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# ON THE LEBESGUE FUNCTION FOR LAGRANGE INTERPOLATION WITH EQUIDISTANT NODES

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**Abstract**

Properties of the Lebesgue function associated with interpolation at the equidistant nodes

$$x_{k,n} = k, \qquad k = 0, 1, 2, \dots, n,$$

are investigated. In particular, it is proved that the relative maxima of the Lebesgue function are strictly decreasing from the outside towards the middle of the interval $[0, n]$, and upper and lower bounds, and an asymptotic expansion, are obtained for the smallest maximum when $n$ is odd.

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## 1. Introduction

Suppose $f$ is a real-valued function defined on the interval $[a, b]$, and let

$$M = \{x_{k,n} : k = 0, 1, 2, \dots, n; n = 1, 2, 3, \dots\}$$

be a triangular matrix such that, for each $n$,

$$a \le x_{0,n} < x_{1,n} < \dots < x_{n,n} \le b.$$

Then, for each $n$, there is a unique polynomial $L_n(f, x)$ of degree $n$ (or less) such that

$$L_n(f, x_{k,n}) = f(x_{k,n}), \qquad k = 0, 1, 2, \dots, n.$$





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We refer to $L_n(f, x)$ as the Lagrange interpolation polynomial of degree $n$. A formula for $L_n(f, x)$ is given by

$$(1) \qquad L_n(f, x) = \sum_{k=0}^{n} f(x_{k,n}) l_{k,n}(x),$$

where

$$(2) \qquad l_{k,n}(x) = \frac{\omega(x)}{(x - x_{k,n})\omega'(x_{k,n})}, \qquad k = 0, 1, 2, \ldots, n,$$

and

$$(3) \qquad \omega(x) = (x - x_{0,n})(x - x_{1,n}) \cdots (x - x_{n,n}).$$

In studying the error $|L_n(f, x) - f(x)|$, an important quantity is the Lebesgue function

$$(4) \qquad \lambda_n(x) = \sum_{k=0}^{n} |l_{k,n}(x)|, \qquad a \le x \le b.$$

This function is important because, for $a \le x \le b$, $\lambda_n(x)$ is the norm of the linear functional

$$L_n(\cdot, x) \colon C([a, b]) \to \mathscr{R} \qquad f \mapsto L_n(f, x).$$

Further,

$$\Lambda_n := \max\{\lambda_n(x) \colon a \le x \le b\},$$

which is known as the Lebesgue constant, is the norm of the linear operator

$$L_n \colon C([a, b]) \to C([a, b]) \qquad f \mapsto L_n(f).$$

The study of Lagrange interpolation polynomials has long had an important place in approximation theory and numerical analysis. However, although it might initially be expected that the matrix of equidistant nodes

$$x_{k,n} = a + k(b - a)/n, \qquad k = 0, 1, 2, \ldots, n,$$

would be the most commonly studied situation, the famous examples of C. Runge [8] and S. N. Bernstein [1] have ensured that the choice of equidistant nodes has been relatively unpopular in the study of interpolation polynomials. More popular are node systems where the $x_{k,n}$ $(k = 0, 1, 2, \ldots, n)$ are the zeros of some classical polynomials such as the Jacobi polynomials. (See, for example, G. Szegö [10].) However, the determination of these zeros can often raise serious computational questions.

A. Schönhage [9] and P. O. Runck [6, 7] have studied the Lebesgue function (4) for equidistant nodes. Since one system of equidistant nodes can be

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obtained from any other such system of nodes by a linear transformation under which the Lebesgue function is invariant, we will, like Schönhage, let

$$x_{k,n} = k, \qquad k = 0, 1, 2, \dots, n.$$

Then, for $j = 0, 1, 2, \dots, n-1$, it is known that

(i) $\lambda_n(x)$ is a polynomial on $[j, j+1]$;
(ii) $\lambda_n(j) = \lambda_n(j+1) = 1$;
(iii) $\lambda_n(x) > 1$ if $j < x < j+1$;
(iv) $\lambda_n(x)$ has precisely one local maximum on $[j, j+1]$ (see [2, p. 271]); we define

$$m_{j,n} := \max\{\lambda_n(x) : j \le x \le j+1\};$$

(v) $\lambda_n(n/2 + t) = \lambda_n(n/2 - t)$ if $0 \le t \le n/2$;
(vi) $\Lambda_n = \max\{\lambda_n(x) : 0 \le x \le n\} = \max\{\lambda_n(x) : 0 \le x \le 1\} = m_{0,n}$.

Properties (i)–(v) are easy to prove; Schönhage proved (vi). In this paper we establish the following two additional properties of $\lambda_n(x)$:

(vii) $\lambda_n(x) > \lambda_n(x+1)$ if $0 < x < [(n-1)/2]$ and $x$ is not an integer;
(viii) $m_{0,n} > m_{1,n} > m_{2,n} > \cdots > m_{[(n-1)/2],n}$.

