Adv. Appl. Prob. **23**, 431–435 (1991)
Printed in N. Ireland

ON THE STARSHAPENESS OF $G/G/c$ QUEUEING SYSTEMS

J. GEORGE SHANTHIKUMAR* AND
COUCHEN WU,** *University of California, Berkeley*

Abstract

In this paper we show that the waiting and the sojourn times of a customer in a single-stage, multiple-server, $G/G/c$ queueing system are increasing and starshaped with respect to the mean service time. Usefulness of this result in the design of the optimal service speed in the $G/G/c$ queueing system is also demonstrated.

SECOND-ORDER PROPERTIES; SINGLE-STAGE QUEUEING SYSTEMS; STOCHASTIC STARSHAPENESS; STOCHASTIC CONVEXITY; DESIGN

1. Introduction

Consider a single-stage multiple-server $G/G/c$ queueing system where customers arrive at times $\{\tau_n(\theta), n=1,2,\cdots\}$ controlled by a (possibly vector) parameter θ. Let $\boldsymbol{A}(\theta)=\{A_n(\theta), n=1,2,\cdots\}$ be the sequence of interarrival times and let $\boldsymbol{S}(\theta)=\{S_n(\theta), n=1,2,\cdots\}$ be the sequence of service times of the customers, again controlled by the parameter θ. Customers are served on a first come–first served basis. Then the waiting time $W_n(\theta)$ and the sojourn time $T_n(\theta)$ of the nth customer are given by

$$\begin{aligned} W_n(\theta) &= V_n^{(1)}(\theta), \\ T_n(\theta) &= W_n(\theta)+S_n(\theta), \quad n=1,2,\cdots, \end{aligned} \tag{1.1}$$

where $(V_n^{(1)}, V_n^{(2)}, \cdots, V_n^{(c)})$ is the increasing rearrangement of the workloads at the c servers seen by the nth arrival. We now define $(V_n^{(1)}, V_n^{(2)}, \cdots, V_n^{(c)})$ more precisely. For a vector $\boldsymbol{x}=(x_1,\cdots,x_c)$ let $x_{(i)}$ denote an ith smallest component, so $x_{(1)}\leqq x_{(2)}\leqq\cdots\leqq x_{(c)}$. Define $\Phi(\boldsymbol{x})=(x_{(1)}, x_{(2)},\cdots,x_{(c)})$. Let $y^+=\max\{y,0\}$. Then for $n=1,2,\cdots$,

$$\begin{aligned} (V_n^{(1)}, V_n^{(2)}, \cdots, V_n^{(c)}) = \Phi((V_{n-1}^{(1)}(\theta)+S_{n-1}(\theta)-A_n(\theta))^+,& \\ (V_{n-1}^{(i)}(\theta)-A_n(\theta))^+, \cdots, (V_{n-1}^{(c)}(\theta)-A_n(\theta))^+).& \end{aligned} \tag{1.2}$$

In the case of a single-server $G/G/1$ queueing system the above equations simplify into (the Lindley equations)

$$\begin{aligned} W_n(\theta) &= (W_{n-1}(\theta)+S_{n-1}(\theta)-A_n(\theta))^+, \\ T_n(\theta) &= W_n(\theta)+S_n(\theta), \quad n=1,2,\cdots. \end{aligned} \tag{1.3}$$

By a simple induction argument it follows that one has the following result (see Shanthikumar and Yao (1988), (1989), (1991).

Lemma 1.1. Let $W_n(\theta)$ and $T_n(\theta)$ be, respectively, the waiting and sojourn times of the nth customer in a $G/G/1$ queueing system (as defined by 1.3). If $\{\boldsymbol{A}(\theta), \theta\in\Theta\}$ is strong

Received 2 October 1990; revision received 19 February 1991.

* Postal address: Walter A. Haas School of Business, University of California at Berkeley, Berkeley, CA 94720, USA.

** Postal address: Department of Industrial Engineering and Operations Research, University of California at Berkeley, Berkeley, CA 94720, USA.

