Can. J. Math., Vol. XXXII, No. 1, 1980, pp. 86–125

CONVERGENCE OF CLASSES OF AMARTS INDEXED BY DIRECTED SETS

ANNIE MILLET AND LOUIS SUCHESTON

Let $(\Omega, \mathscr{F}, P)$ be a probability space, J a directed set filtering to the right. $(X_t)_{t \in J}$ is a family of random variables adapted to an increasing family of σ-algebras $(\mathscr{F}_t)_{t \in J}$. Vitali conditions V and V' on the σ-algebras, abstracting classical assumptions in Lebesgue's derivation theory, were made to insure essential convergence of martingales and submartingales (under proper boundedness assumptions). In reality these conditions, guaranteeing the existence of certain disjoint and properly measurable sets B_i, are better suited for study of amarts, since the sets B_i are a natural habitat and breeding ground for stopping times, thriving, as well known, precisely on disjoint and properly measurable sets. Thus K. Astbury [**1**] showed that the condition V, proved by K. Krickeberg to be sufficient for convergence of martingales (see [**20**] or Neveu [**26**], p. 98) is both necessary and sufficient for convergence of amarts. (We follow Neveu denoting by V the condition Krickeberg denotes by $V_{+\infty}$.) The Vitali condition V', proved by Krickeberg [**21**] to be sufficient for convergence of submartingales, is shown here to be both necessary and sufficient for convergence of *ordered amarts*, defined similarly to amarts, except that the stopping times are ordered. We also introduce the *controlled* Vitali condition V^c, properly weaker than V', and show that V^c is sufficient for convergence of *controlled* amarts, including submartingales. This answers in the negative a question raised by Krickeberg ([**22**], p. 280), whether V' is necessary for convergence of submartingales.

In the direct part of theorems, the amart assumption can be considerably weakened, at the price of some loss of simplicity. If σ, τ are simple stopping times, write $X(\sigma, \tau)$ for the expression $X_\sigma - E^{\mathscr{F}_\sigma} X_\tau$. It is known that (X_t) is an amart if and only if $\lim_{\tau \geqq \sigma \to \infty} X(\sigma, \tau) = 0$ in L^1. Call (X_t) a *pramart* (for *amart in probability*) if $X(\sigma, \tau) \to 0$ $(\tau \geqq \sigma \to \infty)$ in probability. One can go one step further: call (X_t) a *subpramart* if stochastic $\limsup_{\tau \geqq \sigma \to \infty} X(\sigma, \tau) \leqq 0$. Not only amarts and pramarts, but also subpramarts converge essentially under the condition V. This result is new even if $J = N$, and constitutes also in that case a generalization of the amart convergence theorem. Unlike amarts (cf. [**15**]) or pramarts, subpramarts need not be mils (martingales in the limit), and, unlike mils, subpramarts have good optional sampling properties. Thus the generalization of Doob's martingale convergence theorem to sub-

Received February 9, 1978 and in revised form March 20, 1979. The research of the second author was in part supported by the National Science Foundation.

pramarts given below seems of interest. To be sure, subpramarts (or pramarts) cannot in general have the Riesz decomposition, because this decomposition together with the optional sampling theorem is known to characterize the class of amarts [**15**].

Martingale theory has a double origin; it generalizes Paul Levy's approach to sums of independent random variables, and R. de Possel's theory of derivation of set-functions. There exist no applications of amarts to independent random variables; in particular independent r.v.'s and averages of independent positive r.v.'s are amarts only in the uninteresting case when the supremum is integrable [**19**]. Unfortunately, as shown below, the same is true for pramarts. But, for the first time also in case $J = N$, we are able to identify some amarts in the context of the derivation theory. Only preliminary results have been obtained; roughly speaking, derivatives of set-functions that are asymptotically additive are amarts. It seems that in some cases the essential convergence results following from amart theorems cannot be obtained from martingale theorems alone. Thus deriving super-additive set-functions in classical setting, we obtain supermartingales that are also amarts. The supermartingales theorem is not applicable because the ordered Vitali condition V' fails, but the Vitali condition V holds, and therefore the new stopping time results (Astbury's theorem or Theorem 4.2 below) give essential convergence.

Section 1 gives definitions of basic notions. Section 2 establishes optional sampling properties of amarts, pramarts, and subpramarts. Section 3 proves convergence results without Vitali conditions; in particular stochastic convergence of subpramarts in the general case; almost sure convergence of superpramarts in the case of $J = N$. Section 4 proves essential convergence of subpramarts under condition V. Section 5 proves that many different statements are equivalent with V; e.g. the assertion that stochastic convergence of X_τ implies essential convergence of X_t; also the assertion that every amart is a mil. Sections 6 and 7 give the analogous theory for ordered amarts, pramarts, and subpramarts, under the ordered Vitali condition V'. In Section 8 we introduce the controlled Vitali condition V^c and controlled amarts. Section 9 gives examples in the case $J = N$; in particular of subpramarts that are not pramarts, and pramarts that are not amarts. The connection with the classical derivation theory is established in Section 10. In Section 11 we construct further examples and counterexamples. Section 12 sketches various extensions: to the descending case (index set filtering to the left), to infinite measure spaces, and others. The Banach space case is deferred to another paper, but a generalization of Chatterji's martingale theorem to directed sets under the condition V is given, because it follows at once from arguments in earlier sections.

The authors wish to thank G. A. Edgar for his helpful comments.

1. Definitions and basic notions; general case. Let J be a set of indices (partially) ordered by $\leqq$. s, t and u denote elements of J. The set J is assumed *filtering to the right*, i.e., such that for each pair t_1, t_2 of elements in J, there exists an element t_3 in J such that $t_1 \leqq t_3$ and $t_2 \leqq t_3$. A subset K of J is called *terminal* if there exists an index s such that $s \leqq t$ implies $t \in K$. A subset K of J is called *cofinal* if $J - K$ is not terminal. Let $N = \{1, 2, \ldots\}$, $-N = \{\ldots, -2, -1\}$.

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Throughout this paper, functions, sets, and random variables are considered equal if they are equal almost surely. Let $\{X_t\}$ be a set of random variables each taking values in $\overline{\mathbf{R}}$; the *essential supremum* of $\{X_t\}$ is the unique almost surely smallest random variable $e \sup X_t$ such that for every t, $e \sup X_t \geqq X_t$ a.s. The *essential infimum* of $\{X_t\}$, $e \inf_t X_t$, is defined by $e \inf X_t = -e \sup (-X_t)$. Let $\{A_t\}$ be a family of measurable sets; the essential supremum of $\{A_t\}$ is the only set $e \sup A_t$ such that $1_{e \sup A_t} = e \sup 1_{A_t}$, and the essential infimum of $\{A_t\}$ is the only set $e \inf A_t$ such that

$$1_{e \inf A_t} = e \inf_t 1_{A_t}.$$

Let $(X_t)_{t \in J}$ be a family of random variables taking values in $\overline{\mathbf{R}}$; the *stochastic upper limit* of $(X_t)_{t \in J}$, $s \limsup X_t$, is the essential infimum of the set of random variables Y such that $\lim P(\{Y < X_t\}) = 0$. The *stochastic lower limit* of $(X_t)_{t \in J}$, $s \liminf X_t$, is defined by $s \liminf X_t = -s \limsup (-X_t)$. If $s \limsup X_t = s \liminf X_t = X_\infty$, then X_∞ is called the *stochastic limit* of $(X_t)_{t \in J}$, which is then said to *converge stochastically*, or to *converge in probability*, to X_∞. We write $X_\infty = s \lim X_t$. The *essential upper limit* $e \lim_t \sup X_t$ of $(X_t)_{t \in J}$ is defined by:

$$e \limsup X_t = e \inf_s (e \sup_{t \geqq s} X_t).$$

The *essential lower limit* $e \liminf X_t = -e \limsup (-X_t)$. The directed family $(X_t)_{t \in J}$ is said to *converge essentially* if the essential lower and upper limits coincide. Their common value is then called the *essential limit*, $e \lim_t X_t$, of the family $(X_t)_{t \in J}$. In a similar way, if $(A_t)_{t \in J}$ is a directed family of measurable sets, the *essential upper limit*, $e \limsup A_t$ is defined by:

$$e \limsup A_t = e \inf_s (e \sup_{t \geqq s} A_t).$$

A *stochastic basis* $(\mathscr{F}_t)_{t \in J}$, also denoted by $(\mathscr{F}_t)$, is an increasing family of sub σ-algebras of $\mathscr{F}$ (i.e., for every $s \leqq t$, $\mathscr{F}_s \subset \mathscr{F}_t$). Given a stochastic basis $(\mathscr{F}_t)$, we denote by $\mathscr{F}_\infty$ the σ-algebra generated by the algebra $\bigcup_t \mathscr{F}_t$. A *stochastic process* is a triple $(X_t, \mathscr{F}_t, J)$, also simply denoted by (X_t), where J is a directed set, $(\mathscr{F}_t)$ is a stochastic basis, and for each t, $X_t: \Omega \to \mathbf{R}$ and is $\mathscr{F}_t$ measurable. The process is called *integrable* (positive) if every X_t is integrable (positive). The process satisfies *Doob's condition* or is *L^1-bounded* if $\sup E(|X_t|) < \infty$. Given a stochastic basis $(\mathscr{F}_t)$, a family of sets (A_t) is *adapted* if for every t, $A_t \in \mathscr{F}_t$.

A stochastic basis $(\mathscr{F}_t)$ satisfies the *Vitali condition* V if for every adapted family of sets (A_t), for every set A in $\mathscr{F}_\infty$ such that $A \subset e \limsup A_t$, and

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

for every $\epsilon > 0$, there exist finitely many indices $t_1, \ldots, t_n$, and pairwise disjoint sets $B_i \in \mathscr{F}_{t_i}$, $B_i \subset A_{t_i}$, $i = 1, \ldots, n$, such that $P(A \setminus \bigcup_1^n B_i) \leqq \epsilon$.

A stochastic basis $(\mathscr{F}_t)$ satisfies the *Vitali condition* W if for every adapted family of sets $(A_t)_{t \in J}$, for every set A in $\mathscr{F}_\infty$ such that $A \subset e \limsup A_t$, there exists a sequence (t_n) of indices and a sequence (B_n) of pairwise disjoint sets such that $B_n \in \mathscr{F}_{t_n}$, $B_n \subset A_{t_n}$, and $P(A \setminus \bigcup_n B_n) = 0$. Given a stochastic basis $(\mathscr{F}_t)$, the conditions V and W are known to be equivalent. (This equivalence will be proved again in Theorem 5.1).

A *stopping time* (of the stochastic basis $(\mathscr{F}_t)$) is a function $\tau: \Omega \to J$, such that for every $t \in J$, $\{\tau = t\} \in \mathscr{F}_t$. τ is called *simple* if it takes finitely many values. Let $T = T(J)$ denote the set of simple stopping times; under the natural order T is a directed set filtering to the right. σ, τ and ρ denote elements of T. Let $(X_t, \mathscr{F}_t, J)$ be a stochastic process and τ be a simple stopping time; define the random variable X_τ by $X_\tau = X_t$ on $\{\tau = t\}$, and define the σ-algebra $\mathscr{F}_\tau$ by

$$\mathscr{F}_\tau = \{A \in \mathscr{F} \mid \forall t \in J, A \cap \{\tau = t\} \in \mathscr{F}_t\}.$$

X_τ is $\mathscr{F}_\tau$ measurable. If $X_t = 1_{A_t}$, let $A_\tau = \operatorname{supp} X_\tau$. For the order $(\sigma, \tau) \leqq (\sigma', \tau')$ if $\sigma \leqq \sigma'$, the set of ordered pairs of stopping times $\{(\sigma, \tau) | \sigma \leqq \tau\}$ is filtering to the right. Write E^σ for $E^{\mathscr{F}_\sigma}$, and for $\sigma \leqq \tau$ set

$$X(\sigma, \tau) = X_\sigma - E^\sigma X_\tau.$$

The notions of stochastic or essential lower limits, upper limits, and limits of $X(\sigma, \tau)$ are defined for the order mentioned above, and denoted by $s \limsup_{\sigma, \tau \in T} X(\sigma, \tau)$, $e \limsup_{\sigma, \tau \in T} X(\sigma, \tau), \ldots$.

An integrable process $(X_t, \mathscr{F}_t, J)$ is a *martingale* (*submartingale*, *supermartingale*) if $s \leqq t$ implies $E^s X_t = X_s$ ($E^s X_t \geqq X_s$, $E^s X_t \leqq X_s$).

Definition 1.1. An integrable stochastic process $(X_t, \mathscr{F}_t, J)$ is an *amart* if the net $(E(X_\tau))_{\tau \in T}$ converges to a finite limit.

Let us recall the L^1 difference property of amarts ([**1**], Lemma 2.1; in the case $J = N$, the result follows trivially from the Riesz decomposition [**12**]): A stochastic process $(X_t, \mathscr{F}_t, J)$ is an amart if and only if

$$\lim E(|X(\sigma, \tau)|) = 0.$$

A *potential* is an amart (X_t) such that $\lim E(1_A X_t) = 0$ for each $A \in \bigcup_t \mathscr{F}_t$.

Definition 1.2. An integrable stochastic process $(X_t, \mathscr{F}_t, J)$ is a *pramart* if

$$s \lim_{\sigma, \tau \in T} X(\sigma, \tau) = 0,$$

i.e., for every $\epsilon > 0$ there exists $\sigma_0 \in T$ such that $\sigma_0 \leqq \sigma \leqq \tau$ implies

$$P(\{|X_\sigma - E^\sigma X_\tau| > \epsilon\}) \leqq \epsilon.$$

Definition 1.3. An integrable stochastic process $(X_t, \mathscr{F}_t, J)$ is a *subpramart*

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

if $s \limsup_{\sigma,\tau\in T} X(\sigma,\tau) \leqq 0$, i.e., for every $\epsilon > 0$ there exists $\sigma_0 \in T$ such that $\sigma_0 \leqq \sigma \leqq \tau$ implies

$$P(\{X_\sigma - E^\sigma X_\tau > \epsilon\}) \leqq \epsilon.$$

(X_t) is a *superpramart* if $(-X_t)$ is a subpramart.

Definition 1.4. An integrable stochastic process $(X_t, \mathscr{F}_t, J)$ is a *mil* (*martingale in the limit*) if $e \limsup_{s,t\in J}|X(s,t)| = 0$, i.e.,

$$e \lim_s (e \sup_{t\geqq s}|X_s - E^s X_t|) = 0.$$

(In the case $J = N$ the notion of mil was introduced by A. Mucci; cf. [**25**].)

2. The optional sampling properties. In this section we consider optional sampling theorems for (general) simple stopping times. We use the following notation. If $\mathscr{C}$ is a class of stochastic processes $(X_t, \mathscr{F}_t, J)$, given $\tau \in T(J)$, we denote by $X_{J,\tau}$ the random variable taking value $X_{\tau(\omega)}(\omega)$ at the point ω. As in Section 1, we write X_τ instead of $X_{J,\tau}$ if no misunderstanding is possible. Analogously we define $\mathscr{F}_{J,\tau} = \mathscr{F}_\tau$. A class $\mathscr{C}$ of stochastic processes $\{(X_t, \mathscr{F}_t, J)\}$ has the *cofinal optional sampling property* if given an element $(X_t, \mathscr{F}_t, J)$ of $\mathscr{C}$, for every cofinal subset $\mathscr{T}$ in $T(J)$ the process $(X_\tau, \mathscr{F}_\tau, \mathscr{T})_{\tau\in\mathscr{T}}$ belongs to $\mathscr{C}$.

