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POLAR LOCALLY CONVEX TOPOLOGIES AND ATTOUCH-WETS CONVERGENCE

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Let X be a Hausdorff locally convex space. We show that convergence of a net of continuous linear functionals on X with respect to a given polar topology on its continuous dual X' can be explained in terms of the convergence of the corresponding net of its graphs in $X \times R$, and the corresponding net of level sets at a fixed height in X, with respect to a natural generalisation of Attouch-Wets convergence in normable spaces.

1. INTRODUCTION

Let X be a real normed linear space and let $y, y_1, y_2, y_3, \ldots$ be continuous linear functionals on X. It has been long known that convergence of $\langle y_n \rangle$ to y in the norm topology can be explained in terms of the convergence of the associated sequence of graphs $\langle \mathrm{Gr}\, y_n \rangle$ to $\mathrm{Gr}\, y$ in $X \times R$, with respect to a metric $\widehat{\delta}$ on the closed linear subspaces of a normed space given by Kato [**10**, p.197–204]. For closed subspaces M and N, this distance may be described by

$$\widehat{\delta}(M, N) = \inf\{\varepsilon > 0 \colon M \cap U \subset N + \varepsilon U \text{ and } N \cap U \subset M + \varepsilon U\},$$

where U is the solid unit ball of the space. Evidently, this notion of distance is not appropriate for the more general class of closed convex sets. For one thing, such sets need not hit the unit ball. Furthermore, even for convex sets that do hit the unit ball, the triangle inequality may fail with respect to this notion of distance.

Recently, it has been shown [4] that norm convergence of a sequence of linear functionals to a nonzero limit can also be tied to the convergence of the associated sequence of level sets $\langle y_n^{-1}(1) \rangle$ to $y^{-1}(1)$ with respect to a certain metrisable topology that may be defined on all closed convex subsets of X. This Attouch-Wets topology [**1**], which reduces to the Kato metric topology for closed subspaces, is the topology of uniform convergence of distance functions on bounded subsets of X. The Attouch-Wets topology has arguably become the topology of choice for convex sets and convex functions, identified with their epigraphs. One reason for this is its stability with respect

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to duality: the polar map for closed convex sets, and the conjugate map for proper lower semicontinuous convex functions, are both continuous [**5, 14**].

In the setting of a normed linear space, other weaker forms of convergence of sequences of continuous linear functionals correspond to weaker convergence notions for graphs and level sets [**4, 6**]. The purpose of this article is to show how convergence with respect to *any* polar topology on the continuous dual of a locally convex space can be explained in terms of convergence of graphs and level sets at a fixed height. Remarkably, a suitable adaption of Attouch-Wets convergence to locally convex spaces yields a unified approach.

2. Preliminaries and notation

In the sequel X will be a (real Hausdorff) locally convex space with continuous dual X'. We denote the origin of X (respectively X') by θ (respectively θ'), and $\mathcal{U}$ will be the family of convex balanced neighbourhoods of θ. If $x \in X$ and $y \in X'$, we write $\langle x, y\rangle$ for the value of the linear functional y at x. For $y \in X'$, we write $\operatorname{Gr} y$ for the graph of y in $X \times R$: $\operatorname{Gr} y = \{(x, \alpha): x \in X \text{ and } \alpha = \langle x, y\rangle\}$.

All seminorms will be assumed nontrivial, that is, not identically zero. If p is a continuous seminorm on X, and if A is a nonempty subset of X, we write $p(x, A)$ for $\inf\{p(x,a): a \in A\}$. Thus, $p(x, A)$ is just the distance of x to A with respect to the seminorm p. When p is a norm, we write the more usual $d(x, A)$ instead. For a continuous seminorm p, define $\widehat{p}: X' \to [0, \infty]$ by

$$\widehat{p}(y) = \sup_{p(x)<1} \langle x, y\rangle.$$

Each $\widehat{p}$ is an infinite valued norm on X', that is,

(i) for all y, $0 \leqslant \widehat{p}(y) \leqslant \infty$;
(ii) $\widehat{p}(y) = 0$ if and only if $y = \theta'$;
(iii) $\widehat{p}(y_1 + y_2) \leqslant \widehat{p}(y_1) + \widehat{p}(y_2)$;
(iv) $\widehat{p}(\alpha y) = |\alpha| \widehat{p}(y)$.

We shall call $\widehat{p}(y)$ the *norm of y with respect to the seminorm p*. The relation between $\widehat{p}(y)$ and distances is as follows:

Lemma 2.1. *Let X be a locally convex space, and let y be a nonzero element of X'. Then if p is a nontrivial continuous seminorm for X', $x \in X$, and $\alpha \in R$, we have*

$$p\big(x, y^{-1}(\alpha)\big) = \frac{|\langle x, y\rangle - \alpha|}{\widehat{p}(y)}.$$

Given a family of (nontrivial) continuous seminorms $\{p_i: i \in I\}$ on X, we say that the family *determines the topology of X* provided the family $\{\alpha U_i: i \in I, \alpha > 0\}$

forms a neighbourhood base for the topology of X at θ, where for each $i \in I$, $U_i = \{x: p_i(x) < 1\}$. Note that this differs somewhat from standard usage (see, for example, [**15**, p.15]), where only finite intersections of such seminorm balls are required to form a neighbourhood base for the topology of X at θ. Each seminorm p_i gives rise to the seminorm q_i on $X \times R$ defined by $q_i(x, \alpha) = \max\{p_i(x), |\alpha|\}$. Evidently, if $\{p_i: i \in I\}$ determines the topology of X, then $\{q_i: i \in I\}$ determines the usual associated locally convex topology on $X \times R$.