Note that (viii), which is an immediate consequence of (vii), states that the relative maxima of $\lambda_n(x)$ are strictly decreasing from the outside towards the centre of the interval $[0, n]$. This situation is analogous to the behaviour of the Lebesgue function for Lagrange interpolation based on the Čebyšev nodes (see Brutman [3]). Property (vii) follows from Theorem 1 below. In the statement of the theorem, and elsewhere, the symbol $(a)_k$ is defined by

$$(a)_k := \begin{cases} 1, & k = 0, \\ a(a+1)(a+2)\cdots(a+k-1), & k = 1, 2, 3, \dots. \end{cases}$$

THEOREM 1. *Suppose* $0 \le x < [(n-1)/2]$, *and write* $x = m + h$, *where* $m$ *is an integer, and* $0 \le h < 1$. *Then*

$$\lambda_n(x) - \lambda_n(x+1) = \frac{(h)_{m+1}(1-h)_{n-m-1}}{n!} \sum_{k=m+2}^{n-m-1} \binom{n+1}{k}. \tag{5}$$

In his paper [9], Schönhage proved that, if $\gamma$ denotes Euler's constant $0.577\dots$, then

$$\Lambda_n = m_{0,n} \sim \frac{2^{n+1}}{en(\log n + \gamma)}, \quad \text{as } n \to \infty, \tag{6}$$

and

$$m_{r,n} < \frac{2}{\pi}(\log(r + 3/2) + 2\log 2 + \gamma), \qquad n = 2r + 1. \tag{7}$$

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While it seems to be difficult to improve (6) (see Trefethen and Weideman [11] for an account of attempts to estimate the Lebesgue constant for equidistant nodes), we have been able to improve (7) considerably. The result is stated in Corollary 1 below. In addition, in Corollary 2 we present an asymptotic expansion for $m_{r,2r+1}$. Our results depend on the following representation.

THEOREM 2. *Denote the generalised hypergeometric function by*

$$ {}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}\middle| z\right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_q)_k} \frac{z^k}{k!} $$

*and the logarithmic derivative of the gamma function by*

$$\psi(x) := \Gamma'(x)/\Gamma(x),$$

*where $\Gamma(\cdot)$ is the gamma function. Then, if $n = 2r+1$,*

$$m_{r,n} = \sum_{k=0}^{r} \left[\frac{(1/2)_k}{(1)_k}\right]^2 \tag{8}$$

$$= \frac{1}{\pi}\left[\psi(r+3/2) + (\gamma + 4\log 2) - \frac{1}{4(r+3/2)} {}_4F_3\left(\begin{matrix}3/2, 3/2, 1, 1 \\ r+5/2, 2, 2\end{matrix}\middle| 1\right)\right]. \tag{9}$$

COROLLARY 1. *If $n = 2r+1$, then*

$$\frac{1}{\pi}(\log(r+3/2)+2+\log(8/3)-\pi^2/60) < m_{r,n} < \frac{1}{\pi}(\log(r+1)+2+2\log 2). \tag{10}$$

COROLLARY 2. *As $n = 2r+1 \to \infty$, then*

$$m_{r,n} \sim \frac{1}{\pi}\left(\log(r+3/2) + (\gamma + 4\log 2) + \sum_{k=1}^{\infty} \frac{c_k}{(r+3/2)^k}\right), \tag{11}$$

*where $c_1 = -3/4$, $c_2 = -43/192$, $c_3 = -7/128$, $c_4 = 619/122880$, etc.*

## 2. Proofs of the results

PROOF OF THEOREM 1. By Schönhage [9, p. 62], we know that if $p$ is an integer, and $p-1 \le x \le p$, then $\lambda_n(x) \equiv E_p(x)$, where $E_p(x)$ is the polynomial of degree $n$ (or less) which satisfies

$$E_p(\nu) = \begin{cases} (-1)^{p-1-\nu}, & \nu = 0, 1, \ldots, p-1; \\ (-1)^{p-\nu}, & \nu = p, p+1, \ldots, n. \end{cases}$$

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From Lagrange's formulae (1)–(3) it follows that

$$E_p(x) = \sum_{k=0}^{p-1} \frac{(-1)^{n+p-1}}{k!(n-k)!} \frac{(x-n)_{n+1}}{x-k} + \sum_{k=p}^{n} \frac{(-1)^{n+p}}{k!(n-k)!} \frac{(x-n)_{n+1}}{x-k}.$$