LEMMA 2.1. *Let $(X_t, \mathscr{F}_t, J)$ be a stochastic process, and let $\mathscr{T}$ be a cofinal subset of the set $T = T(J)$ of simple stopping times taking values in J. Denote by $T(\mathscr{T})$ the set of simple stopping times for $(\mathscr{F}_\tau)_{\tau\in\mathscr{T}}$ taking values in $\mathscr{T}$. Given $s \in J$, for every element $\theta \geqq s$ in $T(\mathscr{T})$ there exists an element $\sigma \geqq s$ in T such that $X_{\mathscr{T},\theta} = X_{J,\sigma}$, and $\mathscr{F}_{\mathscr{T},\theta} = \mathscr{F}_{J,\sigma}$.*

Proof. Let θ be in $T(\mathscr{T})$, $\theta \geqq s$, and define $\sigma: \Omega \to J$ by $\sigma(\omega) = [\theta(\omega)](\omega)$. Clearly $\sigma \geqq s$, and it is easy to check that $\sigma \in T$ and has the stated properties.

THEOREM 2.2. *The class of amarts, the class of pramarts, and the class of subpramarts have each the cofinal optional sampling property.*

Proof. More generally, let $\mathscr{C}$ be a class of stochastic processes defined by the following asymptotic property of $X(\sigma,\tau)$: There exits a function $f: L^1 \to \mathbf{R}$ such that $(X_t, \mathscr{F}_t, J) \in \mathscr{C}$ if and only if given $\epsilon > 0$, there exists $s \in J$ such that $s \leqq \sigma \leqq \tau$ implies $f[X(\sigma,\tau)] \leqq \epsilon$. Let $s \in J$, $s \leqq \theta \leqq \theta'$ be in $T(\mathscr{T})$, where $\mathscr{T}$ is a cofinal subset of $T(J)$. By Lemma 2.1, $s \leqq \sigma \leqq \sigma'$, and

$$X_{\mathscr{T},\theta} - E^{\mathscr{F}_{\mathscr{T},\theta}} X_{\mathscr{T},\theta'} = X_{J,\sigma} - E^{\mathscr{F}_{J,\sigma}} X_{J,\sigma'}.$$

Hence $\mathscr{C}$ has the cofinal sampling property. If $\mathscr{C}$ is the class of amarts, set $f(X) = E(|X|)$; if $\mathscr{C}$ is the class of pramarts, set

$$f(X) = \inf\{\alpha > 0 | P(|X| \geqq \alpha) \leqq \alpha\};$$

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

if $\mathscr{C}$ is the class of subpramarts, set

$$f(X) = \inf \{\alpha > 0 | P(X \geqq \alpha) \leqq \alpha\}.$$

LEMMA 2.3. *Let X be an integrable random variable and let $(\mathscr{F}_t)_{t\in J}$ be a stochastic basis. Then for every $\sigma \in T$,*

$$E^\sigma X(\omega) = E^{\sigma(\omega)} X(\omega).$$

Proof. For every $t \in J$, $1_{\{\sigma = t\}} E^t X$ is $\mathscr{F}_t$ measurable and for every $A \in \mathscr{F}_t$, $A \cap \{\sigma = t\} \in \mathscr{F}_\sigma$. Hence for every $A \in \mathscr{F}_t$,

$$E(1_A 1_{\{\sigma=t\}} E^t X) = E(1_A 1_{\{\sigma=t\}} E^\sigma X),$$

so that $E^t X = E^\sigma X$ on $\{\sigma = t\}$.

The case $J = N$. A class $\mathscr{C}$ of stochastic processes $(X_n, \mathscr{F}_n)$ is said to have the *monotone optional sampling property* if for every sequence $(X_n, \mathscr{F}_n)$ in $\mathscr{C}$ and for every increasing sequence (τ_n) of simple stopping times for $(\mathscr{F}_n)$, $(X_{\tau_n}, \mathscr{F}_{\tau_n})$ is in $\mathscr{C}$.

THEOREM 2.4. *The classes of amarts, pramarts and subpramarts indexed by N each have the monotone optional sampling property.*

Proof. This result has been proved for amarts in [**12**], Proposition 1.6. Since $(X_n, \mathscr{F}_n)$ is a pramart if and only if $(X_n, \mathscr{F}_n)$ and $(-X_n, \mathscr{F}_n)$ are subpramarts, we only need to show that the class of subpramarts has the monotone optional sampling property. Let $(X_n, \mathscr{F}_n)$ be a subpramart and let (τ_k) be an increasing sequence of simple stopping times for $(\mathscr{F}_n)$. Let $Y_k = X_{\tau k}$, $\mathscr{G}_k = \mathscr{F}_{\tau k}$ and $\tau_\infty = \lim_k \tau_k$. If σ and σ' are two simple stopping times for $(\mathscr{G}_k)$, then τ_σ and $\tau_{\sigma'}$ are two simple stopping times for $(\mathscr{F}_n)$, and $\mathscr{G}_\sigma = \mathscr{F}_{\tau_\sigma}$, so that

$$E^{\mathscr{G}_\sigma} Y_{\sigma'} = E^{\mathscr{F}_{\tau_\sigma}} X_{\tau_{\sigma'}}.$$

Given $\epsilon > 0$, choose $M \in N$ such that $M \leqq \tau \leqq \tau'$ implies $P(\{X(\tau, \tau') > \epsilon\}) \leqq \epsilon$. Then choose K such that

$$P(\{\tau_\infty > M\} \backslash \{\tau_K > M\}) \leqq \epsilon.$$

Let σ and σ' be two simple stopping times for $(\mathscr{G}_k)$, such that $K \leqq \sigma \leqq \sigma'$. Then

$$\begin{aligned} P(\{Y(\sigma, \sigma') > \epsilon\}) &\leqq \epsilon + P(\{X(\tau_\sigma, \tau_{\sigma'}) > \epsilon\} \cap \{\tau_\sigma > M\}) \\ &\quad + P(\{X(\tau_\sigma, \tau_{\sigma'}) > \epsilon\} \cap \{\tau_\infty \leqq M\}) \leqq \epsilon + P(\{X(M \vee \tau_\sigma, \\ &\quad M \vee \tau_{\sigma'}) > \epsilon\}) + \textstyle\sum_{n=1}^M P(\{X(\tau_\sigma, \tau_{\sigma'}) > \epsilon\} \cap \{\tau_\infty = n\}). \end{aligned}$$

For every $n \leqq M$ there exists $K_n \in N$ such that

$$P(\{\tau_\infty = n\} \Delta \textstyle\bigcap_{k \geqq K_n} \{\tau_k = n\})) \leqq \epsilon/M.$$

Let σ, σ' be such that $\sup(K, K_1, \ldots, K_M) \leqq \sigma \leqq \sigma'$. On the set $\bigcap_{k \geqq K_n} \{\tau_k = n\}$ in $\mathscr{F}_n$, we have $\tau_\sigma = \tau_{\sigma'} = n$. Therefore, by Lemma 2.3, on this set $X(\tau_\sigma, \tau_{\sigma'}) = 0$. Hence $P(\{Y(\sigma, \sigma') > \epsilon\}) \leqq 3\epsilon$.

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

3. Convergence without Vitali conditions. We at first prove a.s. convergence of positive superpramarts in the case $J = N$, by a direct argument in part similar to that given for positive amarts in [**2**]; the "upcrossing" method of the proof goes back to J. L. Doob's early papers. We then reestablish the Riesz decomposition of amarts ([**1**], [**12**]). Then stochastic convergence of subpramarts of class (d), i.e., such that $\liminf EX_t^+ + \liminf EX_t^- < \infty$, is derived.

LEMMA 3.1. *Positive superpramarts* $(X_n, \mathscr{F}_n, N)$ *converge a.s.*

Proof. Let $F = \{\liminf X_n = \infty\}$; suppose $P(F) > 0$. Given ϵ, $0 < \epsilon < P(F)/2$, choose n and M such that for every $k > 0$, $P(G_k) < \epsilon$, where

$$G_k = \{E^n X_{n+k} - X_n > \epsilon\} \cup \{X_n > M\}.$$

Given K, choose k such that $P(\{X_{n+k} > K\}) > P(F) - \epsilon$. Since

$$K 1_{G_k{}^c} E^n 1_{\{X_{n+k}\} \geqq K\}} \leqq 1_{G_k{}^c} E^n X_{n+k} \leqq M + \epsilon,$$

$K[P(F) - 2\epsilon] \leqq M + \epsilon$. Hence $P(F) = 0$, and if X_n does not converge a.s., there exist real numbers $\alpha < \beta$ such that $P(A) > 0$, where

$$A = \{\liminf X_n < \alpha < \beta < \limsup X_n\}.$$

Given $\epsilon > 0$ and an integer M, there exists a set B and an integer $M_1 \geqq M$ such that $B \in \mathscr{F}_{M_1}$, and $P(A \Delta B) \leqq \delta$, where $\delta = \epsilon^2/8\beta$. There exist integers M_2, M_3 such that $M_1 < M_2 < M_3$, and

$$P(A \backslash \{\inf_{M_1 \leqq n \leqq M_2} X_n < \alpha < \beta < \sup_{M_2 \leqq n \leqq M_3} X_n\}) \leqq \delta.$$

Define

$$C_1 = \{\inf_{M_1 \leqq n \leqq M_2} X_n < \alpha\} \cap B,\ C_2 = \{\sup_{M_2 \leqq n \leqq M_3} X_n > \beta\} \cap C_1.$$

Define two stopping times τ_1 and τ_2 by:

$$\tau_1(\omega) = \begin{cases} M_2, & \omega \notin C_1 \\ \inf\{n | M_1 \leqq n \leqq M_2, X_n(\omega) < \alpha\}, & \omega \in C_1. \end{cases}$$

$$\tau_2(\omega) = \begin{cases} M_2, & \omega \notin C_1 \\ M_3, & \omega \in C_1 \backslash C_2 \\ \inf\{n | M_2 \leqq n \leqq M_3, X_n(\omega) > \beta\}, & \omega \in C_2. \end{cases}$$

We have

$$X_{\tau_2} - X_{\tau_1} \geqq (\beta - \alpha) 1_{C_2} + 1_{C_1 \backslash C_2} X_{\tau_2} - \alpha 1_{C_1 \backslash C_2}.$$

Hence, neglecting the positive term $1_{C_1 \backslash C_2} X_{\tau_2}$, and applying $E^{\mathscr{F}_{\tau_1}}$, we obtain

$$E^{\tau_1} X_{\tau_2} - X_{\tau_1} \geqq (\beta - \alpha) E^{\tau_1}(1_{C_2}) - \alpha E^{\tau_1}(1_{C_1 \backslash C_2}) \geqq (\beta - \alpha) E^{\tau_1}(1_A) - (\beta - \alpha) E^{\tau_1}(1_{A \Delta C_2}) - \alpha E^{\tau_1}(1_{C_1 \backslash C_2}).$$

Since for every measurable set D, every $\eta > 0$, $P[E^{\tau_1}(1_D) > \eta] \leqq P(D)/\eta$,

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

choosing $D = A\Delta C_2$, or $D = C_1 \backslash C_2$, and appropriate η's we have

$$E^{\tau_1}X_{\tau_2} - X_{\tau_1} \geqq (\beta - \alpha)E^{\tau_1}(1_A) - \epsilon$$

outside of a set of measure $\leqq \epsilon$. Hence using the definition of a superpramart, we can choose M so big that if $\tau_2 \geqq \tau_1 \geqq M$, then $E^{\tau_1}X_{\tau_2} - X_{\tau_1} \leqq \epsilon$ outside of a set of measure $\leqq \epsilon$. Hence we can define an increasing sequence (σ_n) of stopping times such that for each n,

$$(\beta - \alpha)E^{\sigma_n}(1_A) \leqq 1/n$$

outside of a set of measure $1/n$. The sequence $E^{\sigma_n}(1_A)$ converges to 0 stochastically. On integrating we get $P(A) = 0$; this contradicts the assumption $P(A) > 0$.

LEMMA 3.2. *Let (X_t) be an integrable positive stochastic process. For every $t \in J$ set $R_t = e \inf_{\tau \geqq t} E^t X_\tau$. Then for every simple stopping time σ*

$$R_\sigma = e \inf_{\tau \geqq \sigma} E^\sigma X_\tau.$$

Proof. Fix σ and denote by $R(\sigma)$ the right hand side of the last equality. For every $\tau \in T$, if $\tau \geqq \sigma$ we have $\tau \geqq t$ on the set $\{\sigma = t\}$. Since τ takes finitely many values, there exists t' in J such that $\tau \leqq t'$. Define $\tau' = \tau$ on $\{\sigma = t\}$, and $\tau' = t'$ on $\{\sigma \neq t\}$. Then $\tau' \geqq t$ and $E^{\mathscr{F}_\sigma}X_\tau = E^{\mathscr{F}_t}X_{\tau'}$ on $\{\sigma = t\}$, which implies that $R_t \leqq R(\sigma)$ a.s. on $\{\sigma = t\}$. Conversely if t belongs to the range of σ, let τ in T be such that $\tau \geqq t$, and choose $s \in J$ such that $\sigma \leqq s$. Define $\tau' = \tau$ on $\{\sigma = t\}$, and $\tau' = s$ on $\{\sigma \neq t\}$. Then $\sigma \leqq \tau'$, and $E^\sigma X_{\tau'} = E^t X_\tau$ on $\{\sigma = t\}$. Hence $R(\sigma) \leqq R_t$ a.s. on $\{\sigma = t\}$.

PROPOSITION 3.3. *Let (X_t) be a positive integrable process. Then (X_t) is a subpramart if and only if there exists a positive submartingale $(R_\tau, \mathscr{F}_\tau, T)$ such that for every t, $R_t \leqq X_t$ a.s., and $s \lim(X_\tau - R_\tau) = 0$.*

Remark. We at first observe that to say that $(R_\tau, \mathscr{F}_\tau, T)$ is a submartingale is the same as saying that $(R_t, \mathscr{F}_t, J)$ is a submartingale with the optional sampling properties. In the case $J = N$, since (R_n) is a submartingale if and only if (R_τ) is, a positive integrable process is a subpramart if and only if there exists a positive submartingale (R_n) such that $R_n \leqq X_n$, and $s \lim(X_\tau - R_\tau) = 0$.

