On the Pascal Hexagram.

By Prof. J. Jack.

§ 1. *Proof in case of the circle.*

Figure 9.

ABCDEF is any cyclic hexagon ;

AB, DE meet in G,

BC, EF „ „ H,

CD, FA „ „ K ; then G, H, K are in a straight line.

Draw KLNM parallel to BC and produce DE, EF, AB to meet it in L, N, M. Join DA, DM and let BC, DE meet in P.

$\angle$ PCD = $\angle$ BAD ∵ ABCD is a cyclic quadrilateral,

and $\angle$ PCD = $\angle$ DKM ∵ KM is parallel to CB ;

∴ $\angle$ BAD = $\angle$ DKM ; ∴ DAMK is a cyclic quadrilateral.

Again $\angle$ DEH = $\angle$ DAK ∵ AFED is cyclic ;

and $\angle$ DMN, *i.e.*, $\angle$ DMK = $\angle$ DAK ∵ DAMK is cyclic ;

∴ $\angle$ DEH = $\angle$ DMN and ∴ DMNE is cyclic.

Again $\frac{PB}{PE}=\frac{PD}{PC}=\frac{DL}{LK}$

and $\frac{PE}{PH}=\frac{LE}{LN}=\frac{LM}{LD}$; ∴ $\frac{PB}{PH}=\frac{LM}{LK}$;

∴ G, H, K are in a straight line.

Figures 10 and 11.

§ 2. O is any point on a chord AB of a conic ; the focus S, the directrix and *e* are given ; the eccentric circle of O is described. Through O, radii O*a*, O*b* are drawn parallel to SA, SB, in opposite

https://doi.org/10.1017/S0013091500033320 Published online by Cambridge University Press

sense when O, S are on the *same* side of the directrix (Fig. 10) and in the same sense when on *opposite* sides (Fig. 11).

Then ab passes through S.

For $$\frac{Oa}{OK\sin\theta} = \frac{SA}{AK\sin\theta};$$

$\therefore$ S, a, K are in a straight line. So S, b, K are in a straight line; $\therefore$ ab goes through S.

§ 3. *Pascal's Theorem for the Conic (generally).*

ABCDEF is any hexagon inscribed in a conic.

AB, DE meet in O_1; BC, EF meet in O_2, and CD, FA meet in O_3.

Then O_1, O_2, O_3 are in a straight line.

The focus S, the directrix and e are given.

Draw the eccentric circles of the points O_1, O_2, O_3 and draw in each of the circles the six radii parallel to SA, SB, SC, SD, SE and SF, in the opposite sense when O and S are on the same side of the directrix and in the same sense when O, S are on opposite sides of the directrix.

Let the radii be O_1a_1,O_1b_1, etc.; O_2a_2, O_2b_2, etc., etc.

Join the points a_1b_1, b_1c_1, c_1d_1, d_1e_1, e_1f_1, f_1a_1;
a_2b_2, b_2c_2, c_2d_2, etc.,
a_3b_3, b_3c_3, c_3d_3, etc.

Then the three cyclic hexagons $(abcdef)_1$, $(abcdef)_2$, $(abcdef)_3$ are similar and similarly situated.

Let a_1b_1, d_1e_1 meet in g_1,
b_1c_1, e_1f_1 ,, ,, h_1,
and c_1d_1, f_1a_1 ,, ,, k_1,

and similarly for the other two hexagons let the corresponding sides meet in g_2, h_2, k_2 and g_3, h_3, k_3.

Then $g_1h_1k_1$, $g_2h_2k_2$, $g_3h_3k_3$ are straight lines by the proof of the theorem in the case of a circle.

Now from the nature of the eccentric circle, a_1b_1, d_1e_1 meet in S, that is, the point g_1 is S.

Similarly „ „ h_2 is S

and „ „ k_3 is S.

Hence the three straight lines, $g_1h_1k_1$, $g_2h_2k_2$, $g_3h_3k_3$ have one point S common and they are parallel, because the figures are similar and similarly situated;

∴ the three lines are coincident.

Now taking the three triangles $O_1a_1g_1$, $O_2a_2g_2$, $O_3a_3g_3$ which are similar, we have

$$\frac{O_1g_1}{O_1a_1} = \frac{O_2g_2}{O_2a_2} = \frac{O_3g_3}{O_3a_3}$$

and if O_1m_1, O_2m_2, O_3m_3 are the perpendiculars from O_1, O_2, O_3 to the directrix, we have

$$\frac{O_1a_1}{O_1m_1} = \frac{O_2a_2}{O_2m_2} = \frac{O_3a_3}{O_3m_3} = e;$$

it therefore follows that

$$\frac{O_1g_1}{O_1m_1} = \frac{O_2g_2}{O_2m_2} = \frac{O_3g_3}{O_3m_3};$$

∴ O_1, O_2, O_3 are in a straight line, and it passes through the point in which the Pascal line of the cyclic hexagons meets the directrix.

On Newton's Theorem in the Calculus of Variations.

By J. H. Maclagan-Wedderburn, M.A.

https://doi.org/10.1017/S0013091500033320 Published online by Cambridge University Press