Supported in part by NSF Grant ECS–8811234.

stochastically concave (that is, $\{A(\theta),\ \theta \in \Theta\} \in SS - CV$) and $\{S(\theta),\ \theta \in \Theta\}$ is strong stochastically convex (that is $\{S(\theta),\ \theta \in \Theta\} \in SS - CX$), then for each n, $\{(W_n(\theta),\ T_n(\theta)),\ \theta \in \Theta\}$ is strong stochastically convex.

Note that a collection $\{X(\theta),\ \theta \in \Theta\}$ of random vectors parameterized by (possibly a vector parameter) θ in a convex subset Θ of $\mathbb{R}^{|\theta|}$, (where $\mathbb{R} = (-\infty, \infty)$) is said to be strong stochastically convex [concave] if for any $\theta^{(i)} \in \Theta$, $i = 1, 2$ and $\alpha \in (0, 1)$ there exist on a common probability space $\hat{X}(\theta^{(i)}), i = 1, 2$, and $\hat{X}(\alpha\theta^{(1)} + (1-\alpha)\theta^{(2)})$ such that $\hat{X}(\theta^{(i)}) \stackrel{\mathrm{d}}{=} X(\theta^{(i)})$, $i = 1, 2$, $\hat{X}(\alpha\theta^{(1)} + (1-\alpha)\theta^{(2)}) \stackrel{\mathrm{d}}{=} X(\alpha\theta^{(1)} + (1-\alpha)\theta^{(2)})$ and $\alpha\hat{X}(\theta^{(1)}) + (1-\alpha)\hat{X}(\theta^{(2)}) \geqq [\leqq] \hat{X}(\alpha\theta^{(1)} + (1-\alpha)\theta^{(2)})$.

Suppose for the two given sequences of random variables $\{\hat{A}_n, n = 1, 2, \cdots\}$ and $\{\hat{S}_n, n = 1, 2, \cdots\}$, we set $A_n(\theta) = \hat{A}_n$, $n = 1, 2, \cdots$ and $S_n(\theta) = \hat{S}_n/\theta$, $n = 1, 2, \cdots$. That is, the service times are parameterized by the service rate θ. From Lemma 1.1 one sees that the waiting and sojourn times are convex in the service rate θ. This result was first derived by Tu and Kumin (1983) and Weber (1983). The sample path construction used in Shanthikumar and Yao (1988), (1989), (1991) for Lemma 1.1 is essentially the same as that used by Weber (1983). In Shanthikumar and Yao (1988), (1989), (1991) a formal theory of strong stochastic convexity is developed making it possible to consider more than one parameter and the (vector) parameter being more general than the service rate. For example, if the service times are of phase type, one may use the set of rates of all phases as the (vector) parameter. As an application of Lemma 1.1 with vector parameter one may, for example, consider $\theta = (\theta_1, \theta_2)$ and set $A_n(\theta) = \theta_1\hat{A}_n$, $n = 1, 2, \cdots$ and $S_n(\theta) = \theta_2\hat{S}_n$, $n = 1, 2, \cdots$. That is θ_1 $[\theta_2]$ is the scale parameter of the interarrival [service] times. Now applying Lemma 1.1 one sees that the waiting and sojourn times are jointly convex in the scale parameter of the interarrival and service times. This special case of Lemma 1.1 (reported first in Shanthikumar and Yao (1988)) is the main result of Harel (1990).