Proof of proposition. For every $t \in J$, set $R_t = e \inf_{\tau \geqq t} E^t X_\tau$. By Lemma 3.2, for every $\sigma \in T$ we have $R_\sigma = e \inf_{\tau \geqq \sigma} E^\sigma X_\tau$. We show that $(R_\sigma, \mathscr{F}_\sigma, T)$ is a submartingale (a similar argument appears in [**16**]). Since for every $\sigma \in T$, $0 \leqq R_\sigma \leqq X_\sigma$ a.s., R_σ is integrable. It follows from known properties of $e \inf$ that there exists a sequence (τ_n), $\tau_n \geqq \sigma$ and $\tau_n \in T$ for every n, such that $R_\sigma = \inf_n E^\sigma X_{\tau_n}$ (see e.g. [**26**], p. 121). We show that one can assume that the sequence $E^\sigma X_{\tau_n}$ decreases to R_σ. Suppose that $\tau_1, \ldots, \tau_n$ have been properly chosen, and replace τ_{n+1} by τ_{n+1}' defined by $\tau_{n+1}' = \tau_n$ on $A = \{E^\sigma X_{\tau_n} \leqq E^\sigma X_{\tau_{n+1}}\}$, and $\tau_{n+1}' = \tau_{n+1}$ on A^c. Since A is $\mathscr{F}_\sigma$ measurable and $\tau_n \geqq \sigma$,

$\tau_{n+1} \geqq \sigma$, it follows that $\tau_{n+1}' \in T$. Let $\sigma \leqq \sigma'$ be simple stopping times. Applying E^σ to the relation: $E^{\sigma'} X_{\tau_n} \searrow R_{\sigma'}$ yields

$$E^\sigma R_{\sigma'} = \lim_n E^\sigma X_{\tau_n} \geqq e \inf_{\tau \geqq \sigma} E^\sigma X_\tau .$$

Hence $(R_\sigma, \mathscr{F}_\sigma, T)$ is a submartingale. It is easy to see that if $(R_\sigma', \mathscr{F}_\sigma, T)$ is another submartingale such that $X_\sigma \geqq R_\sigma'$ for each σ, then $R_\sigma \geqq R_\sigma'$. Assume that $s \lim\sup (X_\sigma - R_\sigma) \neq 0$. There exists $\epsilon > 0$ and a sequence (σ_n) in T such that $P[\{X_{\sigma_n} - R_{\sigma_n} > \epsilon\}] > \epsilon$. Fix n; since R_{σ_n} is the limit in probability of a sequence $(E^{\sigma_n} X_{\tau_k})_k$, we can choose $\tau_n \geqq \sigma_n$ such that

$$P(\{X(\sigma_n, \tau_n) > \epsilon/2\}) > \epsilon/2.$$

This contradicts the assumption that (X_t) is a subpramart. Conversely, assume that there exists a submartingale $(R_\sigma', \mathscr{F}_\sigma, T)$ such that

$$R_t' \leqq X_t, \quad \text{and} \quad s \lim\sup (X_\sigma - R_\sigma') = 0.$$

Since $R_t' \leqq R_t \leqq X_t$, we have $s \lim\sup (X_\sigma - R_\sigma) = 0$. If (X_t) is not a subpramart, there exists $\epsilon > 0$ and two sequences (σ_n) and (τ_n) such that σ_n increases, $\sigma_n \leqq \tau_n$, and $P[\{X(\sigma_n, \tau_n) > \epsilon\}] > \epsilon$. Since $X(\sigma_n, \tau_n) \leqq X_{\sigma_n} - R_{\sigma_n}$, we get a contradiction.

We now give a characterization of amarts in terms of martingales and supermartingales having the optional sampling property. This result is a generalization of Theorem 1 [**16**] to directed sets, and a refinement of the Riesz decomposition of amarts [**12**], [**1**]. A positive supermartingale $(X_t, \mathscr{F}_t, J)$ is called a *Doob potential* if $(X_\tau, \mathscr{F}_\tau, T)$ is a supermartingale and $X_t \to 0$ in L^1.

PROPOSITION 3.4. *A stochastic process (X_t) is an amart if and only if X_t can be written as a sum $X_t = Y_t + Z_t$ where (Y_t) is a martingale, and there exist $s \in J$ and a Doob potential $(S_t)_{t \geqq s}$, such that for $\tau \geqq s$, $|Z_\tau| \leqq S_\tau$. Furthermore, Y_t is the essential limit and L^1 limit of the net $(E^t X_\tau)_{\tau \geqq t}$.*

Proof. By the difference property of amarts (see Section 1), given $\sigma \in T$, the net $(E^\sigma X_\tau)_{\tau \in T}$ is Cauchy in L^1. Denote by Y_σ its L^1-limit; (Y_t) is a martingale. Furthermore, given a fixed σ in T the stochastic process $E^\sigma X_t$ defined for $t \geqq \sigma$ is an amart for the constant stochastic basis $\mathscr{F}_\sigma$. Given $\epsilon > 0$, let

$$A_t = \{|E^\sigma X_t - Y_\sigma| > \epsilon\}$$

and assume that

$$P[e \lim\sup A_t] = a > 0.$$

We show that the net $(E^\sigma X_\tau)_\tau$ does not converge to Y_σ in probability. For each $s \in J$ there exists a sequence (t_n), $t_n \geqq s$, such that

$$e \sup_{t \geqq s} A_t = \bigcup A_{t_n}.$$

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

Choose M such that $P(\bigcup_{n \leqq M} A_{t_n}) \geqq a/2$. Let

$$B_1 = A_{t_1},\ B_2 = A_{t_2} \backslash A_{t_1},\ B_i = A_{t_i} \backslash \bigcup_{j<i} A_{t_j}.$$

Define a stopping time τ by $\tau = t_i$ on B_i, $i = 1, \ldots, M$, and $\tau = t$ on $(\bigcup_{i \leqq M} B_i)^c$, where $t \in J$, $t \geqq t_i$, $i = 1, 2, \ldots, M$. Then $\tau \geqq s$, and

$$P(\{|E^\sigma X_\tau - Y_\sigma| > \epsilon\}) \geqq a/2.$$

This is a contradiction. Now the difference property gives $\lim E(|Z_\sigma|) = 0$, where $Z_t = X_t - Y_t$. Choose s such that $\sup_{\sigma \geqq s} E(|Z_\sigma|) < \infty$, and given $\sigma \geqq s$ set

$$S_\sigma = e \sup_{\tau \geqq \sigma} E^\sigma(|Z_\tau|).$$

Since $S_\sigma = \lim \uparrow_{n, \tau_n \geqq \sigma} E^\sigma(|Z_{\tau_n}|)$, a proof similar to that of Proposition 3.3 shows that $(S_\sigma, \mathscr{F}_\sigma, T)$ is a supermartingale, and hence that (S_t) is a Doob potential.

PROPOSITION 3.5. *Let (X_t) be a stochastic process for a constant stochastic basis $\mathscr{F}_t = \mathscr{F}$. The following assertions are equivalent:*
(1) *$(X_t, \mathscr{F})$ is an amart.*
(2) *X_t converges essentially, and there exists $s \in J$ such that*

$$E(e \sup_{t \geqq s} |X_t|) < \infty.$$

Proof. In the case $J = N$, a direct proof of the implication (1) $\Rightarrow$ (2) has been given in [**12**] Proposition 2.4.

(1) $\Rightarrow$ (2). Define s, (Y_t), (Z_t), and (S_t) as in the previous proposition. For $t \geqq s$, the net (Y_t) is constant, and the net $(|Z_t|)$ is dominated by the net (S_t) which decreases to 0.

(2) $\Rightarrow$ (1). Since X_t converges essentially, X_τ converges essentially, hence stochastically. By the dominated convergence theorem EX_{τ_n} converges if $s \leqq \tau_1 \leqq \tau_2 \leqq \ldots$.

LEMMA 3.6. *Let (X_t) be a subpramart. Then (X_t^+) is a subpramart. More generally, for every constant λ, $(X_t \vee \lambda)_t$ is a subpramart.*

Proof. Given $\epsilon > 0$, choose $t \in J$ such that

$$t \leqq \sigma \leqq \tau \text{ implies } P(\{X(\sigma, \tau) > \epsilon\}) \leqq \epsilon.$$

Set $A = \{X_\sigma < 0\}$, and define $\tau' = \sigma$ on A, and $\tau' = \tau$ on A^c. We have

$$X_\sigma^+ - X_\tau^+ \leqq X_\sigma - X_{\tau'}.$$

Applying E^σ, we get $X_\sigma^+ - E^\sigma(X_\tau^+) \leqq X(\sigma, \tau')$. Hence $P[\{X_\sigma^+ - E^\sigma X_\tau^+ > \epsilon\}] \leqq \epsilon$, and it follows that (X_t^+) is a subpramart. Given λ, if (X_t) is a subpramart, so is $(\lambda + X_t)_t$; since $Y \vee \lambda = \lambda + (Y - \lambda)^+$, also $(X_t \vee \lambda)_t$ is a subpramart.

For the amart case of this lemma see [**2**] and [**11**].

A particular case of the following theorem was proved in [**12**], p. 206.

THEOREM 3.7. *Let* (X_t) *be a subpramart which satisfies Doob's condition, or only the properly weaker assumption* (d):

(d) $\liminf E(X_t^+) + \liminf E(X_t^-) < \infty$.

Then the net $(X_\tau)_{\tau \in T}$ *converges stochastically to an integrable random variable.*

Remark. In the case $J = N$, one obtains under the same assumptions the almost sure convergence; see Section 4, Theorem 4.3.

Proof. We at first deduce from our results the (well-known) a.s. convergence of positive L^1-bounded submartingales indexed by integers. Let (S_n) be a positive L^1-bounded submartingale. Clearly (S_n) is an amart; its Riesz decomposition is $S_n = Y_n + Z_n$, where (Y_n) is the martingale part given by

$$Y_n = \lim \uparrow_p E^n S_p,$$

and (Z_n) is the potential part. Clearly (Y_n), and $(-Z_n)$ both are positive L^1-bounded superpramarts. Hence the a.s. convergence of S_n follows from Lemma 3.1. Let now (X_t) be a positive subpramart such that $\liminf E(X_t) < \infty$. By Fatou's lemma, the approximating submartingale $(S_\tau, \mathscr{F}_\tau, T)$ considered in Proposition 3.3 is L^1-bounded. Given any increasing sequence (σ_n) of elements of T, the submartingale $(S_{\sigma_n}, \mathscr{F}_{\sigma_n}, N)$ converges in probability. Since the convergence in probability can be defined by the distance of a complete metric space (cf. [**26**], p. 97), the net $(S_\tau)_{\tau \in T}$ converges stochastically (see e.g. [**26**], p. 96). The relation $s\lim(X_\tau - S_\tau) = 0$ yields the theorem in the case $X_t \geqq 0$. The stochastic convergence of $(X_\tau)_{\tau \in T}$ follows under the assumption: there exists a constant λ such that for every t, $\lambda \leqq X_t$. Let now (X_t) be a subpramart for which the boundedness assumption (d) holds, and assume that X_t oscillates: there exist $a < b$ such that on a set of positive measure,

$$s \liminf X_\tau < a < b < s \limsup X_\tau.$$

By Lemma 3.6, $(X_t \vee a)$ is a subpramart; furthermore, $\liminf E(X_t \vee a) < \infty$, and $(X_\tau \vee a)$ does not converge stochastically. It follows that X_t cannot oscillate. By Fatou's lemma, $s \liminf X_\tau > -\infty$, hence the net $(X_\tau)_{\tau \in T}$ converges stochastically.

We finally show that the assumption (d) is properly weaker that Doob's condition in the case of pramarts (it follows from the Riesz decomposition that the two boundedness assumptions are equivalent for amarts). Let $(\Omega, \mathscr{F}, P)$ be the unit interval with Lebesgue measure and let $\mathscr{F}_n = \mathscr{F}\ \forall\, n \in N$. Let

$$X_n = (-1)^n n^2 1_{[0,1/n]}.$$

(X_n) is a pramart of class (d), i.e., satisfying the condition (d), but (X_n) does not satisfy Doob's assumption.

Remark. Let (X_t) be a process of class (d) and such that

$$s \limsup_{s \leqq \tau}(X_s - E^s X_\tau) \leqq 0.$$

The proof of Theorem 3.7 shows that X_t converges stochastically. However, (X_t) need not be a subpramart, because Example 9.6 below shows that (X_τ) need not converge stochastically, even if (X_t) is L^1-bounded.

4. Essential convergence under the Vitali condition V. In the present section we prove various convergence theorems under the Vitali condition V. The first result asserts that stochastic convergence implies essential convergence for fairly general functions f of σ and τ. Other convergence results follow as immediate corollaries. It is also shown that if V holds then pramarts are mils. In the next section the converse implication is proved.

THEOREM 4.1. *Let* $(\mathscr{F}_t)$ *be a stochastic basis which satisfies the Vitali condition* V, *and let* $f(\sigma, \tau)$ *be a family of* $\mathscr{F}_\sigma$ *measurable random variables defined for* $\sigma, \tau \in T$, $\sigma \leqq \tau$. *Assume that for every* $t \in J$,

$$1_{\{\sigma = t\}} f(\sigma, \tau) = 1_{\{\sigma = t\}} f(t, \tau),$$

and $f(\sigma, \tau)$ *converges stochastically to* f_∞; *then* $f(\sigma, \tau)$ *converges essentially to* f_∞.

Proof. We first prove that $f(t, \tau)$ converges essentially to f_∞. Assume the contrary; then there exists $\epsilon > 0$ such that

$$P(e \limsup_{t \leqq \tau}\{|f(t, \tau) - f_\infty| > \epsilon\}) \geqq \epsilon.$$

There exists an index t_0 in J and an $\mathscr{F}_{t_0}$ measurable random variable f such that

$$P(\{|f_\infty - f| > \epsilon/4\}) \leqq \epsilon/8.$$

For every $s \geqq t_0$, set $g_s = e \sup_{\tau \geqq s}|f(s, \tau) - f|$. Fix $s_0 \geqq t_0$ and define $A_s = \{g_s \geqq 3\epsilon/4\}$ if $s \geqq s_0$, $A_s = \emptyset$ otherwise, so that

$$P(e \lim_s \sup A_s) \geqq 7\epsilon/8.$$

By the Vitali condition V, there exist indices $s_1, \ldots, s_n$ greater than s_0, and pairwise disjoint sets $B_i \in \mathscr{F}_{s_i}$, $B_i \subset A_{s_i}$, $i = 1, \ldots, n$, such that if $B = \bigcup_{i=1}^n B_i$,

$$P(e \limsup A_s \backslash B) \leqq \epsilon/8.$$

Choose r bigger than s_i, $i = 1, \ldots, n$, and define $\sigma \in T$ by $\sigma = s_i$ on B_i, $i = 1, \ldots, n$, and $\sigma = r$ on B^c. Fix i, $1 \leqq i \leqq n$; there exists a sequence (τ_k) in T such that

$$e \sup_{\tau \geqq s_i}|f(s_i, \tau) - f| = \sup_k|f(s, \tau_k) - f|.$$

Choose K such that

$$P[B_i \cap (\bigcup_{k=1}^K \{|f(s_i, \tau_k) - f| \geqq 3\epsilon/4\})] \geqq \tfrac{1}{2}P[B_i].$$

On the set B_i, define $\tau = \tau_k$ where k is the first integer $\leqq K$ such that $|f(s_i, \tau_k) - f| \geqq 3\epsilon/4$ if such an integer exists, and $\tau = \tau_K$ elsewhere. On B^c, set $\tau = r$. τ is a $\mathscr{F}_\sigma$ measurable simple stopping time. Since

$$f(\sigma, \tau) - f = \sum_{s \in J} 1_{\{\sigma = s\}}(f(s, \tau) - f),$$

we have $|f(\sigma, \tau) - f| \geqq 3\epsilon/4$ on a set of probability $\geqq \frac{1}{2}P(B)$. Hence given s_0, there exist σ, τ such that $s_0 \leqq \sigma \leqq \tau$, and

$$P(\{|f(\sigma, \tau) - f_\infty| \geqq \epsilon/2\}) \geqq \epsilon/4,$$

which contradicts the assumption $s \lim_{\sigma, \tau \in T} f(\sigma, \tau) = f_\infty$. Since for every $\sigma \leqq \tau$ and for every s,

$$1_{\{\sigma = s\}} f(\sigma, \tau) = 1_{\{\sigma = s\}} f(s, \tau),$$

it follows easily from the definition of essential convergence that $f(\sigma, \tau)$ converges essentially to f_∞.

THEOREM 4.2. *Let $(\mathscr{F}_t)$ be a stochastic basis which satisfies the Vitali condition V, and let $(X_t, \mathscr{F}_t, J)$ be a stochastic process. Then $s \lim X_\tau = X_\infty$ implies $e \lim X_\tau = X_\infty$ (and therefore $e \lim X_t = X_\infty$).*

Proof. Apply Theorem 4.1 with $f(\sigma, \tau) = X_\sigma$.

