LETTER TO THE EDITOR

Dear Editor,

In the December 1988 issue of the *Journal of Applied Probability* the paper [1] by V. Sharma about stability of slotted ALOHA systems was published. The same result was published in my paper [2] in March 1988 issue of the *Siberian Mathematical Journal*. Although the results of these papers are identical, the methods are different. In [2] general theorems due to Borovkov [3] are used, whereas in [1] in fact the author proves these general theorems in a very special situation. The approach used in [2] allows the result to be established in a shorter and more natural way. I think that this is of interest to the readership of JAP, and so I shall describe this approach briefly here.

Let M be the number of users, $a_i(t)$ the number of new packets generated in the slot $(t, t+1)$ at the ith station, $b_i(t)$ the control variable of the ith user in the slot $(t, t+1)$ (if $b_i(t)=1$ then the ith user can transmit one packet in the slot $(t, t+1)$; if $b_i(t)=0$ then it cannot do this). As the main process I study a vector-valued queue length process $\boldsymbol{q}(t)=(q_1(t), \cdots, q_M(t))$, where $q_i(t)$ is the number of packets in the queue at the ith station at time t (including new packets generated in the slot $(t-1, t)$).

It is easy to see that the following recursive relation is valid ($\delta(n)$ is the indicator of the set of positive integers):

$$(1) \qquad q_i(t+1)=q_i(t)+a_i(t)-b_i(t)\cdot\delta(q_i(t))\cdot\prod_{j\neq i}[1-b_j(t)\cdot\delta(q_j(t))].$$

With an initial condition $\boldsymbol{q}(0)=(q_1(0), \cdots, q_M(0))$, this formula allows us to obtain $\boldsymbol{q}(t)$ for all $t \geqq 1$. It should be noted that (1) is a pure algebraic relation and does not depend on assumptions about the probabilistic structure of the sequences $\boldsymbol{a}(t)$ and $\boldsymbol{b}(t)$.

Now suppose that $a_i(t)$ and $b_i(t)$, $i=1, \cdots, M$ ($t=0, 1, 2, \cdots$) are random variables on some probability space. Suppose also that the vectors $(\boldsymbol{a}(t), \boldsymbol{b}(t))$, $t \geqq 0$, form a stationary and metrically transitive sequence. Without loss of generality we can assume that this sequence is defined for $t<0$ as well.

Theorem 1. If $\boldsymbol{q}(0)=\boldsymbol{0}$ then a stationary sequence $\boldsymbol{Q}(t)$, $t \in Z$, exists (perhaps not proper) such that

1. $\boldsymbol{Q}(t)$ satisfies Equation (1).
2. Distribution of $\boldsymbol{q}(t)$ monotonically converges to the distribution of $\boldsymbol{Q}(0)$ as $t \to \infty$.

Proof. Let $\boldsymbol{x}=(x_1, \cdots, x_M)$, $\boldsymbol{y}=(y_1, \cdots, y_M)$, $\boldsymbol{z}=(z_1, \cdots, z_M) \in Z_+^M$ (and $z_i=0$ or 1) and the function $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})=(f_1, \cdots, f_M) \in Z_+^M$ is given by the formula $f_i = x_i+y_i-z_i\delta(x_i)\prod_{j\neq i}(1-Z_j\delta(x_j))$. It is easy to show that $\boldsymbol{f}$ does not decrease with respect to $\boldsymbol{x}$, i.e. $f(\boldsymbol{x}', \boldsymbol{y}, \boldsymbol{z}) \leqq f(\boldsymbol{x}'', \boldsymbol{y}, \boldsymbol{z})$ if $\boldsymbol{x}' \leqq \boldsymbol{x}''$ (all vectors are compared componentwise). Because f is continuous with respect to $\boldsymbol{x}$ our statement follows immediately from Lemma 1, §26 [3].

Theorem 2. If $Ea_i(t) < E[b_i(t) \Pi_{j \neq i} (1 - b_j(t))$ for all $i = 1, \cdots, M$ then $Q(t) < \infty$ with probability 1.

Proof. Let us define a new sequence $\boldsymbol{r}(t) = (r_1(t), \cdots, r_M(t))$ by the formulas (writing $(a)^+ = \max(a, 0)$) : $r_i(0) = 0$, $r_i(t + 1) = [r_i(t) - b_j(t) \Pi_{j \neq i} (1 - b_j(t))] + a_i(t)$, $t \geqq 0$. By means of induction with respect to t we can show that $\boldsymbol{q}(t) \leqq \boldsymbol{r}(t)$ for all $t \geqq 0$. If $u_i(t) = r_i(t) - a_i(t - 1)$, then this new sequence satisfies the relations: $u_i(1) = 0$, $u_i(t + 1) = [u_i(t) + a_i(t - 1) - b_i(t) \Pi_{j \neq i} (1 - b_j(t))]^+$, $t \geqq 1$. Thus $u_i(t)$ is identical to the waiting time of the call $\# t$ arriving in the classical $G/G/1/\infty$ queueing system.

The interval between arrivals of the calls $\# t$ and $\# t + 1$ in this queueing system is equal to $b_i(t) \Pi_{j \neq i} (1 - b_j(t))$, and the service time of the call $\# t$ is equal to $a_i(t - 1)$. Using a well-known result for the ergodicity of this simplest queue we can guarantee that $u_i(t)$ as well as $r_i(t)$ converge to proper random variables (i.e. finite a.s.). This implies that $Q_i(t) < \infty$ a.s.

In a similar way one can use general theorems due to Borovkov [3] in order to consider general initial conditions, establish continuity theorems, estimations of speed of convergence to stationary regime, etc.

Department of Probability, Mechanics and Mathematics Faculty,
Moscow State University, Moscow 119899, USSR

Yours sincerely,

GENNADIJ FALIN

References

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