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ASYMPTOTIC SPECTRUM OF MULTIPARAMETER EIGENVALUE PROBLEMS

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Results are given for the asymptotic spectrum of a multiparameter eigenvalue problem in Hilbert space. They are based on estimates for eigenvalues derived from the minimum–maximum principle. As an application, a multiparameter Sturm–Liouville problem is considered.

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1. Introduction

We consider the k-parameter eigenvalue problem

$$T_r u_r + \sum_{s=1}^{k} \lambda_s V_{rs} u_r = 0, \quad 0 \neq u_r \in D(T_r), \quad r = 1, \ldots, k, \tag{1.1}$$

where k is a positive integer. The selfadjoint operators T_r and $V_{r1}, \ldots, V_{rk}$ act in a separable infinite-dimensional Hilbert space H_r for each $r = 1, \ldots, k$. The operator T_r: $H_r \supset D(T_r) \to H_r$ has compact resolvent and is bounded below, and V_{rs} is bounded for all $r, s = 1, \ldots, k$. A k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbf{R}^k$ for which there exist vectors $u_1, \ldots, u_k$ satisfying (1.1) is an *eigenvalue* of (1.1). The study of multiparameter eigenvalue problems of this type was initiated by Atkinson [**1**]. Since then many results on existence of eigenvalues (generalizing the classical Klein oscillation theorem) and on expansion into eigenvectors have been obtained; see [**12**] for an overview.

The study of the asymptotic behaviour of the *spectrum* of (1.1) (i.e., its set of eigenvalues) has found considerable interest [**2, 6, 8, 9, 10, 11**]. One question is to find the *asymptotic directions* of the spectrum of (1.1), i.e., those unit vectors $\omega \in \mathbf{R}^k$ for which there exists a seqence λ^n of eigenvalues of (1.1) such that $\lambda^n / \|\lambda^n\|$ converges to ω while $\|\lambda^n\|$ converges to infinity as $n \to \infty$. The set of all such asymptotic directions forms the *asymptotic spectrum*, and it is denoted by AS. In the special case of a multiparameter Sturm–Liouville problem (when the T_r-operators are of Sturm–Liouville type and V_{rs} are multiplication operators in a L^2 space), the asymptotic spectrum was determined in an unpublished lecture by Atkinson [**2**]. For the abstract two-parameter problem, the asymptotic spectrum was investigated by Binding, Browne and Seddighi [**6**] (see also [**7**]) by geometrical methods involving eigencurves in the plane. The object of the present paper is to generalize some of the results of the latter paper to general dimension k (when we cannot use figures anymore), and to reprove Atkinson's theorem within the abstract setting.

It is easy to show (see Theorem 2.1) that the asymptotic spectrum of (1.1) is contained in the set $C^+=\bigcap_{r=1}^k C_r^+$, where C_r^+ is the cone (a cone is a subset of $\mathbf{R}^k$ closed under multiplication with positive scalars) defined by

$$C_r^+=\{\lambda\in\mathbf{R}^k\colon \text{there is } \mathbf{a}_r\in\operatorname{cl}(V_r) \text{ such that } \mathbf{a}_r\lambda\leqq 0\}. \tag{1.2}$$

We use the notation

$$V_r=\{((V_{r1}u,u),\ldots,(V_{rk}u,u))\colon u\in U_r\}; \tag{1.3}$$

$\operatorname{cl}(V_r)$ is the closure of V_r and U_r is the unit sphere in H_r. These cones are a standard tool in multiparameter spectral theory; see [5]. In fact, Atkinson [2] showed that $AS=C^+\cap S^{k-1}$ in the uniformly right definite Sturm–Liouville case. It was noted in [6] that, in general, this equality does not hold in the situation of (1.1). This can be seen by the following simple example (compare [6, Example 4.1]) that also shows us the nature of the problem we are considering in this paper.

Let $H_1=\cdots=H_k$ be a Hilbert space with orthonormal basis $e_1,e_2,\ldots$ Let T_r be the operator defined by $T_re_n=t_{r,n}e_n$ where each sequence $t_{r,1}t_{r,2},\ldots$ consists of positive numbers that increase to infinity. The operators V_{rs} are zero for $r\neq s$ and equal to the negative identity operator if $r=s$. Of course, such an uncoupled multiparameter problem is fairly trivial and usually not of much interest. However, the determination of its asymptotic spectrum is not entirely trivial. The eigenvalues are given by tuples $(t_{1,i_1},\ldots,t_{k,i_k})$ where $\mathbf{i}=(i_1,\ldots,i_k)$ is any multiindex of positive integers. The cone C^+ consists of all vectors $(\lambda_1,\ldots,\lambda_k)$ with nonnegative components. The inclusion $AS\subset C^+\cap S^{k-1}$ is obvious. It is also clear that, in general, not every unit vector in C^+ is an asymptotic direction of the spectrum (for example, if $t_{r,n}=2^n$ and $k\geqq 2$). A natural additional assumption to ensure that $AS=C^+\cap S^{k-1}$ is given by

$$\lim_{n\to\infty} t_{r,n+1}/t_{r,n}=1 \text{ for all but at most one index } r=1,\ldots,k; \tag{1.4}$$

see Section 3 for a proof.

In fact, we will show in Sections 3 and 4 that this type of additional assumption is also important in the general case of (1.1). More precisely, a given unit vector $\omega=(\omega_1,\ldots,\omega_k)$ is an asymptotic direction of a uniformly right or left definite problem (1.1) provided that there exist sequences $\alpha_{r,n}$ of eigenvalues of the one-parameter problems

$$(T_r+\alpha_rW_r)u_r=0,\ 0\neq u_r\in D(T_r), \tag{1.5}$$

where

$$W_r = \sum_{s=1}^{k} \omega_s V_{rs} \tag{1.6}$$

that converge monotonically to infinity and satisfy (1.4) with $\alpha_{r,n} = t_{r,n}$.