We denote by $C(X)$ the nonempty closed convex subsets of X. If $A \in C(X)$, the *polar* of A is this subset of X': $A^0 = \{y \in X' : \forall a \in A, \langle a, y\rangle \leqslant 1\}$. Let us write $BC(X)$ for the family of all closed, bounded, balanced convex subsets B of X. We call a subfamily $\mathcal{B}$ of $BC(X)$ a *distinguished class of bounded sets* if

(i) it has union X;
(ii) it is directed by inclusion;
(iii) $\alpha\beta \in \mathcal{B}$ whenever $B \in \mathcal{B}$ and α is real.

By a *polar topology* on X' [**15**, p.47], we mean a locally convex topology having as a neighbourhood base at the origin θ' all sets of the form $\{B^0 : B \in \mathcal{B}\}$, where $\mathcal{B}$ is a distinguished class of bounded convex sets. Such a topology is simply the topology of uniform convergence on elements of $\mathcal{B}$. Following [**15**] we call this the *topology of $\mathcal{B}$-convergence.* When $\mathcal{B} = BC(X)$, we have the so-called *strong topology*, that is a generalisation of the norm topology on X' when X is normable. When $\mathcal{B}$ = the balanced polytopes, we get the *weak topology* $\sigma(X', X)$. When $\mathcal{B}$ = the $\sigma(X, X')$-compact balanced convex sets, we get the *Mackey topology* $\tau(X', X)$.

3. The Attouch-Wets convergence in a locally convex space

As mentioned in the introduction, if X is a normed space, then the Attouch-Wets topology τ_{AW} on the nonempty closed convex subsets $C(X)$ is the topology of uniform convergence of distance functionals (as determined by the norm) on bounded subsets of X [**1, 2, 4, 5, 3, 7, 14, 8**]. There are two standard presentations for this topology as a uniform space. The first most closely reflects the description just given. A base for the first compatible uniformity for τ_{AW} consists of all sets of the form

$$\{(A, C) \in C(X) \times C(X): \sup_{x \in B} |d(x, A) - d(x, C)| < \varepsilon\}$$

where $B \in BC(X)$, and $\varepsilon > 0$. Evidently, a countable base for this uniformity for τ_{AW} consists of all sets of the form

$$\Delta_n = \{(A, C) \in C(X) \times C(X): \sup_{\|x\| \leqslant n} |d(x, A) - d(x, C)| < \frac{1}{n}\} \qquad (n \in Z^+).$$

Our second description is more intuitive, and indicates the Attouch-Wets topology's connection to the stronger Hausdorff metric topology [**9, 12**]. The topology τ_{AW} is also determined by a weaker uniformity with base consisting of all sets of the form

$$\{(A, C) \in C(X) \times C(X) \colon A \cap B \subset C + \varepsilon U \text{ and } C \cap B \subset A + \varepsilon U\}$$

where U is the closed unit ball of the normed space X, $B \in BC(X)$, and $\varepsilon > 0$. A countable base for this second uniformity consists of all sets of the form

$$\Omega_n = \{(A, C) \in C(X) \times C(X) \colon A \cap nU \subset C + \frac{1}{n}U \text{ and } C \cap nU \subset A + \frac{1}{n}U\} \quad (n \in Z^+).$$

It is also from this perspective that one can see most easily that the Attouch-Wets topology agrees with the Kato metric topology when restricted to closed subspaces. For a proof that these two uniformities determine the same topology, the reader may consult [**4**, Lemma 3.1] or [**3**, Proposition 2.1]. Evidently, τ_{AW} is metrisable, and the standard proof that the Hausdorff metric topology is completely metrisable when X is complete goes through with minor modifications to show that τ_{AW} is completely metrisable [**2**]. Also, when restricted to bounded convex sets, τ_{AW} agrees with the usually stronger Hausdorff metric topology [**8**, Lemma 3.1]. It is easy to verify that equivalent norms produce the same hyperspace topologies (see more generally [**7**]).

There are natural generalisations of both of these constructions to the locally convex setting producing divergent results. Viewed as a locally convex space, the norm topology is determined by the single seminorm $p(x) = \|x\|$. Our first uniformity above for τ_{AW} has as a base all sets of the form

$$\Delta(B, \varepsilon) = \{(A, C) \in C(X) \times C(X) \colon \sup_{x \in B} |p(x, A) - p(x, C)| < \varepsilon\},$$

where $B \in BC(X)$, and $\varepsilon > 0$. This motivates the following definition in the locally convex setting.

DEFINITION: Let X be a locally convex space, with a defining family of seminorms $\{p_i \colon i \in I\}$. Let $\mathcal{B} \subset BC(X)$ be a distinguished class. Then the *Attouch-Wets topology* $\tau_{AW}(\mathcal{B})$ on $C(X)$ is the topology determined by the uniformity with typical subbasic entourage of the form

$$\Delta(i, B, \varepsilon) = \{(A, C) \in C(X) \times C(X) \colon \sup_{x \in B} |p_i(x, A) - p_i(x, C)| < \varepsilon\},$$

where $i \in I$, $B \in \mathcal{B}(X)$, and $\varepsilon > 0$.