THEOREM 4.3. *Let $(\mathscr{F}_t)$ be a stochastic basis which satisfies the Vitali condition V. Every subpramart satisfying Doob's condition converges essentially. More generally, the essential convergence holds if (X_t) is of class (d), i.e., such that*

$$\liminf E(X_t^+) + \liminf E(X_t^-) < \infty.$$

In particular, if $J = N$, then every subpramart (X_n) such that

$$\liminf E(X_n^+) + \liminf E(X_n^-) < \infty$$

converges almost surely.

Proof. Apply Theorems 3.7 and 4.2.

THEOREM 4.4. *Let $(\mathscr{F}_t)$ be a stochastic basis satisfying the Vitali condition V. For every pramart (X_t), $e \lim_{\sigma, \tau \in T} X(\sigma, \tau) = 0$, and hence every pramart is a mil (martingale in the limit).*

Proof. Apply Theorem 4.1 to $f(\sigma, \tau) = X(\sigma, \tau) = X_\sigma - E^\sigma X_\tau$, $\sigma \leqq \tau$.

5. Necessity of the Vitali condition V. The condition V, shown sufficient for various convergence statements in the previous section, is also necessary. For emphasis, we also repeat the direct assertions. Some new conditions are added. If $t_k \geqq s_k$ for all $k \in N$, we write $(t_k) \geqq (s_k)$, or $(s_k) \leqq (t_k)$.

THEOREM 5.1. *The following statements are equivalent:*

(1) *The stochastic basis $(\mathscr{F}_t)$ satisfies the Vitali condition V.*

(2) *The stochastic basis* $(\mathscr{F}_t)$ *satisfies the Vitali condition W.*

(3) *Every stochastic process* (X_t) *for which* $X_{\tau,\tau\in T}$ *converges stochastically, converges essentially.*

(4) *Every subpramart of class* (*d*), *i.e., such that* $\liminf EX_t^+ + \liminf EX_t^- < \infty$, *converges essentially.*

(5) *Every submartingale* $(X_t, \mathscr{F}_t, J)$ *of class* (*d*) *such that* $(X_{\mathscr{T}}, \mathscr{F}_{\mathscr{T}}, T)$ *is a submartingale, converges essentially.*

(6) *Every amart* (1_{A_t}) *such that* $\lim P(A_t) = 0$, *converges essentially* (*to* 0).

(7) *Every pramart is a mil* (*i.e., martingale in the limit*).

(8) *Every amart is a mil.*

(9) *Let* (X_t) *be an arbitrary stochastic process, and let* Y *be any* $\mathscr{F}_\infty$ *measurable random variable, such that for almost every* ω, *the number* $Y(\omega)$ *is a cluster point of the net* $(X_t(\omega))_{t\in J}$. *Given an arbitrary sequence* (s_k) *of elements of* J, *there exists an increasing sequence* (τ_k) *of elements of* T, *such that* $(\tau_k) \geqq (s_k)$, *and* X_{τ_k} *converges* a.s. *to* Y.

(10) *Identical to* (9) *except that* $Y = e \limsup X_t$.

Proof. Obviously (2) ⇒ (1), (4) ⇒ (6), (7) ⇒ (8) and (9) ⇒ (10). Since every L^1-bounded submartingale with the optional sampling property is an amart, (4) ⇒ (5). The implications (1) ⇒ (3), (3) ⇒ (4) and (1) ⇒ (7) are the statements of Theorem 4.2, Theorem 4.3 and Theorem 4.4. We prove below (1) ⇒ (9), (6) ⇒ (2), (10) ⇒ (6), (5) ⇒ (2) and (8) ⇒ (2).

Proof of (1) ⇒ (9). (In the case $J = N$, the assertion (9) is due to [**2**], and (10) to [**6**]). Fix k; choose $s \geqq s_k$ and Y' $\mathscr{F}_s$ measurable such that

$$P(\{|Y - Y'| \geqq 1/4k^2\}) \leqq 1/4k^2.$$

For $t \geqq s$ set $A_t = \{|X_t - Y'| \leqq 1/2k^2\}$; otherwise set $A_t = \emptyset$. Using the Vitali condition V, one chooses pairwise disjoint sets B_i, $i = 1, \ldots, n$, $B_i \in \mathscr{F}_{t_i}$, $B_i \subset A_{t_i}$, such that

$$P(\bigcup_{i\leqq n} B_i) \geqq 1 - 1/2k^2.$$

One then defines $\tau_k = t_i$ on B_i, and $\tau_k = t$ on $(\bigcup_{i\leqq n} B_i)^c$, where $t \geqq t_i$, $i = 1, \ldots, n$. Then $\tau_k \geqq s_k$ and

$$P(\{|X_{\tau_k} - Y| \geqq 1/k^2\}) \leqq 1/k^2.$$

Now X_{τ_k} converges a.s. to Y by the Borel-Cantelli lemma.

Proof of (10) ⇒ (6). Assume that (6) fails, and let (1_{A_t}) be an amart such that $\lim P(A_t) = 0$, but 1_{A_t} fails to converge essentially to 0. Set $X_t = 1_{A_t}$, and $Y = e \limsup 1_{A_t}$, and assume that (10) holds. Then given an arbitrary sequence (s_k) in J we can find a sequence of stopping times (τ_k) such that $(\tau_k) \geqq (s_k)$ and that X_{τ_k} converges a.s. to Y. Since (X_{τ_k}) is bounded by 1, $EX_{\tau_k} \to EY > 0$, which contradicts the amart assumption.

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

The implication (6) $\Rightarrow$ (2) is due to Astbury ([**1**], Theorem 3.1). For the sake of completeness we also give this proof.

Proof of (6) $\Rightarrow$ (2). Let $A \in \mathscr{F}_\infty$, and let (A_t) be an adapted family of sets such that $A \subset e \limsup A_t$. Given a sequence (α_k) of numbers such that $0 < \alpha_k < 1$ and $\prod_k (1 - \alpha_k) = 0$, define an amart (1_{C_t}) as follows. Set

$$\mathscr{D} = \{(t_i, B_i)_{i=1,\ldots,n} | n \in N, B_i \text{ pairwise disjoint}, \quad i = 1, \ldots, n, B_i \in \mathscr{F}_{t_i}, B_i \subset A_{t_i}\}.$$

Given $G \in \mathscr{D}$, write $\cup G$ for $\cup_{(t,B) \in G} B$. Define by induction two sequences (G_k) in $\mathscr{D}$ and (r_k) in $\mathbf{R}$ as follows:

$G_0 = \emptyset$ and

$r_0 = \sup_{G \in \mathscr{D}, G \supset G_0} P[\cup (G \backslash G_0)]$

G_k is any element of $\mathscr{D}$ such that $G_k \supset G_{k-1}$ and

$$P[\cup (G_k \backslash G_{k-1})] \geqq \alpha_{k-1} r_{k-1} \quad \text{and}$$

$r_k = \sup_{G \in \mathscr{D},\ G \supset G_k} P[\cup (G \backslash G_k)]$.

Then $G \supset G_k$, $G \in \mathscr{D}$ and

$$r_{k-1} \geqq P[\cup (G \backslash G_k)] + P(\cup (G_k \backslash G_{k-1})) \geqq P[\cup (G \backslash G_k)] + \alpha_{k-1} r_{k-1}.$$

Hence

$$r_k \leqq (1 - \alpha_{k-1}) r_{k-1} \leqq \prod_{0 \leqq j \leqq k-1} (1 - \alpha_j).$$

Set $\bar{G} = \cup_{k \geqq 0} G_k$, and denote $\cup_{(t,B) \in \bar{G}} B$ by $\cup \bar{G}$. Define

$$C_t = A_t \backslash \cup_{(u,B) \in \bar{G}, u \leqq t} B, \; X_t = 1_{C_t}.$$

We show that for the stochastic process $(X_t, \mathscr{F}_t, J)$, $\lim E(X_\tau) = 0$. Let $k \in N$, and choose $\bar{t} \in J$ such that for all $(t, B) \in G_k$, $t \leqq \bar{t}$. Let $\tau \in T$, $\tau \geqq \bar{t}$; τ takes values $t_1, \ldots, t_n$. Define

$$G = G_k \cup (t_i, \{\tau = t_i\} \cap C_{t_i})_{i=1,\ldots,n}.$$

Since $G_k \subset G \in \mathscr{D}$, and since $X_\tau = 1_{\cup (G \backslash G_k)}$, we have $E(X_\tau) \leqq r_k$. Furthermore, setting $A' = A \backslash \cup \bar{G}$,

$$A' \subset e \limsup (A_t \backslash \cup \bar{G}) \subset e \limsup C_t.$$

If the Vitali condition W fails for $(\mathscr{F}_t)$, there exists $A \in \mathscr{F}_\infty$ such that $P(A') > 0$; hence the amart $(1_{C_t}, \mathscr{F}_t, J)$ does not converge essentially.

Proof of (8) $\Rightarrow$ (2). We keep the notation of the proof that (6) $\Rightarrow$ (2), assume that W fails, and fix s in J. Given n, we choose M_n such that $\prod_{0 \leqq j \leqq 2^{M_n} - 1} (1 - \alpha_j) < P(A')/2^{n+1}$, and then choose t_n' in J such that $(s, B) \in G_{M_n}$ implies $s \leqq t_n'$. Let t be larger than s and t_n'. Since

$$P(\{E^s X_{t_n} > 1/2\}) \leqq P(A')/2^n,$$

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

$P(\liminf_n\{E^s X_{t_n} \leqq 1/2\}) = 1$, so that

$$e\sup_{t \geqq s} |E^s X_t - X_s| \geqq \tfrac{1}{2} 1_{C_s}.$$

Since $P(e \limsup_s C_s) > 0$, $(X_t, \mathscr{F}_t, J)$ is not a mil.

Proof of (5) $\Rightarrow$ (2). Assume that W fails, and let (X_t) be a potential, $0 \leqq X_t \leqq 1$, which does not converge essentially to 0. The supermartingale $S_t = e\sup_{\tau \geqq t} E^t X_\tau$, introduced in Proposition 3.7, satisfies Doob's condition and has the optional sampling property. Furthermore, S_t converges to 0 in L^1, but since $X_t \leqq S_t$, S_t does not converge essentially to 0.

6. Definitions and stochastic convergence for ordered stopping times. Given a stochastic basis $(\mathscr{F}_t)$, an *ordered stopping time* is a simple stopping time τ such that the elements $t_1, t_2, \ldots, t_n$ in the range of τ are (linearly) ordered, say $t_1 < t_2 < \ldots < t_n$. We denote by T' the set of ordered stopping times. We set $\sigma <|< \tau$ if σ and τ are in T', and either $\sigma = \tau$, or there exists an s in J such that $\sigma \leqq s \leqq \tau$. For the order $<|<$, T' is a directed set filtering to the right. An integrable real-valued stochastic process $(X_t, \mathscr{F}_t, J)$ is an *ordered amart* if the net $E(X_\tau)_{\tau \in T'}$ converges for the order $<|<$. (Equivalently, for the order $\leqq$.)

Proposition 6.1. *(X_t) is an ordered amart if and only if it has the following L^1 difference property:*

$$\lim_{\sigma, \tau \in T'} E(|X(\sigma, \tau)|) = 0.$$

Proof. The proof is obtained by a slight modification of the proof of Lemma 2.1 in [**1**]; for the discussion of the difference property see also Section 6 of [**19**]. If (X_t) has the above difference property, then for each $\epsilon > 0$ there is $s \in J$ such that $s \leqq \sigma \in T'$ implies

$$\sup_{A \in \mathscr{F}_s} |E(1_A X_s - 1_A E^s X_\sigma)| < \epsilon.$$

Setting $A = \Omega$, we see that the net $(E(X_\tau))_{\tau \in T'}$ is Cauchy and hence converges. Conversely, let $\epsilon > 0$; choose $s \in J$ such that $\sigma \geqq s$, $\tau \geqq s$, $\sigma, \tau \in T'$ implies

$$|E(X_\sigma) - E(X_\tau)| \leqq \epsilon.$$

Let $s \leqq \sigma <|< \tau$; for any $A \in \mathscr{F}_\sigma$ define $\rho = \sigma$ on A, and $\rho = \tau$ on A^c. As $\sigma <|< \tau$, $\rho \in T'$; furthermore

$$E(1_A(X(\sigma, \tau))) = E(X_\rho) - E(X_\tau).$$

Hence the left hand side converges to zero uniformly in $A \in \mathscr{F}_\sigma$ if (X_t) is an ordered amart.

Theorem 6.2. (Riesz decomposition). *Let (X_t) be an ordered amart. Then X_t can be uniquely written as $X_t = Y_t + Z_t$ where (Y_t) is a martingale, and $(Z_\tau)_{\tau \in T'}$ converges to 0 in L^1 norm.*

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

Proof. The proof is similar to the proof of Proposition 3.4.

A submartingale (supermartingale) (X_t) such that $\sup_t|E(X_t)| < \infty$ is an ordered amart. Indeed, if $\sigma, \tau \in T'$, $\sigma \leqq \tau$ (a fortiori if $\sigma <| < \tau$), then $E(X_\sigma) \leqq E(X_\tau)$ so that $E(X_\tau)_{\tau \in T'}$ is an increasing net.

In analogy to the pramart we now introduce an ordered pramart, generalizing the ordered amart. An integrable stochastic process is an *ordered pramart* if $s \lim_{\sigma,\tau \in T'} X(\sigma, \tau) = 0$, i.e., $\forall\, \epsilon$, $\exists\, s$, such that $s \leqq \sigma <| < \tau$, $\sigma, \tau \in T'$ implies

$$P(\{|X_\sigma - E^\sigma X_\tau| > \epsilon\}) \leqq \epsilon.$$

An integrable stochastic process is an *ordered subpramart* if

$$s \lim_{\sigma,\tau \in T'} X(\sigma, \tau) \leqq 0$$

(i.e., $\forall\, \epsilon > 0$, $\exists\, s$ such that $s \leqq \sigma <| < \tau$, $\sigma, \tau \in T'$ implies

$$P(\{X_\sigma - E^\sigma X\tau > \epsilon\}) \leqq \epsilon).$$

PROPOSITION 6.4. *Let (X_t) be an ordered subpramart and let $\lambda \in \mathbf{R}$ be fixed; then $Y_t = X_t \vee \lambda$ is an ordered subpramart. Let (X_t) be an ordered subpramart of class (d); then the net $(X_\tau)_{\tau \in T'}$ converges stochastically to an integrable random variable.*

Proof. Let (X_t) be an ordered subpramart, and let $\sigma, \tau \in T'$, $\sigma <| < \tau$. Let $\tau' = \sigma$ on $\{X_\sigma < 0\}$, $\tau' = \tau$ on $\{X_\sigma \geqq 0\}$; then τ' is in T'. Therefore the proof of Lemma 3.6 extends showing that (X_t^+), hence (Y_t) are ordered subpramarts. Let (X_t) be an ordered subpramart of class (d). Choose an increasing sequence (s_n) of indices such that $s_n \leqq \sigma <| < \tau$ implies

$$P(\{X(\sigma, \tau) > 1/n\}) \leqq 1/n.$$

Assume that $(X_\tau)_{\tau \in T'}$ does not converge stochastically. Let δ be a metric defining the convergence in probability; there exists $\epsilon > 0$, such that for every $s \in J$, there exists $\sigma, \tau \in T'$, $s \leqq \sigma <| < \tau$, such that $\delta(X_\sigma, X_\tau) > \epsilon$. Define $(Y_n, \mathcal{G}_n)$ as follows: Set $Y_1 = X_{s_1}$, $\mathcal{G}_1 = \mathcal{F}_{s_1}$, and choose σ_1 and τ_1 such that $\delta(X_{\sigma_1}, X_{\tau_1}) > \epsilon$. Set $Y_2 = X_{\sigma_1}$, $\mathcal{G}_2 = \mathcal{F}_{\sigma_1}$, $Y_3 = X_{\tau_1}$, $\mathcal{G}_3 = \mathcal{F}_{\tau_1}$. Then choose an index $t_2 > s_2$, $t_2 > \tau_1$, such that

$$E(X_{t_2}^+) \leqq 2 \liminf E(X_t^+).$$

Choose σ_2 and τ_2 such that $t_2 \leqq \sigma_2 <| < \tau_2$, and such that $\delta(X_{\sigma_2}, X_{\tau_2}) > \epsilon$. Set $Y_4 = X_{t_2}$, $\mathcal{G}_4 = \mathcal{F}_{t_2}$, $Y_5 = X_{\sigma_2}$. $\mathcal{G}_5 = \mathcal{F}_{\sigma_2}$, $Y_6 = X_{\tau_2}$, $\mathcal{G}_6 = \mathcal{F}_{\tau_2}$. Then choose an index t_3 such that

$$t_3 > s_3,\ t_3 > \tau_2, \text{ and } E(X_{t_3}^-) \leqq 2 \liminf E(X_t^-),$$

and so on. The proofs of Lemma 2.1 and Theorem 2.2 show that (Y_n) is an ordered subpramart of class (d) for $(\mathcal{G}_n)$. The remark at the end of Section 3 allows us to deduce that Y_n converges stochastically, which brings a contradiction.

