1, we get $\mathbf{K} = constant$. Every Finsler surface satisfies $\mathbf{Ric} = \mathbf{K}F^2$, which by considering the assumption, we get $\mathbf{K} \leq 0$. For homogeneous Finsler metrics, we find that F is Riemannian (if $\mathbf{K} < 0$) or locally Minkowskian (if $\mathbf{K} = 0$). ∎

The 4-th Hilbert problem is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Such Finsler metrics are called projectively flat Finsler metrics. For homogeneous locally projectively flat Randers surfaces, we prove the following.

Corollary 4.3 *Every homogeneous locally projectively flat Randers surface of constant Ricci curvature is Riemannian surface of negative constant sectional curvature or locally Minkowskian.*

Proof Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on an n-dimensional manifold M. Suppose that it has constant Ricci curvature

$$\mathbf{Ric} = (n-1)cF^2, \tag{4.21}$$

where $c = constant$. In this case, it is proved that $c \leq 0$ (see Theorem 8.1.2 in [5]). However, every projectively flat Finsler metric is of scalar flag curvature – namely, it satisfies $R^i_j = \mathbf{K}F^2 h^i_j$, where $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on TM. Taking a trace of it yields

$$\mathbf{Ric} = (n-1)\mathbf{K}F^2. \tag{4.22}$$

By (4.21) and (4.22), it follows that $\mathbf{K} = c$. If $c = 0$, then by Lemma 4.2, F is locally Minkowskian. If $c < 0$, then by Akbar-Zadeh's theorem, F reduces to a Riemannian metric. ∎

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As the final conclusion of Section 4, we consider three-dimensional homogeneous Randers metrics and prove the following.

Corollary 4.4 *Let (M, F) be a three-dimensional homogeneous negatively curved Randers metric. Then the followings hold:*

(i) *If F is an Einstein metric, then it is Riemannian or locally Minkowskian;*
(ii) *If F is of sectional flag curvature, then it is Riemannian or locally Minkowskian.*

Proof Let $F = \alpha + \beta$ be a homogeneous negatively curved Randers metric on a three-dimensional manifold M. In [13], Robles showed that a three-dimensional Randers metric is an Einstein metric if and only if it is of constant flag curvature. Also, Chen-Zhao proved that an n-dimensional Randers metric $(n \geq 3)$ is of sectional flag curvature if and only if it is of constant flag curvature (see page 88 in [5]). By considering the assumption, in both cases (i) and (ii) of this corollary, F has non-positive constant flag curvature. Then, by Akbar-Zadeh's theorem and Lemma 3.5, F is Riemannian or locally Minkowskian. ∎

It is interesting to characterize Einstein metrics with quadratic Ricci curvature. Then, we prove the following.

Corollary 4.5 *Every n-dimensional Einstein metric is strongly Ricci-quadratic if and only if it is a Riemannian or Ricci-flat metric. In the case of $n = 2$, F is Riemannian or locally Minkowskian.*

Proof Let F be an Einstein metric

$$\mathbf{Ric} = (n-1)\lambda F^2, \tag{4.23}$$

where $\lambda = \lambda(x)$ is a scalar function on M. By (4.23), we have

$$\lambda = \frac{1}{n-1}\frac{\mathbf{Ric}}{F^2},$$

which shows that λ is invariant under isometries. According to the assumption, F is strongly Ricci-quadratic. Thus,

$$\mathbf{Ric} = R^{\ m}_{j\ ml}(x) y^j y^l. \tag{4.24}$$

Let at a point $x_0 \in M$, $\lambda(x_0) = 0$ holds. In this case, the homogeneousness of (M, F) and the invariant property of λ imply that $\lambda(x) = 0$, $\forall x \in M$. In this case, F reduces to a Ricci-flat metric. Now, suppose that $\lambda(x) \neq 0$, $\forall x \in M$. Then by (4.23) and (4.24), we get $F = \sqrt{a_{ij}(x) y^i y^j}$, where

$$a_{ij} := \frac{1}{(n-1)\lambda} R^{\ m}_{i\ mj}(x).$$

It shows that F is Riemannian. In the case of $dim(M) = 2$, by Lemma 4.2, we get the result. ∎

For 4-th root Finsler surfaces of Einstein-type, we get the following.