As a uniform topology, $\tau_{AW}(\mathcal{B})$ is completely regular. Also, this topology is Hausdorff. To see this, let A and C be distinct elements of $C(X)$. Without loss of generality, we may assume $A^c \cap C \neq \emptyset$. Choose $x \in A^c \cap C$, $i \in I$ and $\alpha > 0$ with $(x + \alpha U_i) \cap A = \emptyset$. There exists $B \in \mathcal{B}(X)$ with $x \in B$, and it is clear that $(A, C) \notin \Delta(i, B, \alpha)$.

Given a defining family of seminorms $\{p_i : i \in I\}$ for X, the well known *Hausdorff uniform topology* [**9**, p.44] on $C(X)$ is induced by the uniformity with a subbase consisting of all sets of the form

$$\{(A, C) \in C(X) \times C(X) \colon \forall x \in X, |p_i(x, A) - p_i(x, B)| < \varepsilon\},$$

where $i \in I$ and $\varepsilon > 0$. This topology does not depend on the particular choice of the seminorms, as it is well known that another compatible uniformity has as a base all sets of the form

$$\{(A, C) \in C(X) \times C(X) \colon A \subset C + U \text{ and } C \subset A + U\}$$

where $U \in \mathcal{U}$. Evidently, the Hausdorff uniform topology is finer than each Attouch-Wets $\tau_{AW}(\mathcal{B})$. And with respect to different defining families, we must point out an unpleasant fact of life: the hyperspace topology $\tau_{AW}(\mathcal{B})$ depends on the particular set of seminorms chosen as well as the family $\mathcal{B}$.

EXAMPLE 1. In the plane R^2, if we use $\mathcal{B} = BC(X)$ and the single defining seminorm $p(\alpha_1, \alpha_2) = \left(\alpha_1^2 + \alpha_2^2\right)^{1/2}$, we get ordinary Attouch-Wets topology on $C(X)$. But if we include the continuous seminorm $p'(\alpha_1, \alpha_2) = |\alpha_2|$ and use $\{p, p'\}$, we get a strictly finer topology on the closed convex sets. In fact, this is the case if we restrict our attention to hyperplanes. To see this, let $C = \{(\alpha_1, \alpha_2) \colon \alpha_2 = 1\}$ and $C_n = \{(\alpha_1, \alpha_2) \colon \alpha_2 = \alpha_1/n + 1\}$. Then $\langle C_n \rangle$ is Attouch-Wets convergence to C in the usual norm sense. But $p'(\theta, C) = 1$, whereas for each n, $p'(\theta, C_n) = 0$. Thus, we fail to get even pointwise convergence of distance functions associated with the family of seminorms.

Again, let X be a locally convex space with a given distinguished class of bounded convex sets $\mathcal{B}$. To generalise our second presentation of the Attouch-Wets topology, it is natural to consider all "basic entourages" of the form

$$\Omega[B, U] = \{(A, C) \in C(X) \times C(X) \colon A \cap B \subset C + U \text{ and } C \cap B \subset A + U\}$$

where $B \in \mathcal{B}$ and $U \in \mathcal{U}$. But there is one basic problem with this program: such sets do not in general form a base for a uniformity! We next present a negative result in this direction.

PROPOSITION 3.1. *Let X be an infinite dimensional locally convex space, and let $\mathcal{B}$ be a distinguished class of balanced compact convex sets. Then the filter base of sets $\{\Omega[B, U] \colon B \in \mathcal{B}$ and $U \in \mathcal{U}\}$ does not form a base for a uniformity on $C(X)$.*

PROOF: Let τ be the topology for X. As is well-known, if there is a coarsest uniformity compatible with $\langle X, \tau\rangle$, then the space must be locally compact (see, for example, [**13**, Theorem 6.17]). Let us assume $\{\Omega[B, U]: B \in \mathcal{B}, U \in \mathcal{U}\}$ is a base for a uniformity for $C(X)$. Evidently, its trace on X (as identified with $\{\{x\}: x \in X\}$) is a compatible uniformity for $\langle X, \tau\rangle$. We claim that this trace must give rise to the coarse uniformity on $\langle X, \tau\rangle$.

To verify this claim, fix $B \in \mathcal{B}$, $U \in \mathcal{U}$ and let $\Omega'[B, U] = \{(x_1, x_2) \in X \times X: (\{x_1\}, \{x_2\}) \in \Omega[B, U]\}$. It is clear that

$$\Omega'[B, U] = \{(x_1, x_2): \text{ either } (x_1, x_2) \in (B^c \times B^c) \text{ or } x_1 - x_2 \in U\}.$$

Now let $\mathcal{D}$ be any compatible diagonal uniformity for $\langle X, \tau\rangle$. It follows from the compactness of B that there exists a symmetric entourage $D \in \mathcal{D}$ such that for each $b \in B$, whenever $(x, b) \in D$, we have $x - b \in U$. Suppose $(x_1, x_2) \in D$ is arbitrary. If $(x_1, x_2) \in (B^c \times B^c) \cap D$, then $(x_1, x_2) \in \Omega'[B, U]$. Otherwise, either $x_1 \in B$ and $(x_1, x_2) \in D$ or $x_2 \in B$ and $(x_1, x_2) \in D$. In either case, by the choice of D, we have $x_1 - x_2 \in U$ and so $(x_1, x_2) \in \Omega'[B, U]$. We have shown that $\Omega'[B, U] \supset D$, and so the uniformity generated by $\{\Omega'[B, U]: B \in \mathcal{B}, U \in \mathcal{U}\}$ will be contained in $\mathcal{D}$. We may now assert that the uniformity generated by $\{\Omega'[B, U]: B \in \mathcal{B}, U \in \mathcal{U}\}$ is the coarsest compatible uniformity for $\langle X, \tau\rangle$, and we conclude that $\langle X, \tau\rangle$ is locally compact. This implies that X is finite dimensional. ∎

Still, we may talk about convergence of a net $\langle A_\lambda\rangle$ with respect to this filter base of sets as follows.