7. Essential convergence under the ordered Vitali condition V'. In this section we prove that essential convergence of ordered amarts and ordered subpramarts is equivalent with Krickeberg's ordered Vitali condition V'. We also relate the essential convergence of X_t to the stochastic convergence of $(X_\tau)_{\tau \in T'}$.

A stochastic basis $(\mathscr{F}_t)$ is said to satisfy the *Vitali condition* V' if given $A \in \mathscr{F}_\infty$, and an adapted family (A_t) such that $A \subset e \lim\sup A_t$, for every $\epsilon > 0$ there exist indices $t_1 \leqq t_2 \leqq \ldots \leqq t_n$, and pairwise disjoint sets $B_i \in \mathscr{F}_{ti}$ $B_i \subset A_{t_i}$, $i = 1, \ldots, n$, such that

$$P(A \setminus \cup_{i \leqq n} B_i) \leqq \epsilon.$$

A stochastic basis $(\mathscr{F}_t)$ satisfies the *Vitali condition* W' if given $A \in \mathscr{F}_\infty$ and an adapted family (A_t) such that $A \subset e \lim\sup A_t$, there exists a linearly ordered sequence of indices $t_1 \leqq t_2 \leqq \ldots$, and a sequence of pairwise disjoint sets $B_n \in \mathscr{F}_{t_n}$, $B_n \subset A_{t_n}$, such that $A \subset \cup_n B_n$. We write $(s_k) \leqq (t_k)$ if $s_k \leqq t_k$ for all $k \in N$. We say that (t_k) *is frequently above* (s_k) if (t_k) admits a sequence (t_{n_k}) such that $(s_k) \leqq (t_{n_k})$.

PROPOSITION 7.1. *Let $\mathscr{C}$ be a class of stochastic processes $(X_t, \mathscr{F}_t, J)$ where J is an arbitrary directed set filtering to the right. Denote by $\mathscr{C}_N$ the class of elements of $\mathscr{C}$ for which $J = N$, i.e., of the form $(X_n, \mathscr{F}_n, N)$. Assume that every element of $\mathscr{C}_N$ satisfying Doob's condition converges* a.s. *Fix $(X_t, \mathscr{F}_t, J)$ in $\mathscr{C}$ satisfying Doob's condition and Vitali condition V'. Assume that there is a sequence (s_n) in J such that for every increasing sequence (t_n) which is frequently above (s_n), $(X_{t_n}, \mathscr{F}_{t_n}, N)$ is in $\mathscr{C}_N$. Then X_t converges essentially.*

Proof. Assume the contrary. Since (X_t) satisfies Doob's condition, it follows from Fatou's lemma that $e \lim\inf X_t < \infty$, $e \lim\sup X_t > -\infty$. Hence there exist two real numbers $a < b$ such that

$$A = \{e \lim\inf X_t < a < b < e \lim\sup X_t\}, P(A) = \epsilon > 0.$$

Since $(\mathscr{F}_t)$ satisfies the Vitali condition V', there exists a finite sequence $t_1^{(1)} \leqq \ldots \leqq t_{n_1}^{(1)}$, such that $s_1 \leqq t_1^{(1)}$, and

$$P(A \setminus \cup_{i \leqq n_1} \{X_{t_i^{(1)}} \leqq a\}) \leqq \epsilon/4.$$

We can choose a finite sequence $t_1^{(2)} \leqq \ldots \leqq t_{n_2}^{(2)}$, such that

$$t_1^{(2)} \geqq s_2,\ t_1^{(2)} \geqq t_{n_1}^{(1)}, \quad \text{and}$$

$$P(A \setminus \cup_{i \leqq n_2} \{X_{t_i^{(2)}} \geqq b\}) \leqq \epsilon/8.$$

We define by induction an increasing sequence (t_n) in J which is frequently

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

above (s_n), and such that

$$P(\liminf X_{t_n} \leqq a < b \leqq \limsup X_{t_n}) \geqq \epsilon/2.$$

Since $(X_{t_n}, \mathscr{F}_{t_n}, N)$ belongs to $\mathscr{C}_N$ and satisfies Doob's condition, this yields a contradiction.

THEOREM 7.2. *Let $(\mathscr{F}_t)$ be a stochastic basis satisfying the Vitali condition V'. L^1 bounded mils converge essentially.*

Proof. It suffices to check that the class of mils satisfies the assumptions stated in the previous proposition.

Denote by $\mathscr{C}$ the class of mils. By Mucci's theorem [**25**], every $(X_n, \mathscr{F}_n, N)$ in $\mathscr{C}_N$ satisfying Doob's condition converges almost surely. Fix $(X_t, \mathscr{F}_t, J)$ in $\mathscr{C}$, and for every s in J set

$$g_s = e\sup_{s \leqq s' \leqq t'} |X_{s'} - E^{s'} X_{t'}|.$$

Since there exists a sequence (s_k) in J such that

$$e\inf_s g_s = \lim \downarrow_k g_{s_k} = 0,$$

$(X_{t_n}, \mathscr{F}_{t_n}, N)$ is a mil for every (t_n) which is frequently above (s_n).

THEOREM 7.3. *Let $(\mathscr{F}_t)$ be a stochastic basis satisfying the Vitali condition V', and let (X_t) be a stochastic process such that the net $(X_\tau)_{\tau \in T'}$ converges stochastically to X_∞. Then $(X_\tau)_{\tau \in T'}$ converges essentially to X_∞.*

Proof. We first prove that $e\lim_t X_t = X_\infty$. Let $a > 0$, and set

$$A = e\limsup \{|X_t - X_\infty| > a\}.$$

Given ϵ, $0 < \epsilon < a/3$, there exists an $\mathscr{F}_s$ measurable random variable X such that

$$P(\{|X - X_\infty| > \epsilon\}) \leqq \epsilon.$$

Choose $s' \in J$, $s' \geqq s$, such that $\tau \geqq s'$, $\tau \in T'$ implies $P(\{|X_\tau - X_\infty| \geqq \epsilon\}) \leqq \epsilon$. For every $t \in J$, set $A_t = \{|X_t - X| > a - \epsilon\}$ if $t \geqq s'$, and $A_t = \emptyset$ elsewhere; $P(e\limsup A_t) \geqq P(A) - \epsilon$. By the Vitali condition V', there exist finitely many indices $t_1 \leqq \ldots \leqq t_n$, and finitely many pairwise disjoint sets $B_i \in \mathscr{F}_{t_i}$, $B_i \subset A_{t_i}$, $i = 1, \ldots, n$, such that

$$P(e\limsup A_t \setminus \cup_{i \leqq n} B_i) < \epsilon.$$

Define τ in T' by $\tau = t_i$ on B_i, $i = 1, \ldots, n$, and $\tau = t_{n+1}$ on $(\cup_{i \leqq n} B_i)^c$, where t_{n+1} is an index greater than $t_1, \ldots, t_n$. Since $\tau \geqq s'$, we have

$$\begin{aligned} P(A) - 2\epsilon &\leqq P(e\limsup A_t) - \epsilon \leqq P(\{|X_\tau - X| > a - \epsilon\}) \\ &\leqq P(\{|X_\tau - X_\infty| > a - 2\epsilon) + \epsilon \leqq P(\{|X_\tau - X_\infty| > \epsilon\}) + \epsilon \leqq 2\epsilon. \end{aligned}$$

Since this inequality holds for every ϵ, $0 < \epsilon < a/3$, $P(A) = 0$, and hence $e\lim X_t = X_\infty$. Therefore $(X_\tau)_{\tau \in T'}$ converges essentially to X_∞.

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

As in Section 5, we now prove that the condition V', shown sufficient for several convergence theorems, also is necessary. The following theorem is similar to Theorem 5.1.

THEOREM 7.4. *Let* $(\mathscr{F}_t)$ *be a stochastic basis. The following assertions are equivalent:*

(1) $(\mathscr{F}_t)$ *satisfies the Vitali condition* V'.

(2) $(\mathscr{F}_t)$ *satisfies the Vitali condition* W'.

(3) *Every stochastic process for which* $(X_\tau)_{\tau \in T'}$ *converges stochastically to* X_∞, *is such that* X_t *converges essentially to* X_∞.

(4) *Every ordered subpramart of class* (*d*) *converges essentially.*

(5) *Every ordered amart* (1_{A_t}) *such that* $\lim P(A_t) = 0$ *converges essentially to* 0.

(6) *Let* (X_t) *be an arbitrary stochastic process, and let* Y *be any* $\mathscr{F}_\infty$ *measurable random variable, such that for almost every* ω, *the number* $Y(\omega)$ *is a cluster point of the net* $(X_t(\omega))_{t \in J}$. *Given an arbitrary sequence* (s_k) *in* J, *there exists an increasing sequence* (τ_k) *in* T', *such that* $(\tau_k) \geqq (s_k)$, *and* X_{τ_k} *converges almost surely to* Y.

(7) *Identical to* (6) *except that* $Y = e \limsup X_t$.

Proof. The proof is similar to the proof of Theorem 5.1. In order to prove (5) $\Rightarrow$ (2), given an adapted family (A_t), and $A \in \mathscr{F}_\infty$, $A \subset e \limsup A_t$, one defines

$$\mathscr{D} = \{\{(t_i, B_i)\}_{i=1,\ldots,n} \mid n \geqq 1, t_1 \leqq t_2 \leqq \ldots \leqq t_n, B_i \text{ pairwise disjoint}, \\ B_i \in \mathscr{F}_{t_i}, B_i \subset A_{t_i}, i = 1, \ldots, n\}.$$

Now proceed as in the proof of (6) $\Rightarrow$ (2), Theorem 5.1.

8. Essential convergence under the controlled Vitali condition V^c. In the present section we introduce a new *controlled* Vitali condition V^c, properly weaker than V', and a new class of controlled amarts, including amarts.

A simple stopping time τ is called a *controlled* stopping time if there exists $\tau' \in T'$ such that $\tau' \leqq \tau$ and τ is $\mathscr{F}_{\tau'}$ measurable; we then say that τ' *controls* τ, and write τ' ct τ. Denote by T_c the set of controlled stopping times. If σ and τ are in T_c, write $\sigma <_c \tau$ if there exists τ' controlling τ, such that for each σ' controlling σ, $\sigma' \leqq \tau'$. It is easy to see that $(T_c, <_c)$ is a directed set filtering to the right. A stochastic process $(X_t, \mathscr{F}_t, J)$ is called a *controlled amart* if the net $(E(X_\tau))_{\tau \in T_c}$ converges. This of course means that there exists a number z such that given any $\epsilon > 0$, there exists a σ in T_c such that if $\sigma <_c \tau$, then $|E(X_\tau) - z| < \epsilon$; in fact it is easy to see that one may require $\sigma = s$ to be in J. A stochastic basis $(\mathscr{F}_t)$ satisfies the *controlled Vitali condition* V^c if given an adapted family of sets (A_t) and a set $A \in \mathscr{F}_\infty$ with $A \subset e \limsup A_t$, for every $\epsilon > 0$ there exists $\tau \in T_c$ and $B \subset A_\tau$, $B \in \mathscr{F}_{\tau'}$ where τ' ct τ, such that $P(A \backslash B) \leqq \epsilon$. (As usual $A_\tau = \bigcup [A_t \cap \{\tau = t\}]$.)

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

It is easy to see that V is weaker than V^c which is weaker than V' (cf. Section 12 below). In Section 11 we give examples showing that these three notions are different. Clearly every amart is a controlled amart, and every controlled amart is an ordered amart.

PROPOSITION 8.1. *Every supermartingale (submartingale) such that* $\sup |E(X_t)| < \infty$ *is a controlled amart.*

Proof. Let $X_t = Y_t + Z_t$ be the Riesz decomposition of the supermartingale (X_t) as an ordered amart (cf. Theorem 6.2). Since

$$Y_t = \lim \downarrow_{\tau \in T'} E^t X_\tau \leqq X_t,$$

the supermartingale and ordered potential (Z_t) is positive. Let $\tau \in T_c$, and τ' ct τ; τ takes on values $t_1, \ldots, t_n$. Since

$$E(Y_\tau) \leqq E(X_\tau) = E(Y_\tau) + \sum_{i \leqq n} E(1_{\{\tau = t_i\}} Z_{t_i}) \leqq E(Y_\tau) + E(Z_{\tau'}),$$

and $E(Z_{\tau'}) \downarrow 0$, the net $(E(X_\tau))_{\tau \in T_c}$ converges.

THEOREM 8.2. *Let* $(\mathscr{F}_t)$ *be a stochastic basis satisfying the controlled Vitali condition* V^c. *Then given a stochastic process* (X_t), *the stochastic convergence of the net* $(X_\tau)_{\tau \in T_c}$ *implies the essential convergence of* X_t.

Proof. Set $X_\infty = s \lim_{\tau \in T_c} X_\tau$. Let $d > 0$, and set $D = e \limsup \{|X_t - X_\infty| > d\}$. Given $\epsilon > 0$, there exists an $\mathscr{F}_s$ measurable random variable X such that

$$P(\{|X - X_\infty| > \epsilon\}) \leqq \epsilon.$$

Choose $s_1 \in J$, $s_1 \geqq s$, such that if $\tau \in T_c$ and $s_1 <_c \tau$, then

$$P(\{|X_\tau - X_\infty| \geqq \epsilon\}) \leqq \epsilon.$$

For every $t \in J$, set $A_t = \{|X_t - X| > d - \epsilon\}$ if $t \geqq s_1$, and $A_t = \emptyset$ elsewhere; then

$$P(e \limsup A_t) \geqq P(D) - \epsilon.$$

By the Vitali condition V^c, there is $\tau \in T_c$, $s_1 \leqq \tau'$ ct τ, and $B \in \mathscr{F}_{\tau'}$, $B \subset A_\tau$, such that

$$P(e \limsup A_t \backslash B) \leqq \epsilon.$$

Therefore, we have

$$P(D) - 2\epsilon \leqq P(e \limsup_t A_t) - \epsilon \leqq P(\{|X_\tau - X| > d - \epsilon\})$$
$$\leqq P(\{|X_\tau - X_\infty| > d - 2\epsilon\}) + \epsilon.$$

Since ϵ is arbitrarily small, it follows that $P(D) = 0$, and hence $e \lim X_t = X_\infty$.

