Canad. Math. Bull. Vol. **57** (4), 2014 pp. 821–833
http://dx.doi.org/10.4153/CMB-2013-018-3

Real Hypersurfaces in Complex Two-Plane Grassmannians with Reeb Parallel Structure Jacobi Operator

Imsoon Jeong, Seonhui Kim, and Young Jin Suh

Abstract. In this paper we give a characterization of a real hypersurface of Type (A) in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, which means a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, by means of the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$.

1 Introduction

In the geometry of real hypersurfaces in complex space forms or in quaternionic space forms, there have been many characterizations of homogeneous hypersurfaces. For example, in complex projective space $\mathbb{C}P^m$ we call them real hypersurfaces of type (A_1), (A_2), (B), (C), (D), and (E); in complex hyperbolic space $\mathbb{C}P^m$, of type (A_0), (A_1), (A_2), and (B); in quaternionic projective space $\mathbb{H}P^m$, of type (A_1), (A_2), and (B); and in quaternionic hyperbolic space $\mathbb{H}H^m$, of type (A_0), (A_1), (A_2), and (B). They are completely classified by Kimura [12], Berndt [2, 3], and Martinez and Pérez [15].

Now let us consider a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J (see Berndt and Suh [5, 6]). Accordingly, in $G_2(\mathbb{C}^{m+2})$ we have two natural conditions for a real hypersurface M so that $[\xi] = \operatorname{Span}\{\xi\}$ or $\mathfrak{D}^\perp = \operatorname{Span}\{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. Here $\xi = -JN$, $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$, and N is a local unit normal vector field on M.

Using the two invariant conditions mentioned above, Berndt and Suh proved the following theorem.

Theorem 1.1 (Berndt and Suh [5]) *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, where $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape*

Received by the editors March 25, 2013; revised May 29, 2013.
Published electronically July 22, 2013.
This work was supported by grant Proj. No. NRF-2011-220-1-C00002 from the National Research Foundation of Korea. The first author was supported by grant Proj. No. NRF-2011-0013381, the second by grant Proj. No. NRF-2012-R1A2A2A-01043023, and the third by KNU Research Fund, 2013.
AMS subject classification: **53C40**, 53C15.
Keywords: real hypersurfaces, complex two-plane Grassmannians, Hopf hypersurface, Reeb parallel, structure Jacobi operator.

operator of M if and only if M is one of the following types:

Type (A) *M is an open part of a tube around a totally geodesic Grassmannian* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

Type (B) *m is even, say* $m = 2n$*, and M is an open part of a tube around a totally geodesic quaternionic projective space* $\mathbb{H}P^n$ *in* $G_2(\mathbb{C}^{m+2})$.

Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. By the formulas in Section 3 it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf. In such a case, the Reeb flow of ξ on M is said to be *geodesic*, and we say M is a real hypersurface with *geodesic Reeb flow*.

Remark 1.2 Related to a geodesic Reeb flow, we give an example of a ruled real hypersurface M in $G_2(\mathbb{C}^{m+2})$ that is not Hopf. It is foliated by complex hypersurfaces that include a maximal totally geodesic submanifold $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ (see Choi and Suh [7]). Its integrable distribution is given by $T_0(x) = \{X \in T_x M | X \perp \xi\}$, and the expression of the shape operator A of M is given by

$$A\xi = \alpha\xi + \beta U, \quad AU = \beta\xi, \quad \text{and} \quad AX = 0$$

for any X orthogonal to ξ and U. By virtue of the expression of the shape operator, we know that the distribution $T_0(x)$ is integrable. Then the shape operator never commutes with the structure tensor ϕ. Usually, the function $\alpha = g(A\xi, \xi)$ is not constant along the direction of ξ, because $\xi\alpha = g((\nabla_\xi A)\xi, \xi)$ cannot vanish in general. Of course, the Reeb vector field for a ruled hypersurface M in $G_2(\mathbb{C}^{m+2})$ does not have a geodesic Reeb flow; that is, M is not Hopf.

The Reeb vector field ξ on M is called *Killing* if the Reeb flow on M in $G_2(\mathbb{C}^{m+2})$ is *isometric*. It is denoted by $\mathcal{L}_\xi g = 0$, where $\mathcal{L}$ (resp. g) denotes the Lie derivative (resp. the induced Riemannian metric) of M in the direction of the Reeb vector field ξ. This means that the metric tensor g is *invariant* under the Reeb flow of ξ on M.