The paper is organized as follows. In Section 2 we give some results that are true without assuming a definiteness condition. In Sections 3 and 4, we present results that require uniform right and left definiteness of (1.1), respectively. Finally, in Section 5 we specialize to the Sturm–Liouville case and reprove Atkinson's theorem [2].

2. First results

We start with the announced

Theorem 2.1. *The asymptotic spectrum AS of* (1.1) *is contained in* C^+.

Proof. Let $\omega \in AS$. By definition, there is a sequence $\boldsymbol{\lambda}^n$ of eigenvalues of (1.1) such that $\boldsymbol{\lambda}^n / \|\boldsymbol{\lambda}^n\| \to \omega$ and $\|\boldsymbol{\lambda}^n\| \to \infty$ as $n \to \infty$. Choose corresponding eigenvectors $u_r^n \in U_r$, $r = 1, \ldots, k$, such that

$$T_r u_r^n + \sum_{s=1}^{k} \lambda_s^n V_{rs} u_r^n = 0.$$

We multiply by u_r^n and obtain

$$(T_r u_r^n, u_r^n) + \mathbf{v}_r^n \boldsymbol{\lambda}^n = 0 \tag{2.1}$$

with sequences $\mathbf{v}_r^n \in V_r$. Since the sets V_r are bounded, these sequences have convergent subsequences. Going over to subsequences if necessary, we assume that $\mathbf{v}_r^n \to \mathbf{a}_r \in \mathrm{cl}(V_r)$ as $n \to \infty$ for each r. Since T_r is bounded below, there is a constant c_r such that $(T_r u, u) \geq c_r$ for all $u \in D(T_r) \cap U_r$. Now (2.1) yields

$$-\frac{c_r}{\|\boldsymbol{\lambda}^n\|} \geqq -\frac{1}{\|\boldsymbol{\lambda}^n\|}(T_r u_r^n, u_r^n) = \frac{\mathbf{v}_r^n \boldsymbol{\lambda}^n}{\|\boldsymbol{\lambda}^n\|} \to \mathbf{a}_r \omega.$$

Since $\|\boldsymbol{\lambda}^n\| \to \infty$, this implies $\mathbf{a}_r \omega \leqq 0$ for all r. Thus $\omega \in C^+$. □

As in [5], we introduce cones $C_r^- \subset C_r \subset C_r^+$ by

$$C_r^- = \{\boldsymbol{\lambda} \in \mathbf{R}^k: \text{there is } \mathbf{v}_r \in V_r \text{ such that } \mathbf{v}_r \boldsymbol{\lambda} < 0\} \tag{2.2}$$

and

$$C_r = \{\boldsymbol{\lambda} \in \mathbf{R}^k: \text{there is } \mathbf{v}_r \in V_r \text{ such that } \mathbf{v}_r \boldsymbol{\lambda} \leqq 0\}. \tag{2.3}$$

If T_r is positive definite for every $r = 1, \ldots, k$, i.e., $(T_r u, u) > 0$ for all $u \in D(T_r) \cap U_r$, then

(2.1) shows that every eigenvalue of (1.1) is in $C^- = \bigcap_{r=1}^k C_r^-$. This implies the slightly stronger result $AS \subset \mathrm{cl}(C^-)$.

For every $r = 1, \dots, k$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbf{R}^k$, the operator $T_r + \sum_{s=1}^k \lambda_s V_{rs}$ is selfadjoint with compact resolvent and bounded below. Therefore, we can list its eigenvalues in increasing order and counted according to multiplicity as

$$\rho_r^1(\boldsymbol{\lambda}) \leqq \rho_r^2(\boldsymbol{\lambda}) \leqq \rho_r^3(\boldsymbol{\lambda}) \leqq \dots \tag{2.4}$$

By the minimum-maximum principle [**13**], we can write

$$\rho_r^i(\boldsymbol{\lambda}) = \min_E \max_{u \in E \cap U_r} \left[(T_r u, u) + \sum_{s=1}^k \lambda_s (V_{rs} u, u) \right], \tag{2.5}$$

where the minimum is taken over all i-dimensional subspaces E of $D(T_r)$. For every $r = 1, \dots, k$ and every positive integer i_r, the solutions of the equation $\rho_r^{i_r}(\boldsymbol{\lambda}) = 0$ then define the *eigensurface* $Z_r^{i_r}$. The intersection points of $Z_1^{i_1}, \dots, Z_k^{i_k}$, if they exist, are the eigenvalues of (1.1) corresponding to the *oscillation count* $\mathrm{i} = (i_1, \dots, i_k)$. We now prove the following

Theorem 2.2 *If $\boldsymbol{\lambda} \in Z_r^j$, $\boldsymbol{\mu} \in Z_r^i$ and $j \geqq i$, then $\boldsymbol{\lambda} - \boldsymbol{\mu} \in C_r$.*

Proof. Use $\rho_r^j(\boldsymbol{\lambda}) = 0$ and (2.5) to find a subspace E of $D(T_r)$ with dimension j such that

$$\max_{u \in E \cap U_r} \left[(T_r u, u) + \sum_{s=1}^k \lambda_s (V_{rs} u, u) \right] = 0. \tag{2.6}$$