THEOREM 8.3. *Let* $(\mathscr{F}_t)$ *satisfy the condition* V^c. *Every controlled amart of* *class* (d), *i.e., such that* $\liminf E(X_t^+) + \liminf E(X_t^-) < \infty$, *converges essentially.*

Proof. Given a controlled amart (X_t), let $X_t = Y_t + Z_t$ be its Riesz decomposition as an ordered amart. Assume (X_t) of class (d); then (Y_t) is of class (d). Since V^c implies V, Y_t converges essentially by Krickeberg's theorem (cf. Section 4 above). Let $a > 0$ and let $s \in J$; define $A_t = \{Z_t > a\}$ if $t \geqq s$, $A_t = \emptyset$ elsewhere, and set $A = e \limsup \{Z_t > a\}$. Given $\epsilon > 0$, V^c yields the existence of $\tau \in T_c$, $s \leqq \tau'$ ct τ, and of a set $B \in \mathscr{F}_{\tau'}$, $B \subset A_\tau$, such that $P(A \backslash B) \leqq \epsilon$. Since $1_B E^{\tau'} Z_\tau \geqq a\, 1_B$, we have

$$\|E^{\tau'} Z_\tau\|_1 \geqq a(P(A) \backslash \epsilon).$$

We now prove that given $\epsilon > 0$, there exists $s \in J$ such that if $s <_c \tau$, τ' ct τ, then

$$\|Z_{\tau'} - E^{\tau'} Z_\tau\|_1 \leqq \epsilon.$$

Let F be an arbitrary set belonging to $\mathscr{F}_{\tau'}$; set $\sigma = \tau'$ on F, and $\sigma = \tau$ on F^c. Then for any $t \in J$,

$$\{\sigma = t\} = (\{\tau' = t\} \cap F) \cup (\{\tau = t\} \cap F^c) \in \mathscr{F}_{\tau'},$$

so that $\sigma \in T_c$, τ' ct σ. Furthermore $E(1_F(Z_{\tau'} - Z_\tau)) = E(Z_\sigma - Z_\tau)$. Since (Z_t) is a controlled amart, we can choose s so big that

$$\|Z_{\tau'} - E^{\tau'} Z_\tau\|_1 = \sup_{F \in \mathscr{F}_{\tau'}} |E(1_F(Z_{\tau'} - Z_\tau))| \leqq \epsilon$$

if $s \leqq \tau'$ and τ' ct τ.

(Z_t) is an ordered potential; therefore we also can have $\|Z_{\tau'}\|_1 \leqq \epsilon$ if $s \leqq \tau' \in T'$. Hence for every $\epsilon > 0$,

$$a[P(A) - \epsilon] \leqq \|E^{\tau'} Z_\tau\|_1 \leqq 2\epsilon.$$

We deduce that $P(A) = 0$, and a similar argument shows that

$$P(e \limsup \{Z_t < -a\}) = 0.$$

Hence Z_t converges essentially to 0.

We observe that Theorem 8.2 cannot be used to derive Theorem 8.3 in analogy to the derivation of Theorem 4.3 from Theorem 4.2, because there are L^1 bounded controlled amarts such that $(X_\tau)_{\tau \in T_c}$ does not converge stochastically (see Example 11.5 below).

Finally, we show below in Example 11.4 that the condition V^c is not necessary for convergence of L^1-bounded submartingales.

9. Examples and properties in the case $J = N$. In this case, every increasing stochastic basis $(\mathscr{F}_n)$ satisfies the Vitali conditions V and V'. By Theorem 4.4, every pramart is a mil. In this section, we discuss for $J = N$ the validity of the converse inclusion, further simple properties of pramarts,

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

and elementary convergence theorems obtained for weaker notions. A stochastic process $(X_n, \mathscr{F}_n)$ is a *semiamart* if $(E(X_\tau))_{\tau \in T}$ is bounded.

LEMMA 9.1. *Let (X_n) be a sequence of integrable random variables X_n with disjoint supports A_n; set $A = \bigcup A_n$, and define $\mathscr{F}_n = \sigma(X_1, \ldots, X_n, A)$. Then (X_n) is a pramart. The following assertions are equivalent:*

(1) (X_n) is an amart.

(2) (X_n) is a semiamart.

(3) $\sum E(|X_n|) < \infty$.

Proof. Given ϵ choose M such that $P(\bigcup_{n>M} A_n) < \epsilon$; let $M \leqq \sigma \leqq \tau$. Since $\bigcup_{n \leqq M} A_n \in \mathscr{F}_\sigma$, $E^\sigma X_\tau - X_\sigma = 0$ on the set $(\bigcup_{n \leqq M} A_n) \cup A$, and (X_n) is a pramart. (1) implies (2): see [**12**]. Assume that (2) holds, and for each $n \geqq 1$ set

$$\sigma_n = \inf \{k | 1 \leqq k \leqq n, X_k > 0\} \wedge (n+1), \quad \text{and}$$

$$\tau_n = \inf \{k | 1 \leqq k \leqq n, X_k < 0\} \wedge (n+1).$$

Since A_n are pairwise disjoint,

$$E(X_{\sigma_n}) - E(X_{\tau_n}) = \textstyle\sum_{k=1}^n E(|X_k|),$$

and hence $\sum E(|X_n|) < \infty$. Assume that (3) holds; given $\epsilon > 0$ choose M such that $\sum_{n \geqq M} E(|X_n|) < \epsilon$, and let $\tau \in T$, $\tau \geqq M$. Since $E(|X_\tau|) < \epsilon$, (X_n) is an amart.

Example 9.2. The maximal inequality fails for pramarts. Let (A_n) be a measurable partition of Ω with $P(A_n) = 1/n - 1/(n+1)$. $X_n = n \log n \, 1_{A_n}$, and $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. (X_n) is a uniformly integrable pramart, but $\sup_\lambda \lambda P[\{\sup |X_n| > \lambda\}] = +\infty$.

Example 9.3. There exists a pramart of class (B) (i.e., satisfying $\sup_\tau E(|X_\tau|) < \infty$) which is not an amart.

Let us follow the notations of Proposition 1.13 in [**19**]. Let (α_n) be such that $0 < \alpha_n \leqq 1$, $\prod_1^\infty \alpha_n = 0$, (ρ_n) such that $0 < \rho_n \leqq 1$, $\prod_1^\infty \rho_n > 0$, and (p_n), $p_n \in N$, $p_n \neq 0$, be three sequences of numbers. Given $n \geqq 1$ set

$$h_n = \textstyle\prod_{i=1}^n \rho_i \alpha_i^{-1}, \; l_1 = 1, \quad \text{and} \quad l_{n+1} = l_n + 1 + \prod_{i=1}^n p_i.$$

Define by induction $\prod_{i=1}^n p_i$ disjoint intervals $A_n(i_1, \ldots, i_n)$ $(1 \leqq i_k \leqq p_k$, $k = 1, \ldots, n)$ of equal length $\prod_{i=1}^n \alpha_i p_i^{-1}$, which are subsets of

$A_{n-1}(i_1, \ldots, i_{n-1})$. Set

$$X_{l_n} = 0, \quad \text{and} \quad X_{l_n+\nu} = h_n 1_{A_n(i_1, \ldots, i_n)}, \; 1 \leqq \nu \leqq \textstyle\prod_{i=1}^m p_i,$$

where $(i_1, \ldots, i_n)$ is the ν-th element in the lexicographic order of the set of n-tuples $\{1, \ldots, p_1\} \times \ldots \times \{1, \ldots, p_n\}$, and set $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. By Proposition 1.13 in [**19**], $(X_n, \mathscr{F}_n)$ is of class (B), and is not an amart. Given two stopping times $\tau \geqq \sigma \geqq l_n$, since the support of X_σ is included in the union

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

B_{n-1} of $A_{n-1}(i_1, \ldots, i_{n-1})$, for $1 \leqq i_1 \leqq p_1, \ldots, 1 \leqq i_{n-1} \leqq p_{n-1}$, and since $B_{n-1} \in \mathscr{F}_{t_n} \subset \mathscr{F}_\sigma$, $X(\sigma, \tau) = 0$ on $B_{n-1}{}^c$. Since

$$P(B_{n-1}) = \prod_{i=1}^{n-1} \alpha_i,$$

$(X_n, \mathscr{F}_n)$ is a pramart.

Example 9.4. Let (X_n) be a sequence of independent random variables such that $\liminf E(X_n^-) < \infty$ (or $\liminf E(X_n^+) < \infty$), and set

$$\mathscr{F}_n = \sigma(X_1, \ldots, X_n).$$

Then $(X_n, \mathscr{F}_n)$ is an amart if and only if $(X_n, \mathscr{F}_n)$ is a pramart.

Assume $\liminf E(X_n^-) < \infty$, and suppose that $(X_n, \mathscr{F}_n)$ is not a semiamart. If $\liminf E(X_n^+) = +\infty$, we have $E[\sup X_n^+] = +\infty$. It is easy to see that if (X_n) is not a semiamart, then at least one of the two random variables $\sup X_n$, and $\inf X_n$ is not integrable. Hence clearly we may assume $E(\sup X_n^+) = +\infty$, and $\liminf E(X_n^-) < \infty$, so that for every K,

$$E[\sup_{n \geqq K} X_n^+] = +\infty, \text{ and } \liminf_{n \geqq K} E(X_n^-) < \infty.$$

Fix K, and denote by $\bar{T}_K$ the set of (not necessary bounded) stopping times τ which depend only on $(X_K, X_{K+1}, \ldots)$, i.e., $\{\tau = k\} = \emptyset$ if $k < K$, and $\{\tau = k\} \in \sigma(X_K, \ldots, X_k)$ if $k \geqq K$. Theorem 3.1 in [**19**] implies that $\sup_{\tau \in \bar{T}_K} E(X_\tau) = +\infty$. Since $\liminf_{n \geqq K} E(X_n^-) < \infty$, there exists a simple stopping time τ_K in $\bar{T}_K$ such that $E(X_{\tau_K}) > K + 1$, and hence for which $E^{K-1} X_{\tau_K} > K + 1$. Given $\epsilon > 0$, there exists K_0 such that $K_0 \leqq K$ implies $P[\{X_K \leqq K\}] < \epsilon$, and therefore X_n converges stochastically to $+\infty$. We now show that under the assumption $\liminf E(X_n^-) < \infty$, a pramart does not converge in probability to $+\infty$. Assume the contrary; given $0 < \epsilon < 1/2$ choose η such that $\liminf E(X_n^-) < \epsilon\eta/8$, and K such that $K \leqq n \leqq p$ implies

$$P(\{|X(n, p)| > \epsilon\}) < \epsilon.$$

Fix $n > K$ and choose M such that $P(\{|X_n| \leqq M\}) \geqq 1 - \epsilon/4$, and $4(\epsilon + \eta + M) < \epsilon M^2$. Finally choose $p \geqq n$ such that $P(\{X_p \geqq M^2\}) \geqq 1 - \epsilon/4$, and $E(X_p^-) \leqq \epsilon\eta/4$. Then

$$\begin{aligned} P(\{|X(n, p)| \leqq \epsilon\}) &\leqq \epsilon/4 + P(\{|X(n, p)| \leqq \epsilon\} \cap \{|X_n| \leqq M\}) \\ &\leqq \epsilon/2 + P(\{|E^{\mathscr{F}_n} X_p| \leqq \epsilon + M\} \cap \{X_p \geqq M^2\}) \\ &\leqq 3\epsilon/4 + P(\{E^{\mathscr{F}_n} X_p^+ \leqq \epsilon + \eta + M\} \cap \{X_p \geqq M^2\}) \\ &\leqq 3\epsilon/4 + M^{-2} E(X_p^+ 1_{\{E^{\mathscr{F}_n} X_p^+ \leqq \epsilon + \eta + M\}}) \\ &\leqq \epsilon. \end{aligned}$$

Since $\epsilon < 1/2$, this contradicts the assumption $P(\{|X(n, p)| > \epsilon\}) \leqq \epsilon$. Hence under the assumption $\liminf E(X_n^+) < \infty$, or $\liminf E(X_n^-) < \infty$, a pramart $(X_n, \mathscr{F}_n)$ is a semiamart, and therefore by Theorem 3.3 in [**19**], $\sup X_n$ and $\inf X_n$ are integrable, so that $\sup |X_n|$ is integrable. By Theorem 4.3, X_n converges a.s., and by Theorem 3.3 in [**19**], $(X_n, \mathscr{F}_n)$ is an amart.

Example 9.5. Let (Y_n) be a sequence of positive integrable random variables adapted to a stochastic basis $(\mathscr{F}_n)$, and let $c_n \uparrow, c_1 > 0$. Assume that for every n, Y_{n+1} is independent of $\mathscr{F}_n$, and define

$$X_n = c_n^{-1} \sum_{i=1}^n Y_i.$$

Then (X_n) is an amart if and only if (X_n) is a pramart.

Proof. Suppose that (X_n) is a pramart and is not a semiamart. Since $\sup X_n = \sup_T X_\tau$, $\sup X_n$ is not integrable. Fix M, and for every $k \in N$, set

$$Z_k = c_{M+k}^{-1} \sum_{i=1}^k Y_{M+i} \text{ and } \mathscr{G}_k = \sigma(Y_{M+1}, \dots, Y_{M+k}).$$

Since $E[\sup Z_n] = +\infty$, applying Theorem 4.1 [**19**] we define for each M a stopping time σ_M for the stochastic basis $(\mathscr{G}_n)$, such that $E[Z_{\sigma_M}] \geqq M$. Set $\tau_M = M + \sigma_M$; since $X_{\tau_M} \geqq Z_{\sigma_M}$, we have

$$E^{\mathscr{F}_M}(X_{\tau_M}) \geqq E^{\mathscr{F}_M}(Z_{\sigma_M}) = E(Z_{\sigma_M}).$$

Now using the pramart property of (X_n) we deduce that X_n converges in probability to $+\infty$. This brings a contradiction (cf. the Example 9.4 above). (X_n) is hence a semiamart, and Theorem 4.1 [**19**] implies that $\sup X_n$ is integrable, so that (X_n) is an amart.

Example 9.6. We have shown that for every subpramart $(X_n, \mathscr{F}_n)$ satisfying Doob's condition, the net $(X_\tau)_{\tau \in T}$ converges almost surely. The proof of Theorem 3.7 shows that given a stochastic process $(X_n, \mathscr{F}_n)$ satisfying Doob's condition, such that given any $\epsilon > 0$, there exists M such that $M \leqq n \leqq \tau$ implies $P(\{X(n, \tau) > \epsilon\}) < \epsilon$, one has that X_n converges stochastically. However, the following example shows that X_n need not converge a.s. Let (A_n) be an independent sequence of sets, such that $P(A_n) = 1/n$, and set $X_n = 1_{A_n}$, $\mathscr{F}_n = \sigma(X_1, \dots, X_n)$. Given $\epsilon > 0$ and $n \leqq \tau$ and $P(\{X(n, \tau) > \epsilon\}) \leqq P(A_n)$; hence $(X_n, \mathscr{F}_n)$ has the property mentioned above. By the Borel-Cantelli lemma, $\liminf X_n = 0$ a.s., and $\limsup X_n = 1$ a.s.

Example 9.7. There exists a martingale in the limit which is not a subpramart.