In [6], Berndt and Suh have given a characterization of real hypersurfaces of Type (A) in Theorem 1.1 when the shape operator A of M in $G_2(\mathbb{C}^{m+2})$ commutes with the structure tensor ϕ. This is equivalent to the condition that the Reeb flow on M is isometric.

By using such a notion, Berndt and Suh [6] gave the following characterization of Type (A) in $G_2(\mathbb{C}^{m+2})$.

Theorem 1.3 *Let M be a connected orientable real hypersurface in* $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. *Then the Reeb flow on M is isometric if and only if M is an open part of a tube around a totally geodesic* $G_2(\mathbb{C}^{m+1})$ *in* $G_2(\mathbb{C}^{m+2})$.

On the other hand, for real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$, Lee and Suh [14] recently proved the following theorem.

Theorem 1.4 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, where $m = 2n$.*

Now we introduce the notion of structure Jacobi operator R_ξ defined by

$$R_\xi(X) = R(X, \xi)\xi,$$

where $R(X, Y)Z$ denotes the curvature tensor of M in $G_2(\mathbb{C}^{m+2})$ for any tangent vector fields X, Y, and Z on M. Then the structure Jacobi operator R_ξ for the Reeb vector ξ is said to be *parallel* if the covariant derivative of the structure Jacobi operator R_ξ vanishes, that is, if $\nabla_X R_\xi = 0$ for any vector field X on M.

Related to such a structure Jacobi operator R_ξ, many authors have studied some geometric properties for real hypersurfaces in complex space form $M_n(c)$. In [11], Ki, Pérez, Santos, and Suh investigated the covariant derivative $\nabla_\xi S = 0$ for the Ricci tensor S and the parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$ along the direction of ξ. In [19], Pérez, Santos, and Suh classified real hypersurfaces in $\mathbb{C}P^m$ with a ξ-invariant structure Jacobi operator, that is, $\mathcal{L}_\xi R_\xi = 0$. Also, they proved the non-existence of any real hypersurfaces in $\mathbb{C}P^m$ with a $\mathfrak{D}$-parallel structure Jacobi operator $\nabla_X R_\xi = 0$ for any $X \in \mathfrak{D}$, where the distribution $\mathfrak{D}$ is defined by the subspace $\mathfrak{D}_x = \{\, X\in T_xM \mid X\perp \xi \,\}$, $x \in M$. So the distribution $\mathfrak{D}$ becomes an orthogonal complement of the Reeb vector field ξ on real hypersurfaces in $\mathbb{C}P^m$ (see [20]).

Moreover, Pérez, and Suh [17] classified real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ whose curvature tensor is parallel in the direction of the distribution $\mathfrak{D}^\perp$, that is, $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{H}P^k$ in $\mathbb{H}P^m$, $2 \leq k \leq m-2$.

But in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, if we consider these properties, the situation is quite different from that of $\mathbb{C}P^m$ and $\mathbb{H}P^m$.

Recently, Jeong, Pérez, and Suh [10] proved that there does not exist a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with parallel structure Jacobi operator. Also, Jeong, Machado, Pérez, and Suh [9] obtained the non-existence for real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}^\perp$-parallel structure Jacobi operator $\nabla_X R_\xi = 0$ for any X belonging to the distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$.

Motivated by such a notion of parallel structure Jacobi operators, in this paper, we consider the parallelism of R_ξ on M in $G_2(\mathbb{C}^{m+2})$ in the direction of the Reeb vector field ξ.

We note here that the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$ is weaker than the parallel structure Jacobi operator $\nabla_X R_\xi = 0$ for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$.

In such a case we say that M has a *Reeb parallel* structure Jacobi operator. We can give a characterization of Type (A) hypersurfaces in Theorem 1.1 as follows.

Theorem 1.5 (Main Theorem) *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ on M is non-vanishing and constant along the direction of the Reeb vector field ξ, then M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{\sqrt{8}})$.*

Remark 1.6 When the function $\alpha = g(A\xi, \xi)$ vanishes identically, we know that the ruled hypersurface M in $G_2(\mathbb{C}^{m+2})$ in Remark 1.2 becomes a minimal ruled real hypersurface in $G_2(\mathbb{C}^{m+2})$ like in Kimura [13]and Ahn, Lee, and Suh [1] for real hypersurfaces in complex projective space $\mathbb{C}P^m$ and complex hyperbolic space $\mathbb{C}H^m$, respectively. In this case, the shape operator becomes

$$A\xi = \beta U, \quad AU = \beta\xi, \quad \text{and} \quad AX = 0$$

for any X orthogonal to ξ and U (see [8]). Then the Reeb vector field cannot be Hopf, so we know that the structure Jacobi operator cannot be Reeb parallel.