Now $\rho_r^j(\boldsymbol{\mu}) \geqq \rho_r^i(\boldsymbol{\mu}) = 0$ and (2.5) show that the left hand side of (2.6) with $\boldsymbol{\lambda}$ replaced by $\boldsymbol{\mu}$ is greater than or equal to 0. Thus there exists $u \in E \cap U_r$ such that

$$(T_r u, u) + \sum_{s=1}^k \mu_s (V_{rs} u, u) \geqq 0. \tag{2.7}$$

By (2.6), the inequality is reversed if $\boldsymbol{\mu}$ is replaced by $\boldsymbol{\lambda}$. Subtracting the two inequalities, we obtain

$$\sum_{s=1}^k (\lambda_s - \mu_s)(V_{rs} u, u) \leqq 0.$$

This implies $\boldsymbol{\lambda} - \boldsymbol{\mu} \in C_r$. □

If $i = j$ then the statement of Theorem 2.2 remains true if we replace C_r by

$$Q_r = C_r \cap (-C_r) = \{\lambda \in \mathbf{R}^k\colon \text{there are } \mathbf{a}_r, \mathbf{b}_r \in V_r \text{ such that } \mathbf{a}_r\lambda \leqq 0 \leqq \mathbf{b}_r\lambda\}$$
$$= \{\lambda \in \mathbf{R}^k\colon \text{there is } \mathbf{a}_r \in \operatorname{co} V_r \text{ such that } \mathbf{a}_r\lambda = 0\}, \tag{2.8}$$

where $\operatorname{co} V_r$ denotes the convex hull of V_r. Theorem 2.2 immediately yields the following result on the localization of eigenvalues.

Corollary 2.3. *Let $\mathbf{i} = (i_1, \dots, i_k) \in \mathbf{N}^k$, and let $\mu_r \in Z_r^{i_r}$ for every $r = 1, \dots, k$. Then*

$$\bigcap_{r=1}^{k} Z_r^{i_r} \subset \bigcap_{r=1}^{k} (\mu_r + Q_r).$$

In other words: if, for given oscillation count $(i_1, \dots, i_k)$, we know one point μ_r in each of the eigensurfaces $Z_r^{i_r}$, then we can guarantee that the intersection points of these eigensurfaces, if they exist, lie in

$$Q(\mu_1, \dots, \mu_k) := \bigcap_{r=1}^{k} (\mu_r + Q_r). \tag{2.9}$$

This result is related to Theorem 5 in [4] but it is not a direct consequence of it. It allows a simple geometric interpretation because the vectors $\mathbf{v}_r \in V_r$ constitute the possible normal vectors of the surface $Z_r^{i_r}$. However, the eigensurfaces are usually not smooth enough that we can speak of normal vectors in the usual sense.

3. Uniformly right definite eigenvalue problems

In this section we assume that the multiparameter eigenvalue problem (1.1) is uniformly right definite, i.e. there exists $\varepsilon > 0$ such that

$$\det_{r,s=1,\dots,k} (V_{rs}u_r, u_r) \geqq \varepsilon \text{ for all } u_1 \in U_1, \dots, u_k \in U_k. \tag{3.1}$$

It follows from (3.1) that

$$\det(\mathbf{a}_1, \dots, \mathbf{a}_k) \geqq \varepsilon \text{ for all } \mathbf{a}_1 \in \operatorname{co}\operatorname{cl}(V_1), \dots, \mathbf{a}_k \in \operatorname{co}\operatorname{cl}(V_k). \tag{3.2}$$

It is well known [4, Theorem 2] that uniform right definiteness implies that the intersection $\bigcap_{r=1}^k Z_r^{i_r}$ of eigensurfaces consists of exactly one point $\lambda^{\mathbf{i}}$ for every oscillation count $(i_1, \dots, i_k)$ (abstract Klein's oscillation theorem).

Another consequence of right definiteness is the following

Lemma 3.1. *Under uniform right definiteness, $C^+ = \operatorname{cl}(C^-)$.*

Proof. Since C_r^+ is closed and contains C_r^- for every r, the inclusion "$\supset$" is clear. To show the inclusion "$\subset$", let $\lambda \in C^+$. Then there are vectors $\mathbf{a}_r \in \operatorname{cl}(V_r)$ such that $\mathbf{a}_r\lambda \leqq 0$ for

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$r=1,\dots,k$. By right definiteness, the vectors $\mathbf{a}_1,\dots,\mathbf{a}_k$ form a basis of $\mathbf{R}^k$. Let $\mathbf{b}_1,\dots,\mathbf{b}_k$ be its dual basis, and let $\mathbf{b}=\sum_{r=1}^{k}\mathbf{b}_r$. For every $\varepsilon>0$, the vector $\boldsymbol{\lambda}(\varepsilon)=\boldsymbol{\lambda}-\varepsilon\mathbf{b}$ satisfies $\mathbf{a}_r\boldsymbol{\lambda}(\varepsilon)=\mathbf{a}_r\boldsymbol{\lambda}-\varepsilon<0$ which implies $\boldsymbol{\lambda}(\varepsilon)\in C^-$. Since $\boldsymbol{\lambda}(\varepsilon)\to\boldsymbol{\lambda}$ as $\varepsilon\to 0$ this shows that $\boldsymbol{\lambda}\in\mathrm{cl}(C^-)$. □

Right definiteness implies the following property of the set Q of (2.9).

Theorem 3.2. *There is a constant L (depending only on the given uniformly right definite eigenvalue problem* (1.1)) *such that*

$$\|\boldsymbol{\lambda}-\boldsymbol{\mu}\|\leqq L\sum_{r=1}^{k}\|\boldsymbol{\mu}_r-\boldsymbol{\mu}\|$$

for all $\boldsymbol{\mu},\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k\in\mathbf{R}^k$ and $\boldsymbol{\lambda}\in Q(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k)$.