DEFINITION: Let X be a locally convex space, and let $\mathcal{B} \subset BC(X)$ be a distinguished class. Suppose $A \in C(X)$ and $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ is a net in $C(X)$. We write $A = AW(\mathcal{B})$-lim A_λ provided for each $B \in \mathcal{B}$ and $U \in \mathcal{U}$, there exists $\lambda_0 \in \Lambda$ such that for all $\lambda \geqslant \lambda_0$, we have both $A \cap B \subset A_\lambda + U$ and $A_\lambda \cap B \subset A + U$.

LEMMA 3.2. *Let X be a locally convex space, let $\mathcal{B} \subset BC(X)$ be a distinguished class, and let $\{p_i: i \in I\}$ be a defining family of seminorms for the topology of X. Then in $C(X)$, $\tau_{AW}(\mathcal{B})$ convergence as determined by $\{p_i: i \in I\}$ ensures $AW(\mathcal{B})$ convergence.*

PROOF: Fix $B \in \mathcal{B}$ and $U \in \mathcal{U}$. There exist $\varepsilon > 0$ and $i \in I$ such that $\varepsilon U_i \subset U$. Suppose A and C are closed convex sets with $\sup_{x\in B} |p_i(x, A) - p_i(x, C)| < \varepsilon$, and $x \in A \cap B$. Since $p_i(x, A) = 0$, we get $p_i(x, C) < \varepsilon$ so that $(x + \varepsilon U_i) \cap C \neq \emptyset$. This means that $x \in C + \varepsilon U_i$. Thus, $A \cap B \subset C + \varepsilon U_i \subset C + U$, and, similarly, $C \cap B \subset A + U$. ∎

We note that for the sequence $\langle C_n\rangle$ of Example 1, we have $C = AW(\mathcal{B}) - \lim C_n$, while $C \neq \tau_{AW}(\mathcal{B}) - \lim C_n$ with respect to the defining seminorms p and p' and $\mathcal{B} = BC(X)$.

We now come to a second, very disappointing, fact of life: $AW(\mathcal{B})$ limits of nets of (compact) convex sets need not be unique, even if $\mathcal{B} = BC(X)$!

EXAMPLE 2. Consider the function space $C(R, R)$ with topology of compact convergence, with the usual defining family of increasing seminorms $\{p_i : i \in Z^+\}$, where

$$p_i(x) = \max\{|x(t)| : -i \leqslant t \leqslant i\} \qquad (i \in Z^+).$$

Let x_0 be the zero function θ and let x_1 be the function that is identically equal to one. With $\mathcal{B} = BC(C(R, R))$, we produce a net of compact convex sets that is $AW(\mathcal{B})$-convergent to each subsegment of $\operatorname{conv}\{x_0, x_1\}$. Our directed set will be $\mathcal{B} \times Z^+$, where the bounded sets are ordered by inclusion and Z^+ has the usual order. To define our net $(B, i) \to C_{(B,i)}$ on this directed set, fix $B \in \mathcal{B}$ and $i \in I$. There exists an increasing sequence of integers $\langle n_j \rangle$ such that for each $j \in Z^+$ and for each $t \in [-j, j]$ and each $x \in B$, we have $|x(t)| < n_j$. Clearly we can construct a positive continuous function x_B such that for each $j \in Z^+$ we have

$$\max\{x_B(t) : -j \leqslant t \leqslant j\} > j n_j.$$

We claim that for each function w in $\operatorname{conv}\{x_0, x_1, x_B\}$ that does not lie in $\operatorname{conv}\{x_0, x_1\}$ we have $w \notin B$. We may write $w = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_B$, where $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and where $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$ and $\alpha_3 > 0$. Choose $j \in Z^+$ with $1/j < \alpha_3$. Choose $t \in [-j, j]$ maximising the function x_B restricted to that interval. We have

$$w(t) > \alpha_3 x_B(t) > \frac{1}{j} x_B(t) > \frac{1}{j}(j n_j) = n_j.$$

This shows that $w \notin B$. Now pick points $x_0^* \neq x_0$ in $\operatorname{conv}\{x_0, x_B\}$ and $x_1^* \neq x_1$ in $\operatorname{conv}\{x_1, x_B\}$ such that

$$\operatorname{conv}\{x_0, x_1, x_B\} \subset \operatorname{conv}\{x_0^*, x_1^*, x_B\} + \frac{1}{i} U_i.$$

Finally, set $C_{(B,i)} = \operatorname{conv}\{x_0^*, x_1^*, x_B\}$, and let C be any subsegment of $\operatorname{conv}\{x_0, x_1\}$. We shall show that the net $(B, i) \to C_{(B,i)}$ is $AW(\mathcal{B})$-convergent to C. To this end, fix $B_0 \in \mathcal{B}$ and a neighbourhood U of the origin. There exists $i \in Z^+$ with $(1/i)U_i \subset U$. We have

$$C_{(B,j)} \cap B_0 = \emptyset \subset C + U$$

and

$$C \cap B_0 \subset \operatorname{conv}\{x_0, x_1\} \subset C_{(B,j)} + \frac{1}{j} U_j \subset C_{(B,j)} + U,$$

provided $(B, j) \geqslant (B_0, i)$, that is, provided $B \supset B_0$ and $j \geqslant i$. This establishes convergence of the net to multiple limits as required.

