BULL. AUSTRAL. MATH. SOC.
VOL. 16 (1977), 307-309. 20C20

Abstracts of Australasian PhD theses

A study of certain modular representations

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Let p be a prime number, F the field of p elements, M the semigroup of all 2×2 matrices over F, G the group $\mathrm{GL}(2, p)$ of invertible elements of M, and S the normal subgroup $\mathrm{SL}(2, p)$ of G consisting of the matrices of determinant one. The aim of this thesis is to study the representations of M, G, and S over F.

A certain construction is of great help. Let V be the commutative polynomial algebra in two indeterminants, x and y say, over F. For each positive integer m, the homogeneous polynomials of degree $m-1$ form a subspace V_m, of dimension m, in V, and $V = \bigoplus_{m=1}^{\infty} V_m$. Each V_m may be regarded as an FM-module by considering the elements of M as homogeneous linear substitutions, so an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of M applied to a monomial $\mathrm{x}^i\mathrm{y}^j$ yields $(a\mathrm{x}+b\mathrm{y})^i(c\mathrm{x}+d\mathrm{y})^j$.

The one dimensional FM-modules, other than V_1, are the tensor powers D^n $(n = 1, 2, \ldots, p-1)$ of the module D which affords the determinant representation (so on this each element of M acts as the scalar which is its determinant); we put $D^0 = V_1$. We show that M has precisely p^2 (isomorphism classes of) irreducible modules over F;

Received 30 September 1976. Thesis submitted to the Australian National University, June 1976. Degree approved, November 1976. Supervisor: Dr L.G. Kovács.

namely the $V_m \otimes D^n$ with $1 \leq m \leq p$, $0 \leq n \leq p-1$. These modules, after restriction, yield all the irreducibles of G and S as well, but to keep them pairwise nonisomorphic the upper end of the range of n has to be brought down to $p - 2$ and 0 respectively. For each of M, G , and S the principal indecomposable modules are described in sufficient detail to reveal their complete submodule structure. For G and S , all principal indecomposables occur as direct summands in the restrictions of the V_m , while the same will hold for some, but definitely not all, principal indecomposables in the case of M . For G and S , direct decompositions of restrictions of the V_m have at most one nonprojective summand each; this is also false for M . For G and S the nonprojective indecomposable direct summands of the V_m form periodic sequences with period $p(p-1)$. In the (repeated) initial segment of length $p(p-1)$ of this sequence, every pth term is 0 while the others are nonzero and pairwise nonisomorphic. In the case of S every nonprojective indecomposable is isomorphic to one and only one of these, while in the case of G every nonprojective indecomposable is isomorphic to one and only one of these tensored with a D^n . The nonprojective indecomposables for G and S can therefore be described in sufficient detail to reveal a great deal of their submodule structure. Each G-module can be made into an M-module by making all elements of M outside G annihilate it: call such M-modules singular. Now M has infinitely many isomorphism types of nonsingular indecomposables, but only finitely many of them occur as direct summands of the V_m , and we do not attempt to classify even those which do.

To complete this summary of the results in this thesis, it remains to indicate what we know about the structure of the V_m . As far as the action of G or S is concerned, each V_m is a direct summand of $V_{m+p(p-1)}$ with a complement which depends only on the residue class of $m \bmod p^2-1$. This makes it possible to describe all V_m by dealing, as we do, with the first few. By contrast, with respect to the action of M the V_m do not fit such an arithmetic pattern: for instance, V_2 is not a

https://doi.org/10.1017/S0004972700023340 Published online by Cambridge University Press

direct summand of any other V_m , and is not even embeddable in V_m unless $m - 1$ is a power of p .

https://doi.org/10.1017/S0004972700023340 Published online by Cambridge University Press