Proof. Let $\boldsymbol{\lambda}\in Q(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k)$. Choose $\mathbf{a}_r\in\mathrm{co}\,V_r$ such that $\mathbf{a}_r(\boldsymbol{\lambda}-\boldsymbol{\mu}_r)=0$. It follows that

$$\mathbf{a}_r(\boldsymbol{\lambda}-\boldsymbol{\mu})=\mathbf{a}_r(\boldsymbol{\mu}_r-\boldsymbol{\mu})\text{ for all } r=1,\dots,k.$$

Let A be the matrix with rth row $\mathbf{a}_r$. Then

$$\boldsymbol{\lambda}-\boldsymbol{\mu}=A^{-1}(\mathbf{a}_1(\boldsymbol{\mu}_1-\boldsymbol{\mu}),\dots,\mathbf{a}_k(\boldsymbol{\mu}_k-\boldsymbol{\mu}))^T. \tag{3.3}$$

The entries of all possible matrices A are bounded. By condition (3.2), the entries of all possible matrices A^{-1} are bounded, too. Now (3.3) shows that there is a constant L independent of $\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k$ such that

$$\|\boldsymbol{\lambda}-\boldsymbol{\mu}\|\leqq L\sum_{r=1}^{k}\|\boldsymbol{\mu}_r-\boldsymbol{\mu}\|.$$

This is the desired result. □

The above theorem shows that $Q(\boldsymbol{\mu},\dots,\boldsymbol{\mu})$ consists of the point $\boldsymbol{\mu}$ only. Moreover, if the points $\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k$ are close together, then the set $Q(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k)$ is small. Another consequence is

Corollary 3.3. *The set $Q(\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k)$ is bounded for all $\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k\in\mathbf{R}^k$.*

If the intersection points of the eigensurfaces $Z_r^{i_r}$, $r=1,\dots,k$, with a ray starting at $\mathbf{0}$ are close enough together, then we expect that the corresponding intersection point $\boldsymbol{\lambda}^{\mathbf{i}}$ of these eigensurfaces is close to the ray. This is expressed by the following theorem.

Theorem 3.4. *Let ω be a unit vector in $\mathbf{R}^k$, and let K be a cone in $\mathbf{R}^k$ containing the* *ray $R=\{t\omega\colon t>0\}$ in its interior. Then there exists $\delta>0$ such that the following property holds: whenever $(i_1,\dots,i_k)\in\mathbf{N}^k$ and μ_r is a point of intersection of the ray R and the eigensurface $Z_r^{i_r}$ for $r=1,\dots,k$, and we set*

$$\alpha=\min\{\|\boldsymbol{\mu}_1\|,\dots,\|\boldsymbol{\mu}_k\|\},\ \beta=\max\{\|\boldsymbol{\mu}_1\|,\dots,\|\boldsymbol{\mu}_k\|\},$$

then $\beta/\alpha<1+\delta$ implies that K contains the eigenvalue $\boldsymbol{\lambda}^{\mathbf{i}}$ and $\|\boldsymbol{\lambda}^{\mathbf{i}}\|>\alpha/2$.

Proof. Choose $0<\varepsilon<1/2$ such that the ball $B_\varepsilon(\omega)=\{\boldsymbol{\lambda}\in\mathbf{R}^k\colon\|\boldsymbol{\lambda}-\boldsymbol{\omega}\|<\varepsilon\}$ is contained in K. Set $\delta=\varepsilon/kL$ where L is the constant Theorem 3.2. Let $\boldsymbol{\mu}_1,\dots,\boldsymbol{\mu}_k$ be any vectors according to the statement of the theorem. By Corollary 2.3 and Theorem 3.2,

$$\|\boldsymbol{\lambda}^{\mathbf{i}}-\alpha\boldsymbol{\omega}\|\leqq L\sum_{r=1}^{k}\|\boldsymbol{\mu}_r-\alpha\boldsymbol{\omega}\|\leqq Lk(\beta-\alpha).$$

Since $\beta/\alpha<1+\delta$, we obtain

$$\left\|\frac{1}{\alpha}\boldsymbol{\lambda}^{\mathbf{i}}-\boldsymbol{\omega}\right\|<Lk\delta=\varepsilon.$$

It follows that $\boldsymbol{\lambda}^{\mathbf{i}}/\alpha$ lies in $B_\varepsilon(\omega)$. Hence $\boldsymbol{\lambda}^{\mathbf{i}}\in K$ and $\|\boldsymbol{\lambda}^{\mathbf{i}}\|>\alpha/2$. □

The following simple lemma on sequences is useful in connection with the previous result.

Lemma 3.5. *For $r=1,\dots,k$, let $0<x_r^1\leqq x_r^2\leqq x_r^3\leqq\dots$ be monotonically nondecreasing sequences that converge to infinity. Further assume that there is $\delta>0$ such that*

$$\limsup_{n\to\infty}\frac{x_r^{n+1}}{x_r^n}<1+\delta$$

for all $r=1,\dots,k-1$. Then, for every $A>0$, there exists $t>A$ such that the interval $[t,(1+\delta)t)$ contains at least one member of each of the sequences $x_r^n, r=1,\dots,k$.

Proof. We choose N such that $x_r^{n+1}/x_r^n<1+\delta$ for $n\geqq N$ and $r=1,\dots,k-1$. Then we choose n_k so large that $t:=x_k^{n_k}>A$ and $t>x_r^N$ for all $r=1,\dots,k-1$. By choice of N and t, there are $n_1,\dots,n_{k-1}\geqq N$ such that $x_r^{n_r}\in[t,(1+\delta)t)$ for $r=1,\dots,k-1$. □

We now combine the previous lemma with Theorem 3.4.

Theorem 3.6. *Let ω be a unit vector in $\mathbf{R}^k$. Assume that, for each $r=1,\dots,k$, there is a real sequence $\alpha_r^1,\alpha_r^2,\dots$ such that* (*a*) $\alpha_r^n\omega$ *lies in one of the eigensurfaces* Z_r^i, $i \in \mathbf{N}$, *for every* $n \in \mathbf{N}$;

(*b*) *the sequence* $\alpha_r^1, \alpha_r^2, \ldots$ *is positive, nondecreasing and converges to infinity*;

(*c*) $\lim_{n\to\infty}\alpha_r^{n+1}/\alpha_r^n = 1$ *for all* $r = 1, \ldots, k$ *except possibly one* r.