4. Convergence in a polar topology and Attouch-Wets convergence

Our first goal of this section is to show that convergence of continuous linear functionals with respect to the topology of $\mathcal{B}$-convergence corresponds to the $AW(\mathcal{B})$-convergence of their level sets at fixed heights (provided the limit function is not θ').

Lemma 4.1. *Let X be a locally convex space, let $\langle y_\lambda \rangle$ be a net in X' and let $y \in X'$. Let $B \in \mathcal{B}$, where $\mathcal{B}$ is a distinguished subclass of $BC(X)$. Suppose that $y^{-1}(1) = AW(\mathcal{B}) - \lim y_\lambda^{-1}(1)$. Then there exists $\alpha > 0$ and some index λ_0 such that for all $\lambda \geqslant \lambda_0$, we have both $\sup\{\langle b, y_\lambda \rangle : b \in B\} < \alpha$ and $\sup\{\langle b, y \rangle : b \in B\} < \alpha$.*

Proof: Let U be a neighbourhood of the origin that contains B on which y is bounded. Choose $\rho > 2$ such that $\sup\{\langle x, y \rangle : x \in U\} < \rho$. Choose λ_0 so large that for all $\lambda \geqslant \lambda_0$, we have $y_\lambda^{-1}(1) \cap B \subset y^{-1}(1) + (1/2\rho)U$. We claim that for all $\lambda \geqslant \lambda_0$, $\sup\{\langle b, y_\lambda \rangle : b \in B\} < 2\rho^2$. Suppose this fails for some such λ. Since B is balanced, there exists $x \in (1/2\rho)B \subset B$ such that $x \in y_\lambda^{-1}(1)$. Since $\lambda \geqslant \lambda_0$, there exists $w \in y^{-1}(1)$ with $w \in x + (1/2\rho)U$. Since $x \in (1/\rho)U$, we get $w \in (1/\rho)U$. We conclude that $\rho w \in U$ and $\langle \rho w, y \rangle = \rho$, which violates the bound on the linear functional y restricted to U. ∎

Lemma 4.2. *Let X be a locally convex space, and let $\mathcal{B} \subset BC(X)$ be a distinguished class. Let $\langle y_\lambda \rangle$ be a net of nonzero elements of X' and let $y \neq \theta'$ be in X' with $y^{-1}(1) = AW(\mathcal{B}) - \lim y_\lambda^{-1}(1)$. Then for each $\alpha \neq 0$ we have $y^{-1}(\alpha) = AW(\mathcal{B}) - \lim y_\lambda^{-1}(\alpha)$.*

Proof: Let $B \in \mathcal{B}$ and let U be a balanced convex neighbourhood of θ. We have

$$y^{-1}(\alpha) \cap B \subset y_\lambda^{-1}(\alpha) + U \text{ and } y_\lambda^{-1}(\alpha) \cap B \subset y^{-1}(\alpha) + U$$

if and only if

$$y^{-1}(1) \cap \frac{1}{|\alpha|}B \subset y_\lambda^{-1}(1) + \frac{1}{|\alpha|}U \text{ and } y_\lambda^{-1}(1) \cap \frac{1}{|\alpha|}B \subset y^{-1}(1) + \frac{1}{|\alpha|}U.$$

By the definition of distinguished class, $B \in \mathcal{B}$ if and only if $(1/|\alpha|)B \in \mathcal{B}$. ∎

We now come to the main result of this paper.

Theorem 4.3. *Let X be a locally convex space with continuous dual X'. Let $\mathcal{B} \subset BC(X)$ be a distinguished class. Let $\langle y_\lambda \rangle$ be a net of nonzero elements of X' and let $y \neq \theta'$ be in X'. The following are equivalent:*

(i) $y^{-1}(1) = AW(\mathcal{B}) - \lim y_\lambda^{-1}(1)$;

(ii) *$\langle y_\lambda \rangle$ is convergent to y in the topology of $\mathcal{B}$-convergence for X';*

(iii) $\forall\, \alpha \in R$, $y^{-1}(\alpha) = AW(\mathcal{B}) - \lim y_\lambda^{-1}(\alpha)$.

Proof: (i) ⇒ (ii). Fix $B \in \mathcal{B}$ and let U be a balanced convex neighbourhood of the origin for which $\sup\{\langle x, y \rangle : x \in U\} < 1$. Let p be the seminorm associated with

U, so that $\widehat{p}(y) < 1$. Choose by Lemma 4.1 an index λ_0 in the underlying directed set for the net and $\alpha > 2$ such that for all $\lambda \geqslant \lambda_0$, we have

$$\sup_{b \in B}\langle b,\, y\rangle \leqslant \alpha \text{ and } \sup_{b \in B}\langle b,\, y_\lambda\rangle \leqslant \alpha.$$

Also, by Lemma 4.2, we may assume that λ_0 is also chosen so large that for $\gamma = 1, -1, -1 + (1/2\alpha)$ and $\lambda \geqslant \lambda_0$, we have

$$(*)\qquad y^{-1}(\gamma) \cap 2B \subset y_\lambda^{-1}(\gamma) + \frac{1}{4\alpha}U \text{ and } y_\lambda^{-1}(\gamma) \cap 2B \subset y^{-1}(\gamma) + \frac{1}{4\alpha}U.$$

Fix $\lambda \geqslant \lambda_0$; we claim that $y_\lambda \in y + B^0$. Suppose not. Then for some $b \in B$, we have $\langle b,\, y_\lambda\rangle > \langle b,\, y\rangle + 1$. We consider two cases for this inequality, the first of which is simple whereas the second is rather subtle:

Case (a): $\langle b,\, y_\lambda\rangle > 1/2$;
Case (b): $\langle b,\, y\rangle < -1/2$.