Convexity (and concavity) of the performance measures such as the mean number of customers, average sojourn time and throughput with respect to the system parameters such as the arrival rate, service rate, number of servers and buffer capacity are prevalent in a large class of single-stage queueing systems (see Liyanage and Shanthikumar (1990) and the references there). These properties are useful in the design and in the development of bounds for the performance measures of these queueing systems. All attempts to extend the above results to multiple-server queueing systems have been unsuccessful. In particular, Weber (1983) provides a counterexample to show that the mean waiting time in a $GI/GI/2$ queueing system need not be convex in the service rate, nor need it be convex in the scale parameter of the service time (Harel (1990) gives a simulation example for the latter). Using simulation Harel (1990) also concludes that the mean waiting time need not be convex in the scale parameter of the interarrival times. However, it is easy to see that the waiting times are increasing in these scale parameters. Therefore it is natural to question whether a property stronger than monotonicity, but weaker than convexity, can be established for the waiting time. In Section 2 we answer this by showing that the waiting times are increasing and starshaped in the scale parameter of the service times. The next task would be to see whether such a property can be of any use in developing an optimization approach to designing a $G/G/c$ queueing system. In Section 3 we show that the starshapeness can indeed be used to bound the optimal service speed in the design of a $G/G/c$ queueing system.

2. Starshapeness of waiting and sojourn times

In this section we first define starshapeness of a function $f: \mathbb{R}_+ \to \mathbb{R}$ (where $\mathbb{R}_+ = [0, \infty)$) and present some closure properties of such functions that we need later to prove the starshapeness of the waiting and sojourn times in the $G/G/c$ queueing systems.

Definition 2.1. A function $f: \mathbb{R}_+ \to \mathbb{R}$ is said to be starshaped if $f(x)/x$ is non-decreasing in $x \in \mathbb{R}_+$ (see for example Marshall and Olkin (1979), p. 453).

Remark 2.2. A function is starshaped if and only if it has the property that when one draws its graph and imagines a star placed at the origin then the star illuminates from the left every point of the upper part of the curve.

Remark 2.3. Note that if f is a starshaped function then $f(0) \leqq 0$. Therefore if f is a non-negative function then f is starshaped implies that $f(0) = 0$, f is a non-decreasing function, and that it is increasing on $\{x : f(x) > 0,\ x \in \mathbb{R}_+\}$. Furthermore if $f(0) \leqq 0$ and f is a convex function then f is also a starshaped function. Hence it is obvious that the starshapeness of a non-negative function is stronger than monotonicity, but weaker than convexity. We shall see in Section 3 that starshapeness becomes handy in optimization problems.

Since the maximum, or minimum, or sum, of any two monotone functions is monotone, the following lemma is easily verified.

Lemma 2.4. Let $f_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2,$ be two starshaped functions. Then $f_{[1]}$, $f_{[2]}$ and g defined by $f_{[1]}(x) = \min\{f_1(x) + f_2(x)\}$, $x \in \mathbb{R}_+$, $f_{[2]}(x) = \max\{f_1(x) + f_2(x)\}$, $x \in \mathbb{R}_+$ and $g(x) = f_1(x) + f_2(x)$, $x \in \mathbb{R}_+$ are all starshaped functions.

Let Φ be as defined in Section 1. Then the next lemma follows from the repeated use of Lemma 2.4.

Theorem 2.5. Let $\Phi : \mathbb{R}^c \to \mathbb{R}^c$ be a function that permutes the coordinates into increasing order. If f_i, $i = 1, 2, \cdots, c$ are all starshaped functions, then g_i, $i = 1, 2, \cdots, c$ defined by $(g_1(x), \cdots, g_c(x)) = \Phi(f_1(x), \cdots, f_c(x))$, $x \in \mathbb{R}_+$, are all starshaped functions.

The next definition is needed to characterize the stochastic version of starshapeness.

Definition 2.6. Suppose $\boldsymbol{X}(\theta) = \{X_n(\theta), n = 1, 2, \cdots\}$ for $\theta \in \Theta$ is a collection of real-valued random variables. We assume that Θ is a convex subset of $\mathbb{R}_+$ of the form $[0, a)$. Then $\boldsymbol{X}$ is said to be strongly stochastically starshaped almost everywhere, if there exist a collection $\{\hat{\boldsymbol{X}}(\theta), \theta \in \Theta\}$ of random variables defined on a common probability space such that $\{\hat{\boldsymbol{X}}(\theta), \theta \in \Theta\} \stackrel{\mathrm{d}}{=} \{\boldsymbol{X}(\theta), \theta \in \Theta\}$ and $\hat{X}_n(\theta)$ is almost surely starshaped in θ for each $n = 1, 2, \cdots$. We denote this $\{\boldsymbol{X}(\theta), \theta \in \Theta\} \in SS - \mathrm{Star}\, S(\mathrm{ae})$.