Then ω *lies in the asymptotic spectrum* AS *of* (1.1).

Proof. Let K be any cone containing ω in its interior, and let M be any positive number. Let $\delta > 0$ be the constant according to Theorem 3.4. By Lemma 3.5, there is $t > 2M$ such that the interval $[t, (1+\delta)t)$ contains at least one member of each sequence α_r^n. Now Theorem 3.4 yields that K contains an eigenvalue λ of (1.1) of norm greater than M. Since K and M were arbitrary, this implies that ω is in AS. □

Let us consider the example given in the introduction with $k = 2$. The set C^+ is the closed first quadrant. The eigencurves $Z_1^{i_1}$, $Z_2^{i_2}$ are given by $\lambda_1 = t_{1,i_1}$ and $\lambda_2 = t_{2,i_2}$, respectively. Thus the eigenvalues of the uniformly right definite two-parameter problem are the pairs $\lambda^{\mathbf{i}} = (t_{1,i_1}, t_{2,i_2})$. The asymptotic spectrum AS consists of the cluster points of the double sequence $\lambda^{\mathbf{i}}/\|\lambda^{\mathbf{i}}\|$. If at least one of the two conditions

$$\lim_{n\to\infty} t_{1,n+1}/t_{1,n} = 1, \quad \lim_{n\to\infty} t_{2,n+1}/t_{2,n} = 1$$

holds then the above theorem shows that AS consists exactly of the unit vectors (ω_1, ω_2) with $\omega_1, \omega_2 \geqq 0$. Of course, this could also be proved directly without reference to multiparameter spectral theory. Consider now the case that $t_{1,n}$ lies in the union of the intervals $(2^{4p}, 2^{4p+1})$, $p = 1, 2, \ldots$ and $t_{2,n}$ lies in the union of the intervals $(2^{4p+2}, 2^{4p+3})$, $p = 1, 2, \ldots$ for each n. Then no quotient $t_{1,i_1}/t_{2,i_2}$ lies between 1/2 and 2 and, therefore, $\omega = (1/\sqrt{2}, 1/\sqrt{2})$ does not belong to AS. In addition, we can easily arrange the sequences $t_{1,n}$ and $t_{2,n}$ in such a way that $\lim_{n\to\infty}\sqrt[n]{t_{r,n}} = 1$ for $r = 1, 2$. This shows that we cannot weaken condition (c) of Theorem 3.6 to $\lim_{n\to\infty}\sqrt[n]{\alpha_r^n} = 1$ for all $r = 1, \ldots, k$.

Assumption (a) of Theorem 3.6 can be formulated in a slightly different way as follows. For given unit vector $\omega = (\omega_1, \ldots, \omega_k)$, define W_r by (1.6). Then assumption (a) of Theorem 3.6 is equivalent to the condition that α_r^n is an eigenvalue of the rth one parameter problem (1.5).

The following lemma gives a sufficient condition for assumption (b) of Theorem 3.6.

Lemma 3.7. *Let* T *and* W *be selfadjoint operators in a Hilbert space* H *where* T *is bounded below with compact resolvent and* W *is bounded. For every* $n \in \mathbf{N}$, *let there be an* n*-dimensional subspace* F_n *of* H *such that* $(Wx, x) < 0$ *for all* $x \in F_n$, $x \neq 0$. *Then the eigenvalue problem*

$$(T + \alpha W)x = 0, \ 0 \neq x \in D(T)$$

admits a sequence $0 < \alpha_1 \leqq \alpha_2 \leqq \ldots$ *of eigenvalues which converges to* ∞.

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Proof. Let $\rho_n(\alpha)$ be the nth largest eigenvalue of $T+\alpha W$ counted according to multiplicity. We can assume that F_n is contained in the domain of definition $D(T)$ of T because $D(T)$ is dense in H. Then the minimum–maximum principle for $\rho_n(\alpha)$ shows that $\rho_n(\alpha)\leqq M-\varepsilon\alpha$ for $\alpha>0$ where M and ε are positive constants. If n is sufficiently large, $\rho_n(0)>0$ and it follows that the continuous eigencurve $\rho_n(\alpha)$ has a positive zero. Let α_n be the maximal zero of ρ_n. Then these zeros converge to infinity as $n\to\infty$ because another application of the minimum–maximum principle shows that $\rho_n(\alpha)\geqq\rho_n(0)-K\alpha$ where K is a positive constant such that $(Wu,u)\geqq -K$ for all unit vectors u in H. □

It would be of interest to find additional (verifiable) sufficient conditions for assumption (c) of Theorem 3.6, i.e., conditions on T and W that imply that α_n can be chosen so that α_{n+1}/α_n tends to 1 as $n\to\infty$.

4. Uniformly left definite eigenvalue problems

In this section we assume that the multiparameter eigenvalue problem (1.1) is uniformly left definite, i.e.,

(1) there exists $\varepsilon>0$ such that the cofactors of each entry in the kth column of the matrix

$$(V_{rs}u_r,u_r), r,s=1,\ldots,k, \tag{4.1}$$

are greater than or equal to ε, and

(2) the operators $T_1,\ldots,T_k$ are positive definite.
Moreover, we assume that

(3) for every tuple $(i_1,\ldots,i_k)\in\mathbf{N}^k$ there are subspaces E_r of H_r with dimension i_r, $r=1,\ldots,k$, such that the determinant of the matrix (4.1) is negative for all $u_1\in U_1\cap E_1,\ldots,u_k\cap E_k$.

Under these assumptions, there is exactly one eigenvalue $(\lambda_1,\ldots,\lambda_k)$ of (1.1) with $\lambda_k>0$ for every oscillation count $(i_1,\ldots,i_k)$; see [**12**, Thm. 2.5.3].

A consequence of left definiteness is stated in the following lemma.