2 Riemannian Geometry of $G_2(\mathbb{C}^{m+2})$

In this section we summarize basic material about $G_2(\mathbb{C}^{m+2})$, for details refer to [4–6]. By $G_2(\mathbb{C}^{m+2})$ we denote the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. The special unitary group $G = SU(m+2)$ acts transitively on $G_2(\mathbb{C}^{m+2})$ with stabilizer isomorphic to $K = S(U(2) \times U(m)) \subset G$. Then $G_2(\mathbb{C}^{m+2})$ can be identified with the homogeneous space G/K, which we equip with the unique analytic structure for which the natural action of G on $G_2(\mathbb{C}^{m+2})$ becomes analytic. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebra of G and K, respectively, and by $\mathfrak{m}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Cartan–Killing form B of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\mathrm{Ad}(K)$-invariant reductive decomposition of $\mathfrak{g}$. We put $o = eK$ and identify $T_oG_2(\mathbb{C}^{m+2})$ with $\mathfrak{m}$ in the usual manner. Since B is negative definite on $\mathfrak{g}$, its negative restricted to $\mathfrak{m} \times \mathfrak{m}$ yields a positive definite inner product on $\mathfrak{m}$. By $\mathrm{Ad}(K)$-invariance of B this inner product can be extended to a G-invariant Riemannian metric g on $G_2(\mathbb{C}^{m+2})$. In this way $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

When $m = 1$, $G_2(\mathbb{C}^3)$ is isometric to the two-dimensional complex projective space $\mathbb{C}P^2$ with constant holomorphic sectional curvature eight. When $m = 2$, we note that the isomorphism $\mathrm{Spin}(6) \simeq SU(4)$ yields an isometry between $G_2(\mathbb{C}^4)$ and the real Grassmann manifold $G_2^+(\mathbb{R}^6)$ of oriented two-dimensional linear subspaces of $\mathbb{R}^6$. In this paper, we will assume $m \geq 3$.

The Lie algebra $\mathfrak{k}$ has the direct sum decomposition $\mathfrak{k} = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{R}$, where $\mathfrak{R}$ denotes the center of $\mathfrak{k}$. Viewing $\mathfrak{k}$ as the holonomy algebra of $G_2(\mathbb{C}^{m+2})$, the center $\mathfrak{R}$ induces a Kaehler structure J and the $\mathfrak{su}(2)$-part a quaternionic Kaehler structure $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$. If J_1 is any almost Hermitian structure in $\mathfrak{J}$, then $JJ_1 = J_1J$, and JJ_1 is a symmetric endomorphism with $(JJ_1)^2 = I$ and $\mathrm{tr}(JJ_1) = 0$.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\overline{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis J_1, J_2, J_3 of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\overline{\nabla}_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

The Riemannian curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned} \overline{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3} \big\{ g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z \big\} \\ & + \sum_{\nu=1}^{3} \big\{ g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY \big\}, \end{aligned} \tag{2.1}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$ (see [4]).

3 Some Fundamental Formulas in $G_2(\mathbb{C}^{m+2})$

In this section we derive some basic formulae and the equation of Codazzi and Gauss for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (see [5, 6]).

Let M be a real hypersurface of $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M is denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and let A denote the shape operator of M with respect to N.

The Kähler structure J of $G_2(\mathbb{C}^{m+2})$ on M induces an almost contact metric structure (ϕ, ξ, η, g). More explicitly, we can define a tensor field ϕ of type (1,1), a vector field ξ and its dual 1-form η on M by $g(\phi X, Y) = g(JX, Y)$ and $\eta(X) = g(\xi, X)$ for any tangent vector fields X and Y on M. Then they satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \text{and} \quad \eta(\xi) = 1$$

for any tangent vector field X on M. Furthermore, let J_1, J_2, J_3 be a canonical local basis of $\mathfrak{J}$. Then each J_ν induces an almost contact metric structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M in such a way that a tensor field ϕ_ν of type (1,1), a vector field ξ_ν and its dual 1-form η_ν on M are defined by $g(\phi_\nu X, Y) = g(J_\nu X, Y)$ and $\eta_\nu(X) = g(\xi_\nu, X)$ for any tangent vector fields X and Y on M respectively. Then they also satisfy the following:

$$\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \phi_\nu\xi_\nu = 0, \quad \eta_\nu(\phi_\nu X) = 0, \quad \text{and} \quad \eta_\nu(\xi_\nu) = 1$$

for any tangent vector field X on M and $\nu = 1, 2, 3$.