Remark 2.7. Observe that here we use a definition that is slightly different from that used in Shanthikumar and Yao (1988), (1989), (1991) for strong stochastic convexity. This definition is more along the line of the definition of strong stochastic convexity almost everywhere ($SS - CX(\mathrm{ae})$) given in Meester and Shanthikumar (1990). Since $f(0) = 0$ and f is convex implies f is starshaped one sees that

$$\{\boldsymbol{X}(\theta), \theta \in \Theta\} \in SS - CX(\mathrm{ae}) \quad \text{and} \quad \boldsymbol{X}(0) = 0 \text{ a.s.} \Rightarrow \{\boldsymbol{X}(\theta), \theta \in \Theta\} \in SS - \mathrm{Star}\, S(\mathrm{ae}). \tag{2.1}$$

Therefore, for example, when the service times are scaled by the scale parameter (that is, when $S_n(\theta) = \theta \hat{S}_n$) one sees that $\{\boldsymbol{S}(\theta), \theta \in \mathbb{R}_+\} \in SS - \mathrm{Star}\, S(\mathrm{ae})$.

Now observing that all the operations in the dynamics of the $G/G/c$ queueing system (specified in (1.1) (1.2)) preserve starshapeness one has the following result.

Theorem 2.8. Let $W_n(\theta)$ and $T_n(\theta)$ be, respectively, the waiting and sojourn times of the nth customer in a $G/G/c$ queueing system (as defined by (1.1) and (1.2)). If $\{-\boldsymbol{A}(\theta), \theta \in \Theta\} \in SS - \mathrm{Star}\, S(\mathrm{ae})$ and $\{\boldsymbol{S}(\theta, \theta \in \Theta\} \in SS - \mathrm{Star}\, S(\mathrm{ae})$, then $\{(W_n(\theta), T_n(\theta)), \theta \in \Theta\} \in SS - \mathrm{Star}\, S(\mathrm{ae})$.

From the above result and Remark 2.7, one sees that the waiting and sojourn times in a $G/G/c$ queueing system are starshaped in the scale parameter of the service times.

3. Optimal service speed

Consider the following optimal speed selection problem in a single-stage multiple-server $G/G/c$ queueing system where customers arrive at times $\{\hat{\tau}_n, n = 1, 2, \cdots\}$. Let $\hat{A} = \{\hat{A}_n, n = 1, 2, \cdots\}$ be the sequence of interarrival times and let $\hat{S} = \{S_n, n = 1, 2, \cdots\}$ be the sequence of service times of the customers. Customers are served on a first come–first served basis. There is a linear inventory carrying cost h, $h > 0$, such that if there are n customers in the system a cost of nh units is incurred per unit time. We wish to choose the service speed μ such that the cost of inventory is balanced against the service cost. We assume that the service cost $g:\mathbb{R}_+ \to \mathbb{R}_+$ is decreasing and convex in $\theta = (1/\mu)$. Let $ET(\theta)$ be the expected sojourn time of an arbitrary customer through the $G/G/c$ queueing system in steady state when we use a service rate $1/\theta$. Therefore, if the arrival rate is λ, $\lambda > 0$, we see that the average inventory carrying cost is $\lambda ET(\theta)h$. The optimization problem is then

$$\theta^* = \arg\min\{\psi(\theta) = \lambda ET(\theta)h + g(\theta) : \theta < c/\lambda,\ \theta \in \mathbb{R}_+\}. \tag{3.1}$$

If $ET(\theta)$ is convex in θ then the above problem has a convex objective function and therefore it is sufficient to search for a local optimum (that is if we progressively search over increasing values of θ it would be sufficient to stop the search after obtaining the first local minimum.) As the next theorem demonstrates, even with the weaker property of starshapeness for $ET(\theta)$ the search can be terminated after searching through $[0, \theta_u]$ where, as we shall see, θ_u is an upper bound on the optimal solution that is less than c/λ.