Lemma 4.1. *Let $(\beta_1,\ldots,\beta_{k-1})\in\mathbf{R}^{k-1}$ be a nonzero vector. Then there is $p\in\{1,\ldots,k\}$ such that the operator $\sum_{s=1}^{k-1}\beta_sV_{ps}$ is uniformly positive definite.*

Proof. Assume, if possible, that there are $\mathbf{a}_r\in\operatorname{cl}(V_r)$ such that $c_r:=\mathbf{a}_r\boldsymbol{\beta}\leqq 0$ for $r=1,\ldots,k$, where $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_{k-1},0)$. Let A be the k by k matrix with rows $\mathbf{a}_1,\ldots,\mathbf{a}_k$ and kth column replaced by $(c_1,\ldots,c_k)$. Then $(\beta_1,\ldots,\beta_{k-1},-1)$ is in the kernel of this matrix. It follows that the determinant of A vanishes. If we expand the determinant with respect to the kth column, then assumption (1) and $c_1,\ldots,c_k\leqq 0$ show that $c_r=0$ for each r. Since the $k-1$ by $k-1$ matrix obtained from A by deleting its kth row and kth

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column is regular this implies that $\beta_1 = \cdots = \beta_{k-1} = 0$ which is contrary to the assumption. This proves the lemma. □

We next state a result parallel to Lemma 3.1.

Lemma 4.2. *Let* $\mathbf{R}^{k+} = \{(\lambda_1, \ldots, \lambda_k): \lambda_k \geqq 0\}$. *Then* $C^+ \cap \mathbf{R}^{k+} = \mathrm{cl}(C^- \cap \mathbf{R}^{k+})$.

Proof. The inclusion "$\supset$" is clear. In order to prove the reverse inclusion, let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k) \in C^+$ with $\lambda_k \geqq 0$. Lemma 4.1 implies that $\lambda_k > 0$. Without loss of generality, let $\lambda_k = 1$. We remark that the operators $W_r = \sum_{s=1}^{k} \lambda_s V_{rs}$ cannot be positive semidefinite for all $r = 1, \ldots, k$. In fact, the determinant of (4.1) remains unchanged if we replace V_{rk} by W_r for all $r = 1, \ldots, k$. Then the expansion of this determinant with respect to the new kth column together with assumptions (1) and (3) implies our remark. Since $\boldsymbol{\lambda} \in C^+$, there are $\mathbf{a}_r \in \mathrm{cl}(V_r)$ such that $\mathbf{a}_r \boldsymbol{\lambda} \leqq 0$ for all $r = 1, \ldots, k$. Our remark shows that we can assume that $\mathbf{a}_r \boldsymbol{\lambda} < 0$ for at least one r. The determinant of the matrix with rows $\mathbf{a}_1, \ldots, \mathbf{a}_k$ remains unchanged if we replace its kth column by $\mathbf{a}_r \boldsymbol{\lambda}, r = 1, \ldots, k$. Hence this determinant is negative and thus nonzero. We can now argue as in the proof of Lemma 3.1 and obtain $\boldsymbol{\lambda} \in \mathrm{cl}(C^- \cap \mathbf{R}^{k+})$. □

Theorem 4.3. *Let* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_k), \lambda_k > 0$, *be the (uniquely determined) eigenvalue of* (1.1) *corresponding to the oscillation count* $\mathbf{i} = (i_1, \ldots, i_k)$. *Let* $R = \{\alpha\boldsymbol{\omega}: \alpha > 0\}$ *be the ray generated by the vector* $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$, *where* $\omega_k = 1$. *Assume that the eigensurfaces* $Z_r^{i_r}$ *intersect the ray* R *at* $\alpha_r \boldsymbol{\omega}$ *for* $r = 1, \ldots, k$. *Then*

$$\min_{r=1}^{k} \alpha_r \leqq \lambda_k \leqq \max_{r=1}^{k} \alpha_r.$$

Proof. We first remark that the numbers α_r are uniquely determined because every eigensurface can intersect the ray R at most once in the left definite case. This follows from the fact that an eigenvalue problem of the type $(T + \alpha W)u = 0$ can have only one positive eigenvalue for a given oscillation count if T is positive definite. In particular, if $\boldsymbol{\lambda} \in R$ then $\alpha_r = \lambda_k$ for all r and the statement of the theorem is true. We now assume that $\boldsymbol{\lambda} \notin R$. Then $\boldsymbol{\beta} := \lambda_k \boldsymbol{\omega} - \boldsymbol{\lambda}$ is not zero. Consider the eigenvalue functions

$$f_r(t) := \rho_r^{i_r}(\boldsymbol{\lambda} + t\boldsymbol{\beta}) = i_r\text{th eigenvalue of } T_r + \sum_{s=1}^{k-1} (\lambda_s + t\beta_s) V_{rs}, \ 0 \leqq t \leqq 1.$$

By Lemma 4.1, there are p and q such that the operators $\sum \beta_s V_{ps}$ and $-\sum \beta_s V_{qs}$ are uniformly positive definite. Then the functions f_p and f_q are monotonically increasing and decreasing, respectively. Since these functions vanish for $t = 0$, we obtain

$$f_p(1) = \rho_p^{i_p}(\lambda_k \boldsymbol{\omega}) > 0 \text{ and } f_q(1) = \rho_q^{i_q}(\lambda_k \boldsymbol{\omega}) < 0.$$

We note that

$$\rho_r^{i_r}(s\omega) > 0 \text{ for } 0 \leqq s < \alpha_r \text{ and } \rho_r^{i_r}(s\omega) < 0 \text{ for } s > \alpha_r.$$

This implies $\lambda_k < \alpha_p$ and $\lambda_k > \alpha_q$ which proves the statement of the theorem. □

For $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k \in \mathbf{R}^k$ we define sets

$$P(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k) = Q(\boldsymbol{\mu}_1, \dots \boldsymbol{\mu}_k) \cap \{\boldsymbol{\lambda} \in \mathbf{R}^k \colon \min_{r=1}^{k} \mathbf{e}_k \boldsymbol{\mu}_r \leqq \mathbf{e}_k \boldsymbol{\lambda} \leqq \max_{r=1}^{k} \mathbf{e}_k \boldsymbol{\mu}_r\}$$

where $\mathbf{e}_k = (0, \dots, 0, 1)$. We prove the following result analogous to Theorem 3.2.