Using the above expression (2.1) for the curvature tensor $\overline{R}$ of $G_2(\mathbb{C}^{m+2})$, the equations of Gauss and Codazzi are respectively given by

$$\begin{aligned}
R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\
&\quad + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\
&\quad + \sum_{\nu=1}^{3}\big\{ g(\phi_\nu Y,Z)\phi_\nu X - g(\phi_\nu X,Z)\phi_\nu Y - 2g(\phi_\nu X,Y)\phi_\nu Z\big\} \\
&\quad + \sum_{\nu=1}^{3}\big\{ g(\phi_\nu\phi Y,Z)\phi_\nu\phi X - g(\phi_\nu\phi X,Z)\phi_\nu\phi Y\big\} \\
&\quad - \sum_{\nu=1}^{3}\big\{ \eta(Y)\eta_\nu(Z)\phi_\nu\phi X - \eta(X)\eta_\nu(Z)\phi_\nu\phi Y\big\} \\
&\quad - \sum_{\nu=1}^{3}\big\{ \eta(X)g(\phi_\nu\phi Y,Z) - \eta(Y)g(\phi_\nu\phi X,Z)\big\}\xi_\nu \\
&\quad + g(AY,Z)AX - g(AX,Z)AY
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_X A)Y - (\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\
&\quad + \sum_{\nu=1}^{3}\big\{ \eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X,Y)\xi_\nu\big\} \\
&\quad + \sum_{\nu=1}^{3}\big\{ \eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\big\} \\
&\quad + \sum_{\nu=1}^{3}\big\{ \eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\big\}\xi_\nu,
\end{aligned}$$

where R denotes the curvature tensor of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$.

Then the following identities can be proved in a straightforward way and will be used frequently in subsequent calculations:

$$\begin{gathered}
\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \quad \phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \\
\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \quad \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.
\end{gathered}$$

From this and the above formulae we have

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X\xi = \phi AX,$$
$$\nabla_X\xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX,$$
$$(\nabla_X\phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{3.1}$$

Moreover, from $JJ_\nu = J_\nu J$, $\nu = 1,2,3$, it follows that

$$\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu.$$

On the other hand, by using the fact of $A\xi = \alpha\xi$, $\alpha = g(A\xi,\xi)$, and the Codazzi equation, we have

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y)$$

for any tangent vector field Y on M in $G_2(\mathbb{C}^{m+2})$.

Now let us recall a lemma due to Berndt and Suh [6].

Lemma 3.1 *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then*

$$\begin{aligned}\alpha g\big((A\phi+\phi A)X,Y\big) &- 2g(A\phi AX,Y) + 2g(\phi X,Y)\\ &= 2\sum_{\nu=1}^{3}\big\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X,Y)\eta_\nu(\xi)\\ &\qquad - 2\eta(X)\eta_\nu(\phi Y)\eta_\nu(\xi) + 2\eta(Y)\eta_\nu(\phi X)\eta_\nu(\xi)\big\}\end{aligned}$$

for all vector fields X and Y on M.

On the other hand, we introduce the following lemma due to Jeong, Machado, Pérez, and Suh [9].

Lemma 3.2 *Let M be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$. If the principal curvature α is constant along the direction of ξ, then the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the structure vector field ξ is invariant by the shape operator.*

4 The Reeb Parallel Structure Jacobi Operator

In this section we give some lemmas which will be useful in the proof of Theorem 1.5.