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Corollary 4.6 *Let $F = \sqrt[4]{A}$ be a non-Riemannian homogeneous fourth root surface. Then F is an Einstein metric if and only if it is locally Minkowskian.*

Proof According to the argument used in Corollary 3.3, the Ricci curvature **Ric** of a 4-th root Finsler metric is a rational function in y. By assumption, we have $\mathbf{Ric} = c(x)F^2$, where $c = c(x)$ is a scalar function on M. The left side of this relation is a rational function in y, while the right side is an irrational function in y. Thus, $c = 0$, and F is a Ricci-flat metric. By Lemma 4.2, F is locally Minkowskian. The converse is trivial. ■

An (α, β)-metric is called of polynomial-type if $\phi(s) = c_k s^k + c_{k-1}s^{k-1} + \cdots + c_1 s + 1$, where $2 \le i \le k$ are real constants, $c_k \neq 0$ and $k \ge 2$. The well-known square metric $F = (\alpha + \beta)^2/\alpha^2$ is a special polynomial-type (α, β)-metric. Here, we prove the following.

Corollary 4.7 *Every two-dimensional non-Riemannian homogeneous (α, β)-metric of polynomial-type is an Einstein metric if and only if it is locally Minkowskian.*

Proof Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric on an n-dimensional manifold M with $n \ge 2$. Suppose $\phi = \phi(s)$ is a polynomial in s of degree k ($k \ge 2$). In [6], it is proved that if F is an Einstein metric, then it is Ricci-flat. Then, by Lemma 4.2, we get the proof. ■

Let (M, F) be an n-dimensional Finsler manifold. For a nonzero vector $y \in T_xM$, define the Weyl tensor $\mathbf{W}_y : T_xM \to T_xM$ by $\mathbf{W}_y(u) := W^i_j(y)u^j\partial/\partial x^i\big|_x$, where

$$W^i_j := A^i_j - \frac{1}{n+1}\frac{\partial A^k_j}{\partial y^k}y^i, \tag{4.25}$$

and

$$A^i_j := R^i_j - R\delta^i_j, \quad R := \frac{1}{n-1}R^m_{\ m}.$$

Then, F is called a Weyl metric if $\mathbf{W} = 0$. It is well-known that a Finsler metric is of scalar flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ if and only if it is a Weyl metric. A Finsler metric F is said to be W-quadratic if W^i_k are quadratic in y. Every Weyl metric is a trivial W-quadratic metric. Also, R-quadratic Finsler metrics are W-quadratic. But the converse may not hold. In [11], Li and Shen find the necessary and sufficient condition under which a Randers metric $F = \alpha + \beta$ is W-quadratic.

A Finsler metric is called a generalized Douglas-Weyl metric if its Douglas tensor satisfies $D^i_{\ jkl|m}y^m = T_{jkl}y^i$ for some tensor T_{jkl}. All Douglas and Weyl metrics are generalized Douglas-Weyl metrics [5]. Here, we prove Theorem 1.3.

Proof of Theorem 1.3 Every two-dimensional Finsler metric is of scalar flag curvature and then is a generalized Douglas-Weyl metric. However, by Theorem 5.4.1 and Corollary 6.3.1 in [5], we find that every generalized Douglas-Weyl Randers metric is W-quadratic. Then by Theorems 1.1 and 1.2, we get the proof. ■

In the class of non-homogeneous Finsler metrics, we have many W-quadratic Finsler metrics on a closed manifold. Indeed, by the uniformization theorem, every closed surface with genus >1 admits a Riemannian metric of constant negative sectional curvature. So, by perturbing it, we can get non-Riemannian Randers metrics of negative curvature. Using the same argument in Theorem 1.3, we find that these metrics are W-quadratic metrics.

Considering Cheng-Shen's conjecture, another question arises:

Is there any homogeneous closed W-quadratic surface that is not of Berwald-type?

Here, we show that there is not any pure homogeneous two-dimensional W-quadratic metric on a closed surface. More precisely, we prove Theorem 1.4 as follows.

Proof of Theorem 1.4 Denote the surface as G/H in which G is the connected isometry group. Then G is compact with $2 \leq dim(G) \leq 3$. When $dim(G) = 3$, then it is a standard Riemannian sphere of constant curvature. When $dim(G) = 2$, then G is Abelian and the metric reduces to a locally Minkowskian metric. ∎

5 Counterexamples to the Cheng-Shen conjecture

In [2], Atashafrouz-Najafi constructed a three-dimensional homogeneous W-quadratic Randers metric which is not Berwald-type. Here, by using the navigation problem, we construct a four-dimensional W-quadratic Randers metric on a closed manifold which is nontrivial in the sense that it is not a Weyl metric.