Theorem 2 in [**15**] shows that the class of mils indexed by N does not have the optional sampling property, and hence that the class of mils is different from the class of pramarts. It is easy to check directly that the example (iii) Theorem 2 in [**15**] is a mil which is not a subpramart: (A_n) are independent events, $P(A_n) = 1/n^2$, $X_n = -n 1_{A_n}$, $\mathscr{F}_n = \sigma(A_1, \dots, A_n)$.

Example 9.8. There is an L^1-bounded subpramart that is not a mil. Let A_n be independent events with $P(A_n) = 1/n^2$. Let $\mathscr{F}_n$ be the σ-algebra generated by $A_1, \dots, A_n$; $n \in N$. Let $X_n = n^2 1_{A_n}$. If $n \leqq \sigma \leqq \tau$ then

$$P(X_\sigma - E^\sigma X_\tau > \epsilon) \leqq P(X_\sigma > 0) \leqq \sum_{k \geqq n} P(A_k) \to 0 \ (n \to \infty).$$

Thus (X_n) is a subpramart. Now

$$|X_n - E^{\mathscr{F}_n}X_{n+1}| \rightarrow 1 \text{ a.s.},$$

so that (X_n) is not a mil.

We note that Proposition 1.5 in [**12**] asserts that given an amart $(X_n, \mathscr{F}_n)$, and an increasing stochastic basis $(\mathscr{G}_n)$, $\mathscr{G}_n \subset \mathscr{F}_n$, $(E^{\mathscr{G}_n}X_n, \mathscr{G}_n)$ is an amart. This property fails for pramarts and subpramarts. The sequence (X_n) given in Example 9.7 is such that $(X_n, \mathscr{F})$ is a pramart because $X_n \rightarrow$ a.s., and $(X_n, \mathscr{F}_n)$ is not a subpramart (and hence not a pramart).

Example 9.9. To give a common generalization of subpramart and mil, one can attempt to define a *submil* by the following property:

$$\limsup_n [\sup_{k \geqq n} X(n, k)] \leqq 0 \text{ a.s.}$$

However the following example shows that submils are without interest since they do not have convergence theorems. Let A_n be independent events with $P(A_n) = 1/n^2$; define $X_{2n+1} = 1$, $X_{2n} = n^2\, 1_{A_n}$ for each n, and set $\mathscr{F}_n = \sigma(X_1, \ldots, X_n)$. Then

$$\limsup_n [\sup_{k \geqq n} (X_n - E^{\mathscr{F}_n}X_k)] = \limsup (X_n - 1) = 0.$$

However $P(X_{2n} = 0) \rightarrow 1$, $\limsup X_n = 1$ a.s., and $\liminf X_n = 0$ a.s.

10. Derivation in Euclidean space and Vitali condition V. In the present section, r denotes a positive integer and P denotes Lebesgue measure on $[0, 1]^r$. The following proposition shows that the Vitali condition V is satisfied in the classical setting of differentiation in r-dimensional Euclidean space. A standard argument shows that the Vitali condition V could also be stated as follows (similar lemmas can be proved for V' and V^c).

LEMMA 10.1. *Let $(\mathscr{F}_t)$ be a stochastic basis. Assume that there exists a constant α, $0 < \alpha < 1$, such that for each adapted family of sets (A_t) with $A = e \limsup A_t$, there exist indices $t_1, \ldots, t_n$, and pairwise disjoint sets B_i, $B_i \in \mathscr{F}_{t_i}$, $B_i \subset A_{t_i}$, $i = 1, \ldots, n$, such that*

$$P[A \cap (\cup_{i \leqq n} B_i)] \geqq \alpha P(A).$$

Then $(\mathscr{F}_t)$ satisfies the Vitali condition V.

Proof. Let (A_t) be an adapted family of sets, let $A = e \limsup A_t$. Let $D_1 = \cup_{i \leqq n_1} B_i$ with pairwise disjoint B_i, $B_i \in \mathscr{F}_{t_i}$, $B_i \subset A_{t_i}$ for $i = 1, \ldots, n_1$, and $P(A \cap D_1) \geqq \alpha P(A)$. Let s_2 be greater than $t_1, \ldots, t_{n_1}$, and set $A_t' = A_t \backslash D_1$ if $t \geqq s_2$, $A_t = \emptyset$ otherwise. Since $A \backslash D_1 = e \limsup_t A_t'$, there exist finitely many disjoint sets B_i, $i = n_1 + 1, \ldots, n_2$, such that

$$B_i \in \mathscr{F}_{t_i}, B_i \subset A_{t_i}', \text{ and } P((A \backslash D_1) \cap D_2) \geqq \alpha P(A \backslash D_1), \text{ where}$$
$$D_2 = \cup_{n_1+1 \leqq i \leqq n_2} B_i.$$

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

One defines by induction a sequence of pairwise disjoint sets B_i, $B_i \in \mathscr{F}_{t_i}$, $B_i \subset A_{t_i}$, such that at the end of the kth step,

$$P(A \setminus \bigcup_{i \leqq n_k} B_i) \leqq (1 - \alpha)^k P(A).$$

Given a countable partition t of $[0, 1]^r$, by the diameter $d(t)$ of t, we mean the supremum of the diameters of the elements (atoms) of t.

PROPOSITION 10.2. *Let $\mathscr{C}$ be a collection of open subsets C of $[0, 1]^r$. Assume that $\mathscr{C}$ is a family of substantial sets, i.e., there exists a constant M, such that every C in $\mathscr{C}$ is contained in an open ball B with $P(B) < MP(C)$. Let J be a family of countable partitions (modulo sets of measure 0) of $[0, 1]^r$ into elements of $\mathscr{C}$, such that for every $\epsilon > 0$ there exists t in J with $d(t) \leqq \epsilon$. J is ordered by refinement, i.e., if s, t are in J, $s \leqq t$, then every atom of s is a union of atoms of t. J is assumed filtering to the right. Then the stochastic basis $(\mathscr{F}_t)$ of σ-algebras generated by the partitions t satisfies the Vitali condition V.*

Remark. A simple example of such families J is a family, ordered by refinement, of countable partitions of $[0, 1]^r$ into parallelepipeds such that the ratio between the longest and shortest edges is bounded, and also $\lim d(t) = 0$ (cf. [**23**], p. 538). In the case $r = 1$, J is a family of countable partitions of $[0, 1]$ into intervals such that the length of the greatest interval in t converges to 0. Then Proposition 10.2 follows from a classical lemma due to Vitali (see e.g. [**27**], p. 95).

Proof of the proposition. Let (A_t) be an adapted family of sets, and set $A = e \limsup_t A_t$. Assume that $P(A) \neq 0$. Given $\epsilon > 0$ choose s in J such that

$$P[e \sup_{t \geqq s} A_t \setminus A] \leqq \epsilon P(A),$$

and an increasing sequence (s_k) in J such that $s \leqq s_1$, and $d(s_k) \leqq 1/k$ for each k. For every k, there exists a sequence $(t_{k,n})_n$ in J such that $t_{k,n} \geqq s_k$, hence $d(t_{k,n}) \leqq 1/k$, and $A \subset \bigcup_n A_{t_{k,n}}$. Decompose each $A_{t_{k,n}}$ into its atoms of the partition $t_{k,n}$, and denote by $\mathscr{C}$ the family of all such atoms, each of which is included in $e \sup_{t \geqq s} A_t$. Let $\mathscr{C}'$ be a finite subfamily of $\mathscr{C}$ whose union A' satisfies $P(A') \geqq (1 - \epsilon)P(A)$. We can choose a finite disjoint subfamily of $\mathscr{C}'$ whose union $D \subset e \sup_{t \geqq s} A_t$ has the property $P(A') \leqq M3^r P(D)$ (see e.g. [**28**], p. 154). Hence

$$P(D \cap A) = P(D) - P(D \cap A^c) \geqq M^{-1}3^{-r}P(A') - P(e \sup_{t \geqq s} A_t \setminus A)$$
$$\geqq P(A)[M^{-1}3^{-r}(1 - \epsilon) - \epsilon].$$

The proposition now follows from Lemma 10.1, because ϵ can be chosen so small that $\alpha = M^{-1}3^{-r}(1 - \epsilon) - \epsilon > 0$.

We now produce a general method of constructing amarts, related to the classical setting of derivation theory. The following may be considered as a derivation theorem for not necessarily additive set-functions.

PROPOSITION 10.3. *Let* $(\mathscr{F}_t)$ *be the stochastic basis generated by a family J of partitions t of* $[0, 1]^r$, *satisfying the assumptions of Proposition* 10.2. *Let Q be a finite signed measure absolutely continuous with respect to P on* $\mathscr{F}_\infty$; *denote by X the Radon-Nikodym derivative of Q with respect to P. Let f and g be two real functions such that* $f(0) = g(0) = 0$, $f'(0)$ *and* $g'(0)$ *exist*, $g'(0) \neq 0$. *The stochastic process* (X_t) *defined for every partition t by*

$$X_t = \sum_{[A\in\tau)} \frac{f[Q(A)]}{g[P(A)]} 1_A$$

is an amart which converges essentially to $\frac{f'(0)}{g'(0)}X$. *The martingale* (Y_t) *in the Riesz decomposition of* (X_t) *is given by*

$$Y_t = \sum_{A\in t} \frac{f'(0)}{g'(0)} \frac{Q(A)}{P(A)} 1_A.$$

Proof. Let $f(x) = xf'(0) + x\varphi(x)$, and $g(x) = xg'(0) + x\psi(x)$. Given $\epsilon > 0$, $0 < \epsilon < |g'(0)|$, choose $\alpha > 0$ such that $|x| < \alpha$ implies $|\varphi(x)| < \epsilon$ and $|\psi(x)| < \epsilon$. Then choose s in J such that for every atom A in s, $P(A) < \alpha$ and $|Q(A)| < \alpha$; let τ be in T, $\tau \geqq s$. There exists a partition $\mathscr{P}(\tau)$ of $[0, 1]^r$ whose atoms satisfy $P(A) < \alpha$ and $|Q(A)| < \alpha$, such that

$$X_\tau = \sum_{A\in\mathscr{P}(\tau)} \frac{f[Q(A)]}{g[P(A)]} 1_A$$

Set $Z_t = X_t - Y_t$. Since by an easy computation

$$|E(Z_\tau)| = \left| \sum_{A\in\mathscr{P}(\tau)} \frac{g'(0)\varphi[Q(A)] - f'(0)\psi[P(A)]}{g'(0)[g'(0) + \psi[P(A)]]} Q(A) \right|$$

$$\leqq \sum_{A\in\mathscr{P}(\tau)} \epsilon \frac{|f'(0)| + |g'(0)|}{|g'(0)|(|g'(0)| - \epsilon)} |Q|(A),$$

(X_t) is an amart with the Riesz decomposition $Y_t + Z_t$. Proposition 10.2 implies that $(\mathscr{F}_t)$ satisfies the Vitali condition V, and hence by Astbury's Theorem, i.e., the amart case of Theorem 4.3 below, X_t converges essentially. The relation $Y_t = (f'(0)/g'(0))E^tX$ yields the identification of the limit of X_t.

It is also possible to let the functions f and g depend on t, provided that there is a uniformity of behavior in a neighborhood of zero. More precisely, we have

PROPOSITION 10.4. *Let* $(\mathscr{F}_t)$ *be the stochastic basis generated by a family J of partitions t of* $[0, 1]^r$, *satisfying the assumptions of Proposition* 10.2. *Let Q be a signed finite measure absolutely continuous with respect to P on* $\mathscr{F}_\infty$; *denote by X the Radon-Nikodym derivative of Q with respect to P. Let U be a neighborhood of zero, and let* (f_t) *and* (g_t) *be two families of real-valued functions continuously differentiable in U, such that* $f_t(0) = g_t(0) = 0$ *for every t. Assume that there*

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

exist f and g differentiable in U, such that $\lim f_t = f$, $\lim g_t = g$, $g'(0) \neq 0$, *and such that* f_t' *and* g_t' *converge to* f' *and* g' *uniformly in U. The stochastic process defined for every partition t by*

$$X_t = \sum_{A \in t} \frac{f_t[Q(A)]}{g_t[P(A)]} 1_A,$$

is an amart which converges essentially to $\frac{f'(0)}{g'(0)} X$. *The martingale* (Y_t) *in the Riesz decomposition of* (X_t) *is given by*

$$Y_t = \sum_{A \in t} \frac{f'(0)}{g'(0)} \frac{Q(A)}{P(A)} 1_A.$$

The proof, similar to the argument above, is omitted.

It should be pointed out that the two previous propositions can be derived without too much effort from the essential convergence of martingales. We now give an application of the amart theory to differentiation of superadditive set functions. A non-negative, finitely additive (respectively superadditive) set function defined on an algebra $\mathcal{O}$ is called a *charge* (respectively a *supercharge*). A *pure charge* (respectively *pure supercharge*) on $\mathcal{O}$ is a charge (respectively a supercharge) which does not dominate any non-trivial measure (respectively non-trivial charge) on $\mathcal{O}$. Let λ be a supercharge defined on a field $\mathcal{O}$. λ admits a unique decomposition $\lambda = \lambda_m + \lambda_c + \lambda_s$, where λ_m is a measure, λ_c is a pure charge, and λ_s is a pure supercharge. λ_m is given by

$$\lambda_m(A) = \inf_c \textstyle\sum_i \lambda(A_i),$$

where $\inf_c$ is the infimum taken over all countable partitions (A_i) of A. λ_c is given by

$$\lambda_c(A) = \inf_f \textstyle\sum_i (\lambda - \lambda_m)(A_i),$$

where $\inf_f$ is the infimum taken over all finite partitions (A_i) of A. In the case where λ is a charge, this statement is due to Yosida-Hewitt (cf. [**30**]), and the general decomposition is proved in [**29**].

THEOREM 10.5. *Let J be a set of finite (respectively countable) measurable partitions of* Ω, *ordered by inclusion, and let* $(\mathcal{F}_t)$ *be the stochastic basis of* σ*-algebras generated by the partitions t. Let* (Q_t) *be a decreasing family of supercharges on* $\bigcup \mathcal{F}_t$, *(i.e.,* Q_t *is a supercharge on* $\bigcup \mathcal{F}_t$, *and* $s \leq t$, $A \in \bigcup \mathcal{F}_t$ *implies* $Q_s(A) \geq Q_t(A)$*), and set for each partition t*

$$X_\tau = \sum_{A \in t} \frac{Q_t(A)}{P(A)} 1_A,$$

with the convention $Q_t(A)/P(A) = 0$ *if* $P(A) = 0$. *Then* (X_t) *is an amart. Assume that for every atom A in the partitions t,* $P(A) \neq 0$. *Denote by* λ *the*

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

supercharge on $\bigcup \mathscr{F}_t$ *defined as the limit of* Q_t. *If* J *is a set of finite partitions, the Riesz decomposition of* (X_t) *is* $X_t = Y_t + Z_t$ *with*

$$Y_t = \sum_{A \in t} \frac{(\lambda_m + \lambda_c)(A)}{P(A)} 1_A, \quad Z_t = \sum_{A \in t} \frac{Q_t(A) - (\lambda_m + \lambda_c)(A)}{P(A)} 1_A.$$

If J *is a set of countable partitions such that every countable partition into elements of* $\bigcup \mathscr{F}_t$ *belongs to* J, *then the Riesz decomposition of* (X_t) *is*

$$Y_t = \sum_{A \in t} \frac{\lambda_m(A)}{P(A)} 1_A, \quad Z_t = \sum_{A \in t} \frac{Q_t(A) - \lambda_m(A)}{P(A)} 1_A.$$

Remark. Let $(\mathscr{F}_t)$ be the stochastic basis generated by all the countable partitions of $[0, 1]^r$ into substantial subsets. Applying Theorem 4.3, Proposition 10.2 and Theorem 10.5, we see that if (Q_t) is a decreasing family of supercharges on $(\mathscr{F}_t)$, then $X_t = \sum_{A \in t} Q_t(A)/P(A)\ 1_A$ converges essentially. For this result, the amart property of (X_t) is more important than the supermartingale property which (X_t) also possesses, since arbitrary L^1 bounded supermartingales converge essentially only under Vitali conditions stronger than V (e.g., V' and V^c).