Now we put the structure vector $\xi = -JN$ into the curvature tensor R of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector of M in $G_2(\mathbb{C}^{m+2})$. Then for any tangent vector field X on M in $G_2(\mathbb{C}^{m+2})$ we calculate the structure

Jacobi operator R_ξ in such a way that

$$R_\xi X = R(X,\xi)\xi = X - \eta(X)\xi - \sum_{\nu=1}^{3}\Big\{ \big(\eta_\nu(X) - \eta(X)\eta_\nu(\xi)\big)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\Big\} + \alpha AX - \eta(AX)A\xi, \tag{4.1}$$

where α denotes the function defined by $g(A\xi,\xi)$.

Let us assume that the structure Jacobi operator R_ξ on a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$ satisfies the Reeb parallelism $(\nabla_\xi R_\xi)X = 0$ for any tangent vector field X on M. By differentiating (4.1), we have

$$\begin{aligned}0 &= (\nabla_X R_\xi)Y \\ &= \nabla_X(R_\xi Y) - R_\xi\nabla_X Y \\ &= -g(\phi AX, Y)\xi - \eta(Y)\phi AX \\ &\quad - \sum_{\nu=1}^{3}\Big[g(\phi_\nu AX, Y)\xi_\nu - 2\eta(Y)\eta_\nu(\phi AX)\xi_\nu + \eta_\nu(Y)\phi_\nu AX \\ &\qquad + 3\big\{ g(\phi_\nu AX, \phi Y)\phi_\nu\xi + \eta(Y)\eta_\nu(AX)\phi_\nu\xi \\ &\qquad\quad - \eta_\nu(\phi Y)\eta(AX)\xi_\nu + \eta_\nu(\phi Y)\phi_\nu\phi AX\big\} \\ &\qquad + 4\eta_\nu(\xi)\{\eta_\nu(\phi Y)AX - g(AX,Y)\phi_\nu\xi\} + 2\eta_\nu(\phi AX)\phi_\nu\phi Y\Big] \\ &\quad + \eta\big((\nabla_X A)\xi\big)AY + \alpha(\nabla_X A)Y - \alpha\eta\big((\nabla_X A)Y\big)\xi \\ &\quad - \alpha g(AY, \phi AX)\xi - \alpha\eta(Y)(\nabla_X A)\xi - \alpha\eta(Y)A\phi AX\end{aligned} \tag{4.2}$$

for any tangent vector fields X and Y on M.

If we put $X = \xi$ and $Y = X$ in (4.2), then we have

$$\begin{aligned}0 &= (\nabla_\xi R_\xi)X \\ &= 4\alpha\sum_{\nu=1}^{3}\big\{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi - \eta_\nu(\xi)\eta_\nu(\phi X)\xi + \eta_\nu(\xi)\eta(X)\phi_\nu\xi\big\} \\ &\quad + (\xi\alpha)AX + \alpha(\nabla_\xi A)X - 2\alpha(\xi\alpha)\eta(X)\xi\end{aligned} \tag{4.3}$$

for any tangent vector field X on M.

Remark 4.1 When the function α vanishes, the above equation gives that the structure Jacobi operator is Reeb parallel $\nabla_\xi R_\xi = 0$. Moreover, from Pérez and Suh [18], we know that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

Lemma 4.2 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the distribution $\mathfrak{D}$ or $\mathfrak{D}^\perp$ component of the Reeb vector field ξ is invariant under the shape operator, then ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof In order to prove this lemma, let us put $\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1$ for some unit vector $X_0 \in \mathfrak{D}$ and non-zero functions $\eta(X_0)$ and $\eta(\xi_1)$. By putting $X = X_0$ into (4.3) we have

$$0 = 4\alpha\eta_1(\xi)\eta(X_0)\phi_1\xi + (\xi\alpha)AX_0 + \alpha(\nabla_\xi A)X_0 - 2\alpha(\xi\alpha)\eta(X_0)\xi.$$

Using a method similar to that in [10, Lemma 3.1], we obtain $\phi X_0 = 0$. This gives a contradiction, which completes the proof of our lemma. ■

By Lemmas 3.2 and 4.2, we have the following lemma.

Lemma 4.3 *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature α is constant along the direction of ξ, then the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

5 Proof of Theorem 1.5

In this section, we assume that M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with Reeb parallel structure Jacobi operator. Then by Lemma 4.2 we assume that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

First, let us investigate the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$. Then we have the following lemma, which will be useful in the proof of Theorem 5.3.