In case (a) there exists $\beta \leqslant 2$ such that $\langle \beta b,\, y_\lambda\rangle = 1$. Note that we must have $1/\alpha \leqslant \beta$ because y_λ is bounded by α on B. Using $\langle b,\, y_\lambda\rangle > \langle b,\, y\rangle + 1$, we have

$$\langle \beta b,\, y\rangle < 1 - \beta \leqslant 1 - \frac{1}{\alpha}.$$

Since $x \in U$ implies $|\langle x,\, y\rangle| < 1$, we cannot have $\beta b \in y^{-1}(1) + (1/\alpha)U$. Since $\beta b \in 2B$, this contradicts $y_\lambda^{-1}(1) \cap 2B \subset y^{-1}(1) + (1/4\alpha)U$, finishing case (a).

For case (b), since $-B \subset B$, there exists β with $1/\alpha \leqslant \beta \leqslant 2$ and $\langle \beta b,\, y\rangle = -1$. Clearly, $\langle \beta b,\, y_\lambda\rangle > -1 + 1/\alpha$. By condition (*) with $\gamma = -1$ we get

$$\beta b \in y_\lambda^{-1}(-1) + \frac{1}{4\alpha}U.$$

This means that there exists $x \in y_\lambda^{-1}(-1)$ with $\beta b - x \in (1/4\alpha)U$. Since $\langle \beta b,\, y_\lambda\rangle > -1 + 1/\alpha$ and $\langle x,\, y_\lambda\rangle = -1$, there exists z on the line segment joining βb and x with $\langle z,\, y_\lambda\rangle = -1 + 1/2\alpha$. Of course, $z \in \beta b + (1/4\alpha)U$, and since $\widehat{p}(y) < 1$, we have $\langle z,\, y\rangle < -1 + 1/4\alpha$. Again since $\widehat{p}(y) < 1$, for each $w \in z + (1/4\alpha)U$, we have $\langle w,\, y\rangle < -1 + 1/4\alpha + 1/4\alpha < -1 + 1/2\alpha$. As a result, the point z does not belong to $y^{-1}(-1 + 1/2\alpha) + (1/4\alpha)U$, which contradicts (*) with $\gamma = -1 + 1/2\alpha$. This finishes case (b).

(ii) $\Rightarrow$ (iii). Let $U \neq X$ be a balanced convex neighbourhood of the origin and let p be the seminorm determined by U. We have $\widehat{p}(y) \leqslant \liminf \widehat{p}(y_\lambda)$ (in fact, this is true for convergence in the $\sigma(X',\, X)$-topology which is always contained in the topology of $\mathcal{B}$-convergence). Fix $B \in \mathcal{B}$. Choose an index λ_0 in the underlying directed set for the net and a number $\mu > 0$ such that for each $\lambda \geqslant \lambda_0$

(a) $\widehat{p}(y) > \mu$ and $\widehat{p}(y_\lambda) > \mu$;
(b) $y - y_\lambda \in ((1/\mu)B)^0$.

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Fix $\lambda \geqslant \lambda_0$. Suppose $x \in B \cap y^{-1}(\alpha)$. By (b), we have $|\langle x, y_\lambda\rangle - \alpha| \leqslant \mu$. But by (a), $\sup\{\langle w, y_\lambda\rangle : w \in U\} > \mu$. Thus, there exists $w \in U$ with $\langle w, y_\lambda\rangle = \alpha - \langle x, y_\lambda\rangle$. As a result, $x + w \in y_\lambda^{-1}(\alpha)$ so that $x \in y_\lambda^{-1}(\alpha) + U$. Thus, for all $\lambda \geqslant \lambda_0$, we have

$$y^{-1}(\alpha) \cap B \subset y_\lambda^{-1}(\alpha) + U,$$

and similarly, $y_\lambda^{-1}(\alpha) \cap B \subset y^{-1}(\alpha) + U$.

(iii) $\Rightarrow$ (i). This is trivial. ∎

Corollary 4.4. *Let X be a locally convex space with continuous dual X'. Let $\langle y_\lambda\rangle$ be a net of nonzero elements of X' and let $y \neq \theta'$ be in X'. The following are equivalent:*

(i) $y^{-1}(1) = AW(BC(X)) - \lim y_\lambda^{-1}(1)$;
(ii) *$\langle y_\lambda\rangle$ is convergent to y in the strong topology for X';*
(iii) $\forall \alpha \in R,\ y^{-1}(\alpha) = AW(BC(X)) - \lim y_\lambda^{-1}(\alpha)$.

At the opposite extreme, we get a characterisation of $\sigma(X', X)$ convergence. Weak convergence of sequences in the normed setting was investigated in [4].