Example 4 (four-dimensional W-quadratic Randers Metric) Any Randers metric $F = \alpha + \beta$ on the manifold M is a solution of the following Zermelo navigation problem:

$$\mathbf{h}\Big(x, \frac{y}{F} - \mathcal{W}_x\Big) = 1,$$

where $\mathbf{h} = \sqrt{h_{ij}(x)y^i y^j}$ is a Riemannian metric and $\mathcal{W} = \mathcal{W}^i(x)\partial/\partial x^i$ is a vector field such that

$$\mathbf{h}(x, -\mathcal{W}_x) = \sqrt{h_{ij}(x)\mathcal{W}^i(x)\mathcal{W}^j(x)} < 1.$$

In fact, α and β are given by

$$\alpha = \frac{\sqrt{\lambda h^2 + \mathcal{W}_0}}{\lambda}, \qquad \beta = -\frac{\mathcal{W}_0}{\lambda},$$

respectively, and moreover, $\lambda = 1 - \|\mathcal{W}\|_h^2$ and $W_0 = h_{ij}\mathcal{W}^i y^j$ [5]. Now, F can be written as follows:

$$F = \frac{\sqrt{\lambda \mathbf{h}^2 + \mathcal{W}_0^2}}{\lambda} - \frac{\mathcal{W}_0}{\lambda}. \tag{5.1}$$

In this case, the pair $(\mathbf{h}, \mathcal{W})$ is called the navigation data of F. On the Lie group $G = SU(2) \times \mathbb{S}^1$, we can find a non-Riemannian bi-invariant Randers metric F which is

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induced by navigation from a bi-invariant Riemannian metric $\mathbf{h}$ on G and the vector field $\mathcal{W} = c\partial/\partial t$ from $\mathbb{S}^1$. Then F is a Douglas metric because it is of Berwald-type. Thus, it is W-quadratic. However, F is not of scalar flag curvature – namely,

$$\mathbf{W} \neq 0.$$

At any point $x \in G$, linearly independent vectors $y, z \in T_xG$ are tangent to the $SU(2)$ factor, and the nonzero vector $w \in T_xG$ is tangent to the $\mathbb{R}$-factor. Then, we get

$$\mathbf{K}(x, y, y \wedge z) > 0, \quad \mathbf{K}(x, y, y \wedge w) = 0.$$

It follows that F is not projectively flat.

In the following example, we quote the well-known classification of two-dimensional Lie groups which admits a left invariant Randers metric.

Example 5 (two-dimensional W-quadratic Randers Metrics) It is well-known that there are only two non-isomorphic Lie algebras in dimension two: one is commutative, and the other one has a basis $\{e_1, e_2\}$ such that $[e_1, e_2] = e_1$. Each Lie algebra corresponds to a unique connected and simply connected Lie group. So, there are essentially two Lie groups in dimension two. On any given Lie group G, a left invariant metric can be defined by assigning a Minkowski norm on its Lie algebra T_eG, and then left translating it to other points. Therefore, every Lie group G admits a left invariant Randers metric because one can freely assign a Randers norm to T_eG. These Randers metrics are W-quadratic which are not R-quadratic nor strongly Ricci-quadratic.

Maybe, other than Example 5, there exist nontrivial homogeneous W-quadratic metrics of dimension $n \geq 3$. During the preparation of this paper, Libing Huang informed me that Lun Zhang classified all three-dimensional Lie groups which admit a left invariant Randers metric of scalar flag curvature. His result shows that there are only three possibilities:

(i) The group $E(2)$ of all Euclidean motions on the plane with a locally Minkowski metric;
(ii) The group $SU(2)$ or $SO(3)$ with Bao-Shen metrics;
(iii) The hyperbolic model of a Riemannian space of constant sectional curvature -1, with a Randers metric, is given by this Riemannian metric plus a left invariant closed one-form.

Case (iii) of the above classification certifies our claim. It is remarkable that Zhang has excluded the trivial case of a commutative group $\mathbb{R}^3$, with Minkowski metric. Case (i) can also be found by Huang (for more details, see Section 5.1 in [10]).

By considering Theorem 1.3, an interesting question arises:

Is there any homogeneous W-quadratic surface that is not R-quadratic?

Here, by using Theorems 1.1 and 1.2, we construct a family of two-dimensional homogeneous W-quadratic metrics that are not trivial.

Example 6 (A family of W-quadratic Randers metrics) Let G be a two-dimensional connected and simply connected Lie group which is not Abelian. Then G admits a left invariant Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ with constant curvature $\mathbf{K}_\alpha = -1$. Let $\beta = b_i(x)y^i$ be a nonzero left invariant one-form on G. Then for $t \in \mathbb{R}$ sufficiently close to 0, $F_t = \alpha + t\beta$ is a family of non-Riemannian left invariant Randers metrics on G. When t is sufficiently close to 0, F_t has negative flag curvature, so they are not locally Minkowskian. However, by applying the same argument used in Theorem 1.3, we find that $\{F_t\}_{t\in\mathbb{R}}$ are W-quadratic. Then, by using Theorems 1.1 and 1.2, we conclude that $\{F_t\}_{t\in\mathbb{R}}$ is a family of W-quadratic Randers metrics which are not R-quadratic nor strongly Ricci-quadratic.

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