Lemma 5.1 *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ is non-vanishing and ξ belongs to the distribution $\mathfrak{D}^\perp$, then the shape operator A commutes with the structure tensor field ϕ.*

Proof In order to prove this lemma, we may put $\xi = \xi_1$, because $\xi \in \mathfrak{D}^\perp$. From (4.3), we have $\alpha(\nabla_\xi A)X = 0$ for any tangent vector field X on M.

Since the geodesic Reeb flow α is *non-vanishing*, we have $(\nabla_\xi A)X = 0$. By using the Codazzi equation, we have

$$\begin{aligned}0 &= (\nabla_\xi A)X\\ &= -A\phi AX + (X\alpha)\xi + \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3.\end{aligned}$$

From this, by taking an inner product with ξ, it follows that $X\alpha = 0$ for any tangent vector field X on M.

This gives that the principal curvature α is constant. Then we have

$$A\phi AX = \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3. \tag{5.1}$$

From Lemma 3.1, we have

$$2A\phi AX = \alpha A\phi X + \alpha\phi AX + 2\phi X + 2\phi_1 X + 4\eta_3(X)\xi_2 - 4\eta_2(X)\xi_3 \tag{5.2}$$

for any tangent vector field X on M. Using (5.1) and (5.2), we know that $A\phi = \phi A$. Thus we complete the proof of our lemma. ∎

By Theorem 1.3, we assert that a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with the assumption in Lemma 5.1 is a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. In other words, M is locally congruent to a real hypersurface of Type (A) in Theorem 1.1.

Conversely, let us check whether real hypersufaces of Type (A) satisfy the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$.

We recall a proposition given by Berndt and Suh [5].

Proposition 5.2 *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}^\perp$. Let $J_1 \in \mathfrak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then M has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures*

$$\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0$$

with some $r \in (0, \pi/\sqrt{8})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces are

$$\begin{aligned} T_\alpha &= \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1, \\ T_\beta &= \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3, \\ T_\lambda &= \{X | X \perp \mathbb{H}\xi, JX = J_1 X\}, \\ T_\mu &= \{X | X \perp \mathbb{H}\xi,\ JX = -J_1 X\}, \end{aligned}$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$, and $\mathbb{H}\xi$ respectively denote the real, complex, and quaternionic spans of the structure vector ξ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Now let us check case by case whether real hypersurfaces of Type (A) satisfy formula (4.3).

Case A-1 $X \in T_\alpha$

By using the conditions of $\xi \in \mathfrak{D}^\perp$ and $\xi\alpha = 0$ in (4.3), we assert formula (5.1) (see [10]). Then it can be easily checked by putting $X = \xi$ in (5.1).

Case A-2 $X \in T_\beta$

We put $A\xi_2 = \beta\xi_2$, $A\xi_3 = \beta\xi_3$, where $\beta = \sqrt{2}\cot(\sqrt{2}r)$. By putting $X = \xi_2$ in (5.1), we have

$$\begin{aligned} (\nabla_\xi A)\xi_2 &= -A\phi A\xi_2 + \alpha\phi A\xi_2 + \phi\xi_2 + \phi_1\xi_2 + 2\eta_3(\xi_2)\xi_2 - 2\eta_2(\xi_2)\xi_3 \\ &= -\beta A\phi\xi_2 + \alpha\beta\phi\xi_2 - 2\xi_3 = \beta^2\xi_3 - \alpha\beta\xi_3 - 2\xi_3 \\ &= (\beta^2 - \alpha\beta - 2)\xi_3 = 0. \end{aligned}$$

https://doi.org/10.4153/CMB-2013-018-3 Published online by Cambridge University Press

Similarly, by putting $X = \xi_3$ in (5.1), we obtain

$$(\nabla_\xi A)\xi_3 = -(\beta^2 - \alpha\beta - 2)\xi_2 = 0.$$

Case A-3 $X \in T_\lambda = \{X \mid X \perp \mathbb{H}\xi,\ \phi X = \phi_1 X\}$
For any tangent vector field $X \in T_\lambda$, $\lambda = -\sqrt{2}\tan(\sqrt{2}r)$ we get

$$\begin{aligned}(\nabla_\xi A)X &= -A\phi AX + \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3\\ &= -\lambda A\phi X + \alpha\lambda\phi X + \phi X + \phi_1 X = -\lambda^2\phi X + \alpha\lambda\phi X + 2\phi X\\ &= -(\lambda^2 - \alpha\lambda - 2)\phi X = 0.\end{aligned}$$

Case A-4 $X \in T_\mu = \{X \mid X \perp \mathbb{H}\xi,\ \phi X = -\phi_1 X\}$
For any tangent vector field $X \in T_\mu$, $\mu = 0$ we get

$$\begin{aligned}(\nabla_\xi A)X &= -A\phi AX + \alpha\phi AX + \phi X + \phi_1 X + 2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3\\ &= -\mu A\phi X + \alpha\mu\phi X + \phi X + \phi_1 X = 0.\end{aligned}$$

Summing up all cases, we have formula (4.3). Thus we can assert the following theorem.