Corollary 4.5. *Let X be a locally convex space with continuous dual X'. Let $\langle y_\lambda\rangle$ be a net of nonzero elements of X' and let $y \neq \theta'$ be in X'. Then $y = \sigma(X', X) - \lim y_\lambda$ if and only if for each $\alpha \in R$, for each convex polytope P in X, and for each $U \in \mathcal{U}$, there exists an index λ_0 such that for all $\lambda \geqslant \lambda_0$ we have both*

$$y^{-1}(\alpha) \cap P \subset y_\lambda^{-1}(\alpha) + U \text{ and } y_\lambda^{-1}(\alpha) \cap P \subset y^{-1}(\alpha) + U.$$

We now turn to convergence of graphs. First, a lemma, which illustrates most clearly why we need our class $\mathcal{B}$ of bounded sets to be a distinguished class.

Lemma 4.6. *Let X be a locally convex space, and $\mathcal{B} \subset BC(X)$ be a distinguished class. Suppose that $\langle C_\lambda\rangle$ is a net of nonempty closed convex subsets of X satisfying $C = AW(\mathcal{B}) - \lim C_\lambda$. Then for each $x_0 \in X$ we have $C + x_0 = AW(\mathcal{B}) - \lim (C_\lambda + x_0)$.*

Proof: Fix $B \in \mathcal{B}$ and $U \in \mathcal{U}$. Since $\cup\mathcal{B} = X$, there exists $B_0 \in \mathcal{B}$ containing $-x_0$ (and x_0). Since $\mathcal{B}$ is directed by inclusion, there exists B^* in $\mathcal{B}$ such that B^* contains both $2B$ and $2B_0$. Since B^* is convex, B^* will contain $B - x_0$.

Now choose λ_0 so large that for all $\lambda \geqslant \lambda_0$ we have both

$$C \cap B^* \subset C_\lambda + U \quad \text{and} \quad C_\lambda \cap B^* \subset C + U.$$

Now fix $\lambda \geqslant \lambda_0$, and suppose $x \in (C + x_0) \cap B$. Then $x - x_0 \in C \cap B^*$, so that $x - x_0 \in C_\lambda + U$. As a result $x \in (C_\lambda + x_0) + U$, and this shows that $(C + x_0) \cap B \subset (C_\lambda + x_0) + U$. Similarly, $(C_\lambda + x_0) \cap B \subset (C + x_0) + U$. ∎

Theorem 4.7. *Let X be a locally convex space, and let $\mathcal{B} \subset BC(X)$ be a distinguished class. Let $\langle y_\lambda \rangle$ be a net in X', and let $y \in X'$. The following are equivalent:*

(i) *$\langle y_\lambda \rangle$ is convergent to y in the topology of $\mathcal{B}$-convergence for X';*

(ii) *$\operatorname{Gr} y = AW(\mathcal{B}_0) - \lim \operatorname{Gr} y_\lambda$, where $\mathcal{B}_0$ consists of products of elements of $\mathcal{B}$ with intervals of the form $[-\alpha, \alpha]$.*

Proof: We consider auxiliary continuous linear functionals y' and $\langle y'_\lambda \rangle$ on $X \times R$ defined by $y'_\lambda(x, \alpha) = \langle x, y_\lambda \rangle - \alpha$ and $y'(x, \alpha) = \langle x, y \rangle - \alpha$. Notice that none of the auxiliary linear functionals is the zero functional. If (i) holds, then $\langle y'_\lambda \rangle$ is clearly convergent to y' in the topology of $\mathcal{B}_0$-convergence for $(X \times R)'$; so, by the equivalence of conditions (ii) and (iii) in Theorem 4.3, we get $y'^{-1}(0) = AW(\mathcal{B}_0) - \lim y'^{-1}_\lambda(0)$. This in turn means that $\operatorname{Gr} y = AW(\mathcal{B}_0) - \lim \operatorname{Gr} y_\lambda$. Conversely, if (ii) holds, then since translation by a fixed vector is continuous, we have $\operatorname{Gr} y + (\theta, -1) = AW(\mathcal{B}_0) - \lim \operatorname{Gr} y_\lambda + (\theta, -1)$. This means that $y'^{-1}(1) = AW(\mathcal{B}_0) - \lim y'^{-1}_\lambda(1)$, so that by Theorem 4.3, $\langle y'_\lambda \rangle$ is convergent to y' in the topology of $\mathcal{B}_0$-convergence. Condition (i) now easily follows. ∎

We now turn to a search for a mode of convergence for linear functionals corresponding to $\tau_{AW}(\mathcal{B})$ (respectively $\tau_{AW}(\mathcal{B}_0)$) convergence of level sets (respectively graphs). Theorem 4.3 of [4] points the way. We need this analog of Lemma 4.6, whose proof is similar and is left to the reader.

Lemma 4.8. *Let X be a locally convex space with a prescribed defining family of seminorms. Suppose that $\langle C_\lambda \rangle$ is a net of nonempty closed convex subsets of X with $C = \tau_{AW}(\mathcal{B})\text{–}\lim C_\lambda$. Then for each $x_0 \in X$, we have $C + x_0 = \tau_{AW}(\mathcal{B})\text{–}\lim (C_\lambda + x_0)$.*

Theorem 4.9. *Let X be a locally convex space, and let $\{p_i : i \in I\}$ be a family of seminorms determining the topology. Let $\langle y_\lambda \rangle$ be a net of nonzero elements of X' and let $y \neq \theta'$ be in X'. The following are equivalent:*

(i) $y^{-1}(1) = \tau_{AW}(\mathcal{B}) - \lim y_\lambda^{-1}(1)$;

(ii) *$\langle y_\lambda \rangle$ is convergent to y in the topology of $\mathcal{B}$-convergence, and for all $i \in I$, $\widehat{p}_i(y) = \lim_\lambda \widehat{p}_i(y_\lambda)$;*

(iii) $\forall \alpha \in R$, $y^{-1}(\alpha) = \tau_{AW}(\mathcal{B}) - \lim y_\lambda^{-1}(\alpha)$.