Now, if $x$ is an integer, the theorem is true by the earlier-mentioned property (ii) of $\lambda_n(x)$. On the other hand, if $x = m + h$, where $m$ is an integer and $0 < h < 1$, then

$$\begin{aligned} \lambda_n(x) - \lambda_n(x+1) &= E_{m+1}(x) - E_{m+2}(x+1) \\ &= \frac{(h)_{m+1}(1-h)_{n-m-1}}{n!} \\ &\quad \times \left[ \left( \sum_{k=0}^{m} \frac{n-m-h}{m-k+h} \binom{n}{k} + \sum_{k=m+1}^{n} \frac{n-m-h}{k-m-h} \binom{n}{k} \right) \right. \\ &\quad \left. - \left( \sum_{k=0}^{m+1} \frac{m+1+h}{m+1-k+h} \binom{n}{k} + \sum_{k=m+2}^{n} \frac{m+1+h}{k-m-1-h} \binom{n}{k} \right) \right]. \end{aligned} \tag{12}$$

Now write

$$\sum_{k=0}^{m+1} \frac{m+1+h}{m+1-k+h} \binom{n}{k} = 1 + \sum_{k=0}^{m} \frac{m+1+h}{m-k+h} \binom{n}{k+1}$$

and

$$\sum_{k=m+1}^{n} \frac{n-m-h}{k-m-h} \binom{n}{k} = \sum_{k=m+2}^{n} \frac{n-m-h}{k-m-1-h} \binom{n}{k-1} + 1,$$

and substitute these results in (12). We obtain, after some tidying up,

$$\begin{aligned} \lambda_n(x) - \lambda_n(x+1) &= \frac{(h)_{m+1}(1-h)_{n-m-1}}{n!} \left[ \sum_{k=m+2}^{n} \binom{n+1}{k} - \sum_{k=0}^{m} \binom{n+1}{k+1} \right] \\ &= \frac{(h)_{m+1}(1-h)_{n-m-1}}{n!} \left[ \sum_{k=1}^{n-m-1} \binom{n+1}{k} - \sum_{k=1}^{m+1} \binom{n+1}{k} \right]. \end{aligned}$$

The condition $x < [(n-1)/2]$ ensures that $m + 1 < n - m - 1$, and hence (5) is established.

Proof of Theorem 2. Let $n = 2r + 1$. Then Schönhage [9, p. 64] has proved

$$\begin{aligned} m_{r,n} &= \max\{\lambda_n(x) : r \le x \le r+1\} \\ &= \lambda_n(r + 1/2) \\ &= 2 \prod_{j=0}^{r} (r + 1/2 - j)^2 \sum_{k=0}^{r} \frac{1}{k!(2r+1-k)!} \frac{2}{2r+1-2k}. \end{aligned} \tag{13}$$

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The key step in the proof of the theorem is to obtain the representation (8) of $m_{r,n}$ as a partial sum of the (divergent) hypergeometric series ${}_2F_1\left({1/2,\,1/2 \atop 1}\middle|1\right)$. This is done as follows, where on several occasions we use standard results for the integral or sum of trigonometric functions—these results can be found, for example, in [4].

We begin by rearranging Schönhage's formula (13). We have

$$\begin{aligned}
m_{r,n} &= 2\prod_{j=0}^{r}(r+1/2-j)^2\sum_{k=0}^{r}\frac{1}{k!(2r+1-k)!}\frac{2}{2r+1-2k}\\
&= \frac{((2r+1)!)^2}{2^{4r}(r!)^2}\sum_{k=0}^{r}\frac{1}{k!(2r+1-k)!}\frac{1}{2r+1-2k}\\
&= \frac{(2r+1)!}{2^{4r}(r!)^2}\sum_{k=0}^{r}\binom{2r+1}{k}\frac{1}{2r+1-2k}.
\end{aligned}$$

Since

$$\binom{2r+1}{k} = \frac{2^{2r+2}}{\pi}\int_0^{\pi/2}\cos^{2r+1}\theta\cos(2r+1-2k)\theta\,d\theta,$$

$$k = 0, 1, 2, \ldots, r, \; m_{r,n}$$

can be written as

$$\begin{aligned}
m_{r,n} &= \frac{(2r+1)!}{\pi 2^{2r-2}(r!)^2}\int_0^{\pi/2}\cos^{2r+1}\theta\left(\sum_{k=0}^{r}\frac{\cos(2k+1)\theta}{2k+1}\right)d\theta\\
&= \frac{(2r+1)!}{\pi 2^{2r-2}(r!)^2}\int_0^{\pi/2}\cos^{2r+1}\theta\int_\theta^{\pi/2}\left(\sum_{k=0}^{r}\sin(2k+1)\phi\right)d\phi\,d\theta\\
&= \frac{(2r+1)!}{\pi 2^{2r-2}(r!)^2}\int_0^{\pi/2}\cos^{2r+1}\theta\int_\theta^{\pi/2}\frac{\sin^2(r+1)\phi}{\sin\phi}\,d\phi\,d\theta.
\end{aligned}$$