Theorem 4.4. *There is a constant L (depending only on the given uniformly left definite eigenvalue problem* (1.1)) *such that*

$$\|\boldsymbol{\lambda} - \boldsymbol{\mu}\| \leqq L \sum_{r=1}^{k} \|\boldsymbol{\mu} - \boldsymbol{\mu}_r\|.$$

for all $\boldsymbol{\mu}, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k \in \mathbf{R}^k$ *and* $\boldsymbol{\lambda} \in P(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k)$.

Proof. Let $\boldsymbol{\lambda} \in P(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k)$. Choose $\mathbf{a}_r \in \operatorname{co} V_r$ such that $\mathbf{a}_r(\boldsymbol{\lambda} - \boldsymbol{\mu}_r) = 0$. It follows that

$$\mathbf{a}_r(\boldsymbol{\lambda} - \boldsymbol{\mu}) = \mathbf{a}_r(\boldsymbol{\mu}_r - \boldsymbol{\mu}) \text{ for all } r = 1, \dots, k.$$

Let $A = (a_{rs})$ be the k by k matrix with rth row $\mathbf{a}_r$, and let $\tilde{A}$ be the $k-1$ by $k-1$ matrix obtained from A by deleting its last row and column. For $\boldsymbol{\omega} \in \mathbf{R}^k$ let $\tilde{\boldsymbol{\omega}}$ denote the vector obtained from $\boldsymbol{\omega}$ by deleting its last component. Then we have

$$\tilde{A}(\tilde{\boldsymbol{\lambda}} - \tilde{\boldsymbol{\mu}}) + \mathbf{e}_k(\boldsymbol{\lambda} - \boldsymbol{\mu})(a_{1k}, \dots, a_{k-1,k})^T = (\mathbf{a}_1(\boldsymbol{\mu}_1 - \boldsymbol{\mu}), \dots, \mathbf{a}_{k-1}(\boldsymbol{\mu}_{k-1} - \boldsymbol{\mu}))^T.$$

The entries of all possible matrices A as well as those of all possible matrices $\tilde{A}^{-1}$ are bounded. Therefore there exists a constant M independent of $\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k$ such that

$$\|\tilde{\boldsymbol{\lambda}} - \tilde{\boldsymbol{\mu}}\| \leqq M\left(|\mathbf{e}_k(\boldsymbol{\lambda} - \boldsymbol{\mu})| + \sum_{r=1}^{k} \|\boldsymbol{\mu} - \boldsymbol{\mu}_r\|\right).$$

The definition of $P(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k)$ shows that $|\mathbf{e}_k(\boldsymbol{\lambda} - \boldsymbol{\mu})| \leqq \max_{r=1}^{k} \|\boldsymbol{\mu} - \boldsymbol{\mu}_r\|$. This gives the statement with $L = 2M + 1$. □

Using this theorem together with Theorem 4.3, we can now show that Theorem 3.4 also holds under the assumptions of this section if the kth component of the vector $\boldsymbol{\omega}$ is positive. In the proof we just replace $Q(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k)$ by $P(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_k)$.

As in Section 4, we can then prove that Theorem 3.6 holds as well if the kth component of $\boldsymbol{\omega}$ is positive.

Theorem 4.5. *Let ω be a unit vector in $\mathbf{R}^k$ with $\omega_k>0$. Assume that, for each $r=1,\ldots,k$, there is a real sequence $\alpha_r^1,\alpha_r^2,\ldots$ such that*

(*a*) *$\alpha_r^n\omega$ lies in one of the eigensurfaces Z_r^i, $i\in\mathbf{N}$, for every $n\in\mathbf{N}$;*

(*b*) *the sequence $\alpha_r^1,\alpha_r^2,\ldots$ is positive, nondecreasing and converges to infinity;*

(*c*) $\lim_{n\to\infty}\alpha_r^{n+1}/\alpha_r^n=1$ *for all $r=1,\ldots,k$ except possibly one r.*

Then ω lies in the asymptotic spectrum AS of (1.1).

If we replace V_{rk} by $-V_{rk}$ for every $r=1,\ldots,k$ and assume that the new eigenvalue problem satisfies condition (3) of the beginning of this section, then the above results applied to the new eigenvalue problem yield results for the eigenvalues of the original problem with $\lambda_k<0$.

5. The multiparameter Sturm–Liouville problem

As a special case of (1.1), we consider the multiparameter Sturm–Liouville problem. For given compact intervals $[a_r,b_r]$, the Hilbert spaces are $H_r=L^2(a_r,b_r)$, and the operators are given by

$$T_ry_r=-(p_ry_r')'+q_ry_r,\ V_{rs}y_r=v_{rs}y_r.$$

The functions p_r,q_r,v_{rs} are continuous, real-valued and defined in $[a_r,b_r]$, and p_r is positive and continuous differentiable. The domain of definition $D(T_r)$ of T_r is the usual one involving the boundary conditions

$$\alpha_{1r}y_r(a_r)+\alpha_{2r}y_r'(a_r)=0,\ \beta_{1r}y_r(b_r)+\beta_{2r}y_r'(b_r)=0.$$

The cone C^+ is given by

$$C^+=\left\{(\lambda_1,\ldots,\lambda_k):\ \sum_{s=1}^k\lambda_sv_{rs}(x_r)\leqq 0 \text{ for some } x_r\in[a_r,b_r], r=1,\ldots,k\right\}.$$

Similarly, we obtain C^- by replacing "$\leqq$" by "$<$". The Sturm–Liouville eigenvalue problem is uniformly right definite if the Stäckel determinant $\det v_{rs}$ satisfies

$$\det_{r,s=1,\ldots,k} v_{rs}(x_r)>0 \text{ for all } x_1\in[a_1,b_1],\ldots,x_k\in[a_k,b_k]. \tag{5.1}$$

Condition (1) of Section 4 means that the cofactors of the kth column of the Stäckel determinant are positive functions in $[a_1,b_1]\times\cdots\times[a_{k-1},b_{k-1}]$. Condition (3) means that the Stäckel determinant $\det v_{rs}$ takes on at least one negative value.