Theorem 5.3 *Let M be a connected orientable Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator. If the principal curvature of the Reeb vector field ξ is non-vanishing and $\xi \in \mathfrak{D}^\perp$, then M is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with radius $r \in (0, \frac{\pi}{4\sqrt{2}}) \cup (\frac{\pi}{4\sqrt{2}}, \frac{\pi}{\sqrt{8}})$.*

Next we consider the case that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$. By Theorem 1.4, we see that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with ξ-parallel structure Jacobi operator is of Type (B) in Theorem 1.1. In order to complete the proof of our main theorem let us recall a proposition due to Berndt and Suh [5].

Proposition 5.4 *Let M be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r)\,,\quad \beta = 2\cot(2r)\,,\quad \gamma = 0\,,\quad \lambda = \cot(r)\,,\quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1,\quad m(\beta) = 3 = m(\gamma),\quad m(\lambda) = 4n - 4 = m(\mu),$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R}\xi,\quad T_\beta = \mathfrak{J}J\xi,\quad T_\gamma = \mathfrak{J}\xi,\quad T_\lambda\,,\quad T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp,\quad \mathfrak{J}T_\lambda = T_\lambda,\quad \mathfrak{J}T_\mu = T_\mu,\quad JT_\lambda = T_\mu.$$

https://doi.org/10.4153/CMB-2013-018-3 Published online by Cambridge University Press

Then for $\xi \in \mathfrak{D}$ and $\xi\alpha = 0$ in (4.3), we have

$$0 = 4\alpha \sum_{\nu=1}^{3} \{ \eta_\nu(\phi X)\xi_\nu - \eta_\nu(X)\phi_\nu\xi \} + \alpha(\nabla_\xi A)X.$$

From this, by putting $X = \xi_2$ we have $0 = -4\alpha\phi_2\xi + \alpha(\nabla_\xi A)\xi_2$. By taking the inner product with $\phi_2\xi$ and using (3.1), we have $-4\alpha + \alpha^2\beta = 0$.

Since the principal curvature α is non-zero, it follows that $\alpha\beta = 4$. This gives a contradiction. Then we assert that the structure Jacobi operator R_ξ of real hypersurfaces of Type (B) in Theorem 1.1 does not satisfy $\nabla_\xi R_\xi = 0$. Then from this fact, we assert the following theorem.

Theorem 5.5 *There does not exist any connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with Reeb parallel structure Jacobi operator if the principal curvature of the Reeb vector field ξ is non-vanishing and $\xi \in \mathfrak{D}$.*

Combining Theorems 5.3 and 5.5, we complete the proof of Theorem 1.5. ∎

Remark 5.6 Recently, we have been informed that the Reeb invariant structure Jacobi operator $\mathcal{L}_\xi R_\xi = 0$ for the Lie derivative $\mathcal{L}_\xi$ along the Reeb vector field ξ was studied by Machado and Pérez [16]. But usually the Reeb parallel structure Jacobi operator $\nabla_\xi R_\xi = 0$ for the covariant derivative ∇_ξ along the direction of ξ becomes a condition weaker than the Reeb invariant $\mathcal{L}_\xi R_\xi = 0$.

Acknowledgments The authors would like to express their gratitude to the referee for careful reading of our article, useful comments, and valuable suggestions, which helped us to develop the first version of our manuscript.