Interchanging the order of integration gives

$$m_{r,n} = \frac{(2r+1)!}{\pi 2^{2r-2}(r!)^2}\int_0^{\pi/2}\frac{\sin^2(r+1)\phi}{\sin\phi}\int_0^{\phi}\cos^{2r+1}\theta\,d\theta\,d\phi.$$

Now,

$$\int_0^{\phi}\cos^{2r+1}\theta\,d\theta = \frac{2^{2r}(r!)^2}{(2r+1)!}\sin\phi\sum_{k=0}^{r}\frac{(2k)!}{2^{2k}(k!)^2}\cos^{2k}\phi,$$

and for $k = 0, 1, 2, \ldots, r$, we have

$$\begin{aligned}
\int_0^{\pi/2}\sin^2(r+1)\phi\cos^{2k}\phi\,d\phi &= \frac{1}{2}\int_0^{\pi/2}[1-\cos 2(r+1)\phi]\cos^{2k}\phi\,d\phi\\
&= \frac{1}{2}\int_0^{\pi/2}\cos^{2k}\phi\,d\phi = \frac{(2k)!}{2^{2k}(k!)^2}\frac{\pi}{4}.
\end{aligned}$$

Consequently we can write

$$m_{r,n} = \sum_{k=0}^{r} \frac{((2k)!)^2}{2^{4k}(k!)^4},$$

which is (8).

Now let the truncated ${}_2F_1$ hypergeometric series be denoted by

$$y_m\left(\begin{matrix} a\,,\,b \\ c \end{matrix}\,\middle|\, z\right) := \sum_{k=0}^{m} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

By noting that $m_{r,n} = y_r\left(\begin{smallmatrix} 1/2,\,1/2 \\ 1 \end{smallmatrix}\middle|1\right)$, and using the relations $\psi(1) = -\gamma$ and $\psi(1/2) = -\gamma - 2\log 2$, we see that (9) follows immediately from the equation

$$\begin{aligned} y_m = \left(\begin{matrix} a\,,\,b \\ a+b \end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \Big\{ & \psi(b+m+1) + \psi(1) - \psi(a) - \psi(b) \\ & - \frac{b(1-a)}{b+m+1}\, {}_4F_3\left(\begin{matrix} b+1\,,\,2-a\,,\,1\,,\,1 \\ b+m+2\,,\,2\,,\,2 \end{matrix}\,\middle|\, 1\right)\Big\}, \end{aligned}$$

which can be found in Luke [5, p. 109].

**Proof of the Corollaries.** To prove the corollaries, we note that

$$\psi(r+3/2) = \psi(1/2) + \sum_{k=0}^{r}(k+1/2)^{-1} = -\gamma - 2\log 2 + 2 + \sum_{k=1}^{r}(k+1/2)^{-1}.$$

Because

$$\log\left(\frac{k+3/2}{k+1/2}\right) < \frac{1}{k+1/2} < \log\left(\frac{k+1}{k}\right), \qquad k = 1, 2, \ldots, r,$$

we have

$$\begin{aligned} &-\gamma - 2\log 2 + 2 + \log(r+3/2) + \log(2/3) \\ &\qquad < \psi(r+3/2) < -\gamma - 2\log 2 + \log(r+1). \end{aligned} \tag{14}$$

Furthermore,

$$\begin{aligned} 0 &< \frac{1}{4(r+3/2)}\, {}_4F_3\left(\begin{matrix} 3/2\,,\,3/2\,,\,1\,,\,1 \\ r+5/2\,,\,2\,,\,2 \end{matrix}\,\middle|\, 1\right) \\ &= \frac{1}{4(r+3/2)} \sum_{k=0}^{\infty} \frac{((3/2)_k)^2}{(k+1)^2(1)_k(r+5/2)_k} < \frac{1}{4(r+3/2)} \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} \\ &= \frac{\pi^2}{24(r+3/2)} \le \pi^2/60. \end{aligned} \tag{15}$$

Upon substituting the bounds (14) and (15) into (9), we obtain the inequalities (10) of Corollary 1.

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Finally, we can deduce the asymptotic expansion (11) of $m_{r,n}$ from (9) by using the asymptotic expansion [5, p. 33]

$$\psi(x) \sim \log x - \frac{1}{2x} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2kx^{2k}}$$

(where $B_{2k}$ denotes the $(2k)$-th Bernoulli number), together with the expansion as a series in increasing powers of $(r+3/2)^{-1}$ of ${}_4F_3\left({3/2, 3/2, 1, 1 \atop r+5/2, 2, 2} \middle| 1\right)$.

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