Theorem 5.1. (*a*) *If the k-parameter Sturm–Liouville problem is uniformly right definite then its asymptotic spectrum AS is equal to $C^+\cap S^{k-1}=\mathrm{cl}(C^-)\cap S^{k-1}$.*

(b) *The same result holds if the k-parameter Sturm–Liouville problem is uniformly left definite and the Stäckel determinant* det v_{rs} *takes on positive and negative values.*

Proof. (a) Because of Lemma 3.1 and Theorem 2.1, it is enough to show that $C^- \subset AS$ (AS is closed). So let $\omega \in C^-$. Then, for each r, the function

$$g_r = \omega_1 v_{r1} + \cdots + \omega_k v_{rk}$$

is negative on a (possibly small) subinterval of $[a_r, b_r]$. Therefore, the multiplication operator W_r induced by g_r satisfies the assumption of Lemma 3.7. This lemma then shows that (1.5) admits an increasing sequence of positive eigenvalues that converges to infinity. In the Sturm–Liouville case, it is known [3, Thm. 2.3] that this sequence can be chosen such that condition (c) of Theorem 3.6 is satisfied. Now Theorem 3.6 implies that $\omega \in AS$ and the proof is complete in the right definite case.

(b) Because of Lemma 4.2 and the remark at the end of Section 4 we know that $C^+ = \mathrm{cl}(C^-)$. Therefore, it is again enough to show that $C^- \subset AS$. So let $\omega \in C^-$. By Lemma 4.1, the kth component of ω is not zero. Therefore, using Theorem 4.5 in place of Theorem 3.6, we can argue as in the right definite case to obtain that $\omega \in AS$. □

As an example, consider the k-parameter Sturm–Liouville eigenvalue problem with $v_{rs}(x_r) = x_r^{s-1}, r, s = 1, \ldots, k$, and $a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$. A problem of this type arises when we separate variables in the k-dimensional wave equation transformed to

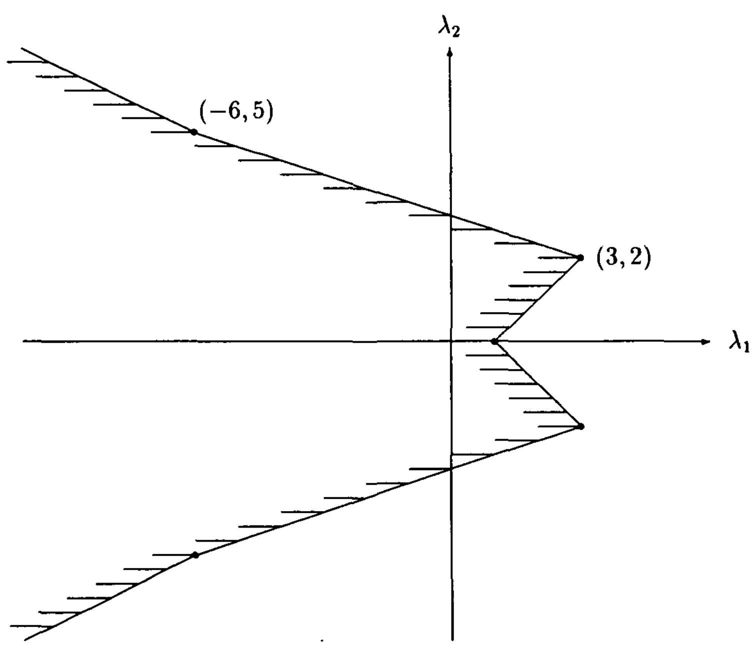

FIGURE 1: Intersection of C^+ with plane $\lambda_3 = -1$

ellipsoidal coordinates; cf. [**12**, Sec. 6.9]. This eigenvalue problem is uniformly right definite because condition (5.1) is satisfied for the Vandermonde determinant $\det_{r,s=1,\dots,k} x_r^{s-1}$. In order to determine the asymptotic spectrum, we have to find the cone $C^+ = \mathrm{cl}(C^-)$. A vector $(\lambda_1,\dots,\lambda_k)$ lies in C^+ if and only if there exist $x_r \in [a_r, b_r]$ such that $p(x_r) \leqq 0$, $r = 1,\dots,k$, where p is the polynomial $p(x) = \sum_{s=1}^{k} \lambda_s x^{s-1}$. If $k=3$ we can compute this set explicitly. For example, let $[a_1,b_1]=[-3,-2]$, $[a_2,b_2]=[-1,1]$ and $[a_3,b_3]=[2,3]$. Then $(\lambda_1,\lambda_2,1)$ belongs to C^+ if and only if $\lambda_1+4 \leqq -2|\lambda_2|$, and $(\lambda_1,\lambda_2,0)$ belongs to C^+ if and only if $\lambda_1 \leqq -2|\lambda_2|$. The set of pairs (λ_1,λ_2) such that $(\lambda_1,\lambda_2,-1) \in C^+$ is given by the shaded set in Figure 1. The lines in this figure have slopes ± 1, $\pm 1/2$, $\pm 1/3$, respectively.

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