Theorem 3.1. The optimal solution to Problem (3.1) is in $[0, \theta_u]$, where

$$\theta_u = \min\{\theta : \lambda ET(\theta)h + \theta g'(\theta) \geqq 0,\ \theta \in \mathbb{R}_+\}, \tag{3.2}$$

where $g'(x)$ is the right derivative of g at x.

Proof. Since $ET(\theta) \to \infty$ as $\theta \to c/\lambda$ and $|g'(\theta)| < \infty$ for $\theta > 0$, it is clear that $\lambda h > 0$ implies that $\theta_u < c/\lambda$. Therefore it will be sufficient to show that for all $\theta > \theta_u$ we have $\psi(\theta) \geqq \psi(\theta_u)$. Observe that for $\theta \geqq \theta_u$,

$$\begin{aligned}
\psi(\theta) &= \lambda ET(\theta)h + g(\theta) \\
&\geqq \frac{\theta}{\theta_u}\lambda ET(\theta_u)h + g(\theta_u) + (\theta - \theta_u)g'(\theta_u) \\
&= \lambda ET(\theta_u)h + (\theta - \theta_u)\frac{\lambda ET(\theta_u)h}{\theta_u} + g(\theta_u) + (\theta - \theta_u)g'(\theta_u) \\
&= \lambda ET(\theta_u)h + g(\theta_u) + (\theta - \theta_u)\left\{\frac{\lambda ET(\theta_u)h}{\theta_u} + g'(\theta_u)\right\} \\
&\geqq \lambda ET(\theta_u)h + g(\theta_u) = \psi(\theta_u).
\end{aligned} \tag{3.3}$$

The first inequality follows from the starshapeness of $ET(\theta)$ and the convexity of $g(\theta)$ and the last inequality follows from the definition of θ_u (see 3.2).

From the above theorem one sees that the starshapeness property can be used in optimization problems to reduce the search space.

Acknowledgement

The authors would like to thank the referee for his helpful comments on an earlier version of this paper.

References

HAREL, A. (1990) Convexity results for single-server queues and for multiserver queues with constant service times. *J. Appl. Prob.* **27,** 465–468.

LIYANAGE, L. AND SHANTHIKUMAR, J. G. (1990) Second order properties of single stage queueing systems. Technical Report, Walter A. Haas School of Business, University of California, Berkeley, CA.

MARSHALL, A. W. AND OLKIN, I. (1979) *Inequalities: Theory of Majorization and its Applications.* Academic Press, New York.

MEESTER, L. E. AND SHANTHIKUMAR, J. G. (1990) Concavity of the throughput of tandem queueing systems with finite buffer storage space. *Adv. Appl. Prob.* **22,** 764–767.

SHANTHIKUMAR, J. G. AND YAO, D. (1988) Stochastic convexity and its applications in parametric optimization of queueing systems. *Proc. 27th IEEE Conf. Decision and Control,* Austin, TX, 657–662.

SHANTHIKUMAR, J. G. AND YAO, D. (1989) Second-order stochastic properties in queueing systems. *Proc. IEEE* **77,** 162–170.

SHANTHIKUMAR, J. G. AND YAO, D. (1991) Strong stochastic convexity: closure properties and applications. *J. Appl. Prob.* **28,** 131–145.

TU, H. AND KUMIN, H. (1983) A convexity result for a class of $GI/G/1$ queueing systems. *Operat. Res.* **31,** 948–950.

WEBER, R. (1983) A note on waiting times in single server queues. *Operat. Res.* **31,** 950–951.