References

[1] S.-S. Ahn, S.-B. Lee, and Y. J. Suh, *On ruled real hypersurfaces in a complex space form*. Tsukuba J. Math. **17**(1993), no. 2, 311–322.
[2] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space.* J. Reine Angew. Math. **395**(1989), 132–141.
[3] ———, *Real hypersurfaces in quaternionic space forms*. J. Reine Angew. Math. **419**(1991), 9–26.
[4] ———, *Riemannian geometry of complex two-plane Grassmannian*. Rend. Sem. Mat. Univ. Politec. Torino **55**(1997), no. 1, 19–83.
[5] J. Berndt and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*. Monatsh. Math. **127**(1999), no. 1, 1–14. http://dx.doi.org/10.1007/s006050050018
[6] ———, *Isometric flows on real hypersurfaces in complex two-plane Grassmannians* Monatsh. Math. **137**(2002), no. 2, 87–98. http://dx.doi.org/10.1007/s00605-001-0494-4
[7] Y. S. Choi and Y. J. Suh, *Real hypersurfaces with η-parallel shape operator in complex two-plane Grassmannians*. Bull. Austral. Math. Soc. **75**(2007), no. 1, 1–16. http://dx.doi.org/10.1017/S0004972700038934
[8] I. Jeong, M. Kimura, H. Lee, and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with Reeb parallel shape operator*. Monatsh. Math., to appear. http://dx.doi.org/10.1007/s00605-013-0475-4
[9] I. Jeong, C. J. Machado, J. D. Pérez, and Y. J. Suh, *Real hypersurface in complex two-plane Grassmannians with $\mathfrak{D}^\perp$-parallel structure Jacobi operator*. Internat. J. Math. **22**(2011), no. 5, 655–673. http://dx.doi.org/10.1142/S0129167X11006957
[10] I. Jeong, J. D. Pérez, and Y. J. Suh, *Real hypersurfaces in complex two-plane Grassmannians with parallel structure Jacobi operator*. Acta Math. Hungar. **112**(2009), no. 1–2, 173–186. http://dx.doi.org/10.1007/s10474-008-8004-y

[11] U-H. Ki, J. D. Pérez, F. G. Santos, and Y. J. Suh, *Real hypersurfaces in complex space forms with ξ-parallel Ricci tensor and structure Jacobi operator*. J. Korean Math. Soc. **44**(2007), no. 2, 307–326. http://dx.doi.org/10.4134/JKMS.2007.44.2.307

[12] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*. Trans. Amer. Math. Soc. **296**(1986), no. 1, 137–149. http://dx.doi.org/10.1090/S0002-9947-1986-0837803-2

[13] ———, *Sectional curvatures of holomorphic planes on a real hypersurface in $P_n(C)$*. Math. Ann. **276**(1987), no. 3, 487–497. http://dx.doi.org/10.1007/BF01450843

[14] H. Lee and Y. J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*. Bull. Korean Math. Soc. **47**(2010), no. 3, 551–561. http://dx.doi.org/10.4134/BKMS.2010.47.3.551

[15] A. Martinez and J. D. Pérez, *Real hypersurfaces in quaternionic projective space*. Ann. Math. Pura Appl. **145**(1986), 355–384. http://dx.doi.org/10.1007/BF01790548

[16] C. J. Machado and J. D. Pérez, *Real hypersurfaces in complex two-plane Grassmannians whose Jacobi operators are ξ-invariant*. Internat. J. Math. **23**(2012), no. 3, 1250002. http://dx.doi.org/10.1142/S0129167X1100746X

[17] J. D. Pérez and Y. J. Suh, *Real hypersurfaces of quaternionic projective space satisfying $\nabla_{\xi_i} R = 0$*. Differential Geom. Appl. **7**(1997), no. 3, 211–217. http://dx.doi.org/10.1016/S0926-2245(97)00003-X

[18] ———, *The Ricci tensor of real hypersurfaces in complex two-plane Grassmannians.* J. Korean Math. Soc. **44**(2007), no. 1, 211–235. http://dx.doi.org/10.4134/JKMS.2007.44.1.211

[19] J. D. Pérez, F. G. Santos, and Y. J. Suh, *Real hypersurfaces in complex projective space whose structure Jacobi operator is Lie ξ-parallel*. Differential Geom. Appl. **22**(2005), no. 2. 181–188. http://dx.doi.org/10.1016/j.difgeo.2004.10.005

[20] ———, *Real hypersurfaces in complex projective space whose structure Jacobi operator is $\mathfrak{D}$-parallel*. Bull. Belg. Math. Soc. Simon Stevin **13**(2006), no. 3, 459–469.

Department of Mathematics, Kyungpook National University, Taegu, 702-701, KOREA
e-mail: imsoon.jeong@gmail.com kimsh0123@hanmail.net yjsuh@knu.ac.kr