Proof: (i) $\Rightarrow$ (ii). We have for each $i \in I$, $p_i(\theta, y^{-1}(1)) = \lim_\lambda p_i(\theta, y_\lambda^{-1}(1))$, which means by Lemma 2.1 that $\widehat{p}_i(y) = \lim_\lambda \widehat{p}_i(y_\lambda)$. By Lemma 3.2, we have $y^{-1}(1) = AW(\mathcal{B}) - \lim y_\lambda^{-1}(1)$, and so by Theorem 4.3, we have $\langle y_\lambda \rangle$ convergent to y in the topology of $\mathcal{B}$-convergence.

(ii) $\Rightarrow$ (iii). This is immediate from Lemma 2.1.

(iii) $\Rightarrow$ (i). This is trivial. ∎

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THEOREM 4.10. *Let X be a locally convex space, and let $\{p_i : i \in I\}$ be a family of seminorms determining the topology. Let $\langle y_\lambda \rangle$ and y be in X'. The following are equivalent:*

(i) *$\langle y_\lambda \rangle$ is convergent to y in the topology of $\mathcal{B}$-convergence, and for all $i \in I$, $\widehat{p}_i(y) = \lim_\lambda \widehat{p}_i(y_\lambda)$;*

(ii) *$\operatorname{Gr} y = \tau_{AW}(\mathcal{B}_0) - \lim \operatorname{Gr} y_\lambda$ with respect to the seminorms $\{q_i : i \in I\}$ on $X \times R$, where for each i, $q_i(x, \alpha) = \max\{p_i(x), |\alpha|\}$.*

PROOF: Again, we consider auxiliary continuous linear functionals y' and $\langle y'_\lambda \rangle$ on $X \times R$ defined by $y'_\lambda(x, \alpha) = \langle x, y_\lambda \rangle - \alpha$ and $y'(x, \alpha) = \langle x, y \rangle - \alpha$. In terms of the family of defining seminorms $\{q_i : i \in I\}$, we have $\widehat{q}_i(y_\lambda) = \widehat{p}_i(y_\lambda) + 1$ and $\widehat{q}_i(y) = \widehat{p}_i(y) + 1$. The proof now proceeds exactly as in the proof of Theorem 4.7, using Lemma 4.8. ∎

Even when $\mathcal{B} = BC(X)$, Condition (i) of Theorem 4.10 does not guarantee that $\lim_\lambda \widehat{p}_i(y_\lambda - y) = 0$ for each defining seminorm p_i. We return to the space of Example 1, with defining seminorms p and p'. Consider $y, y_1, y_2, \ldots$ in X' defined by $\langle (\alpha_1, \alpha_2), y \rangle = \alpha_1 + \alpha_2$ and $\langle (\alpha_1, \alpha_2), y_n \rangle = (1 + 1/n)\alpha_1 + \alpha_2$. Evidently, $\langle y_n \rangle$ converges to y in the strong topology. We also have $\widehat{p}(y) = \sqrt{2} = \lim \widehat{p}(y_n)$ and for each n, $\widehat{p}'(y) = \widehat{p}'(y_n) = \infty$. But for each $n \in Z^+$, $\widehat{p}'(y_n - y) = \infty$. In particular, for each $n \in Z^+$ and $k \in Z^+$, we have

$$\widehat{p}'(y_n - y) \geqslant \langle (kn, 0), y_n - y \rangle = k(n+1) - kn = k.$$

Strong convergence of continuous linear functionals plus convergence of the norms of the functionals as determined by a prescribed family of seminorms on X is not an infrequent occurrence. For example, we get this mode of convergence with $X = C(R, R)$, equipped with the seminorms for the topology of compact convergence, when $y, y_1, y_2, y_3, \ldots$ are defined by

$$\langle x, y \rangle = \int_0^1 x(t)dt, \qquad \langle x, y_n \rangle = \int_{1/n}^1 x(t)dt \qquad (n \in Z^+).$$

On the other hand, Example 1 shows that convergence in this sense is properly stronger than convergence in the strong topology, as $C = y^{-1}(1)$ and $C_n = y_n^{-1}(1)$ where $\langle (\alpha_1, \alpha_2), y \rangle = \alpha_2$ and $\langle (\alpha_1, \alpha_2), y_n \rangle = (1/n)\alpha_1 + \alpha_2$.

The Attouch-Wets topology can be defined for the space of all closed subsets of a metric space, using either the truncation approach or the function space approach (see, for example, [7]). As in the less general normed linear space setting, both paths yield the same (metrisable) topology. In particular, a "metric" Attouch-Wets topology can be defined for the closed convex subsets of a metrisable locally convex space, equipped with a translation invariant metric. Recently, Holà [**10**] has shown that metric Attouch-Wets convergence of graphs of linear functionals is stronger than convergence of the functionals in the strong topology, and that the two notions coincide if and only if X is normable. The divergence between these results and ours is easily explained: metrically bounded sets and bounded sets in the topological vector space sense do not in general coincide.

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