Proof. Let $\sigma \leqq \tau$ be in T, σ taking values (s_i), and τ taking values (t_j). Since a supercharge is automatically monotone and countably superadditive,

$$\begin{aligned} E^\sigma X_\tau &= \sum_i \sum_{A \in s_i, A \subset \{\sigma = s_i\}} (P(A))^{-1} \left(\int_A X_\tau dP \right) 1_A \\ &= \sum_i \sum_{A \in s_i, A \subset \{\sigma = s_i\}} (P(A))^{-1} \left(\sum_j \sum_{B \in t_j, B \subset \{\tau = t_j\} \cap A} Q_{t_j}(B) \right) 1_A \\ &\leqq \sum_i \sum_{A \in s_i, A \subset \{\sigma = s_i\}} (P(A))^{-1} \left(\sum_{t_j \geqq s_i} Q_{t_j}(\{\tau = t_j\} \cap A) \right) \\ &\leqq X_\sigma. \end{aligned}$$

Let J be a set of finite partitions. Since λ_s is a pure supercharge, given any $\epsilon > 0$ there exists a finite partition $A_1, \ldots, A_n$ into sets of $\bigcup \mathscr{F}_t$ such that $\sum_{i \leqq n} \lambda_s(A_i) \leqq \epsilon$. Choose t such that $A_i \in \mathscr{F}_t$, $i = 1, \ldots, n$, and such that for every $i = 1, \ldots, n$, $Q_t(A_i) - \lambda(A_i) < \epsilon/n$; let t' be greater than t. From the inequalities

$$\begin{aligned} 0 &\leqq \textstyle\sum_{A \in t'} (Q_{t'}(A) - (\lambda_m + \lambda_c)(A)) \\ &\leqq \textstyle\sum_{i \leqq n} (Q_t(A_i) - \lambda(A_i)) + \sum_{i \leqq n} \lambda_s(A_i) \leqq 2\epsilon, \end{aligned}$$

we deduce the Riesz decomposition of (X_t). A similar argument applies in the case of countable partitions.

Remark. Theorem 10.5 can also be derived from the implications (1) $\Rightarrow$ (5) or (1) $\Rightarrow$ (3) in Theorem 5.1.

11. Examples for general index set J. In the present section we show that the conditions V, V^c and V' are all different. Krickeberg's remarkable work earlier showed that V is different from V', and that V is not necessary for convergence of L^∞-bounded martingales. The examples also give some information about the connections between the maximal theorem and essential convergence. The condition V^c, sufficient for the essential convergence of supermartingales (cf. Section 8), is shown to be not necessary.

Example 11.1. (A modification of an example in [**22**]). The stochastic basis satisfies V but does not satisfy V^c. (X_t) is a supermartingale satisfying Doob's condition, and hence a controlled amart. $(X_\tau)_{\tau \in T}$ is not a supermartingale, X_t does not converge essentially (and hence neither (X_t) nor $(-X_t)$ are subpramarts); in fact, $e \limsup X_t = +\infty$.

Let (c_i) be a sequence of integers, $0 < c_i \uparrow \infty$, such that for every i, c_{i+1} is a multiple of c_i. Let J be the set of ordered pairs (i, j) for $1 \leqq j \leqq c_i$, and denote by $\mathscr{F}(i, j)$ the σ-algebra generated by the partition

$$[(k-1)c_i^{-1}, kc_i^{-1}[, \; 1 \leqq k \leqq c_i.$$

On the set J we define the order $(i, j) \leqq (k, l)$ if $i < k$ or $(i, j) = (k, l)$. Then $(\mathscr{F}_t)$ satisfies the Vitali condition V (see the remark following Proposition 10.2). Let $0 \leqq \alpha_i \uparrow$, $0 \leqq \beta_i \downarrow$, $\alpha_i \geqq \beta_i$, $i \in N$, and set

$$X(i, j) = \begin{cases} \alpha_i & \text{on } [(j-1)c_i^{-1}, jc_i^{-1}[\\ \beta_i & \text{elsewhere.} \end{cases}$$

$(X(i, j))$ is a supermartingale if and only if $\forall i$, $\forall j \leqq c_i$, $\forall k \leqq c_{i+1}$, $\forall A \in \mathscr{F}(i, j)$, $E(1_A X(i, j)) \geqq E(1_A X(i+1, k))$. If suffices to consider sets A atoms of $\mathscr{F}(i, j)$. Hence if for every i,

$$c_{i+1}^{-1}\alpha_{i+1} \leqq c_i^{-1}(\beta_i - \beta_{i+1}),$$

$(X(i, j))$ is a supermartingale. This condition is e.g. satisfied for $c_i = 2^{i^2}$, $\beta_i = i^{-1}$, and $\alpha_i = i$. Obviously $\liminf X(i, j) = 0$, $\limsup X(i, j) = +\infty$, and $X(i, j)$ converges to 0 in L^1. By Proposition 8.1 and Theorem 8.3, V^c fails. (Furthermore, the condition V^d fails; cf. Example 11.4 below.)

Example 11.2. In the following example the stochastic basis $(\mathscr{F}_t)$ satisfies the Vitali condition V', (X_t) is a pramart and an ordered amart, but not an amart, submartingale or supermartingale, and the maximal theorem fails. Let $\Omega = [0, 1[$, and let $r_k \in N$, $1 = r_1$, $r_k \uparrow$ and $s_k = r_k/2^k \in N$, $k = 2, 3, \ldots$. J is the set of pairs (i, j) with $1 \leqq j \leqq s_i$, $i \in N$, ordered by $(i, j) \leqq (k, l)$ if either $(i, j) = (k, l)$ or $i < k$. $\mathscr{F}(1, 1)$ is the σ-algebra generated by the partition $P_2 = \{[0, 2^{-1}[, [2^{-1}, 1[\}$. For $1 \leqq j \leqq s_2$, the σ-algebras $\mathscr{F}(2, j)$ are the same, and generated by the partition P_2 composed of $[0, 2^{-2}[$, $[2^{-1}, 1[$, and s_2 intervals of equal length dividing $[2^{-2}, 2^{-1}[$. At the next step, the interval $[0, 2^{-2}[$ is divided into $[0, 2^{-3}[$, $[2^{-3}, 2^{-2}[$, and the second interval is again subdivided. (A similar construction proving a different point appears in [**1**]).

More generally, for i fixed and $1 \leqq j \leqq s_i$, the σ-algebras $\mathscr{F}(i, j)$ are identical and generated by the partition P_i of Ω which agrees with P_{i-1} on the set $[2^{-i+1}, 1[$, contains $[0, 2^{-i}[$, and s_i subintervals of equal length of $[2^{-i}, 2^{-i+1}[$. Since the σ-algebras $\mathscr{F}(k, l)$ agree on $D_i = [2^{-i+1}, 1]$ if $(k, l) \geqq (i, j)$, and $D_i \uparrow \Omega$, it is easy to see that the condition V' holds. Let α_i and β_i be positive numbers such that $\sum \alpha_i/r_i < \infty$, $\beta_i \downarrow 0$ and let

$$X(i, j) = \begin{cases} \alpha_i & \text{on } [2^{-i} + (j-1)/r_i, 2^{-i} + j/r_i[\\ \beta_i & \text{elsewhere.} \end{cases}$$

Then if $\tau \in T'$, $\tau \geqq (i, j)$, $E(X_\tau) \leqq \sum_{k \geqq i} (\alpha_k/r_k) + \beta_i \rightarrow 0$ $(i \rightarrow \infty)$; hence $X(i, j)$ is an ordered potential. Now suppose in addition that α_i are chosen so that $\alpha_i 2^{-i} \rightarrow \infty$. (e.g. $r_i = i^2 2^{2i}$, $\alpha_i = 2^{2i}$, $\beta_i = i^{-2}$). Letting $\tau = (i, j)$ on $[2^{-i} + (j-1)/r_i, 2^{-i} + j/r_i[$, $1 \leqq j \leqq s_i$, and $\tau = (i, 1)$ outside of $[2^{-i}, 2^{-i+1}[$, we have a simple stopping time $\tau \in T$, such that $X_\tau \geqq \alpha_i 1_{[2^{-i}, 2^{-i+1}[}$; hence $\limsup_{\tau \in T} E(X_\tau) = +\infty$. $(X(i, j))$ is not an amart; however, it is easy to see that $X(i, j)$ is a pramart. Since for $(i, j) < (k, l)$, $X(i, j) = \beta_i$ and $X(k, l) = \beta_k$ on $[2^{-i+1}, 1[$, $X(i, j)$ is not a submartingale. $X(i, j)$ is a supermartingale if and only if

$$\alpha_{i+1}/r_{i+1} + \beta_{i+1}(2^{-i} - 1/r_{i+1}) \leqq \beta_i 2^{-i} \text{ for every } i.$$

With the particular values assigned above to α_i, β_i and r_i, $X(i, j)$ is not a supermartingale, and for every i, $\alpha_i P[\sup X(k, l) > \alpha_i] = 2^i$.

Example 11.3. In the following example, the stochastic basis satisfies the controlled Vitali condition V^c, but does not satisfy the ordered Vitali condition V'. Let $\Omega = [0, 1[$, let P be Lebesgue measure on $[0, 1[$, and let $J = \{(i, j) | j = 1, 2, \ldots, 2^i; i = 0, 1, \ldots\}$. On J define an order $\leqq$ by $(i, j) \leqq (k, l)$ if $(i, j) = (k, l)$ or $i < k$, and for every (i, j) let $\mathscr{F}(i, j)$ be the σ-algebra of Borel sets of $[0, 1[$. The controlled Vitali condition V^c is satisfied because $(0, 1)$ controls each $\tau \in T$, hence V^c is equivalent with V which holds (cf. Section 10). However, let $A(i, j) = [(j-1)2^{-i}, j2^{-i}[$. Clearly $\limsup A(i, j) = [0, 1[$. For every finite increasing sequence $(i_1, j_1) \leqq \ldots \leqq (i_n, j_n)$ with $i_1 > k$ we have

$$P(\bigcup_{r \leqq n} A(i_r, j_r)) \leqq \sum_{p > k} 2^{-p} = 2^{-k},$$

and hence V' fails.

Vitali condition V^d. The following example shows that V^c is not necessary for the essential convergence of L^1 bounded supermartingales. We at first notice that the essential convergence of L^1 bounded supermartingales holds under a condition V^d, which is then shown by example to be strictly weaker than V^c. V^d is the *logical union* of $V^d(\alpha)$, $\alpha > 0$, defined as follows. Let $\alpha > 0$; $(\mathscr{F}_t)$ *satisfies the condition* $V^d(\alpha)$ if given an adapted family of sets (A_t) and a set $A \in \mathscr{F}_\infty$ with $A \subset e \limsup A_t$, for every $\epsilon > 0$ there exist $\tau \in T_c$,

τ' ct τ such that

$$P(A\backslash\{E^{\tau'}1_{A_\tau} \geqq \alpha\}) \leqq \epsilon.$$

(The condition V^c is obtained for $\alpha = 1$). Let $(\mathscr{F}_t)$ satisfy V^d, and let (Z_t) be a positive controlled amart such that $\lim_{\tau\in T_c} E(Z_\tau) = 0$. A slight modification of the proof of Theorem 8.3 shows that Z_t converges essentially to 0. Hence the implication (6) $\Rightarrow$ (1) in Theorem 5.1 proves that V^d implies V. Now let (X_t) be an L^1 bounded supermartingale. The proof of Proposition 8.1 shows that we can write $X_t = Y_t + Z_t$, where (Y_t) is an L^1 bounded martingale which converges essentially because V^d implies V, and Z_t is a positive controlled amart such that $\lim_{\tau\in T_c} E(Z_\tau) = 0$; hence Z_t converges essentially.

We now show that the condition V^d is strictly weaker than V^c.

Example 11.4. Let $\Omega = [0, 1[$, let P be Lebesgue measure on Ω, and given a positive integer M, let $J = \{(i, j)| i \in N, 1 \leqq j \leqq M^i\}$. On J define the order $(i, j) \leqq (k, l)$ if $i < k$ or $(i, j) = (k, l)$. Let $\mathscr{F}(i, j)$ be the σ-algebra generated by the partition $[(k - 1)M^{-i}, kM^{-i}[$, $1 \leqq k \leqq M^i$. For every atom B of $\mathscr{F}(i + 1, j)$, the atom B' of $\mathscr{F}(i, 1)$ containing B satisfies the relation $E^{(i,1)}1_B = M^{-1}1_{B'}$. Set $A(i, j) = [(j - 1)M^{-i}, jM^{-i}[$. Fix i and consider $\tau \in T_c, \tau' \in T'$, $(i, 1) < \tau'$ ct τ. Set $D = A_\tau \cap \{\tau = \tau'\} \in \mathscr{F}_{\tau'}$; then

$$P(D) \leqq M^{-i}(M - 1)^{-1} \text{ and } E^{\tau'}1_{A_\tau} \leqq 1_D + M^{-1}1_{D^c}.$$

Hence if $M > 1$, for every $\alpha > M^{-1}$, $V^d(\alpha)$ fails and $V^c = V^d(1)$ fails. We now show that $V^d(M^{-1})$ holds. For every $B \in \mathscr{F}(i + 1, j)$, the smallest element $B' \in \mathscr{F}(i, 1)$ such that $B' \supset B$ satisfies $E^{(i,1)}1_B \geqq M^{-1}1_{B'}$. Let $B(i, j)$ be an adapted family of sets. Given $\epsilon > 0$ choose

$$\mathscr{P} = \{(i_{,1}, j_1^{(1)}), \ldots, (i_1, j_{n_1}^{(1)}), \ldots, (i_k, j_1^{(k)}), \ldots, (i_k, j_{nk}^{(k)})\}$$

such that $i_1 < i_2 < \ldots < i_k$, and

$$P(\limsup B(i, j)\backslash \cup_{(i,j)\in\mathscr{P}} B(i, j)) \leqq \epsilon.$$

Let B_1' be the smallest element of $\mathscr{F}(i_1 - 1, 1)$ which contains $B(i_1, j_1^{(1)})$. Set $\tau' = (i_1 - 1, 1)$ and $\tau = (i_1, j_1^{(1)})$ on B_1'. We now consider the first pair (i, j) listed in $\mathscr{P}$ after $(i_1, j_1^{(1)})$ such that $B(i, j) \cap B_1' = \emptyset$ (if such a pair exists), say (k, l). Let B_2' the smallest element of $\mathscr{F}(k - 1, 1)$ such that $B(k, l) \subset B_2'$, and set $\tau' = (k - 1, 1)$ and $\tau = (k, l)$ on B_2'. If for every element (i, j) in $\mathscr{P}$ listed after $(i_1, j_1^{(1)})$ we have $B(i, j) \subset B_1'$, set $\tau = \tau' = (i_1 + 1, 1)$ on B_1^c. In a finite number of steps we define stopping times $\tau \in T_c$, τ' ct τ such that

$$\{E^{\tau'}1_{B_\tau} \geqq M^{-1}\} \supset \cup_{(i,j)\in\mathscr{P}} B(i, j).$$

Hence $\mathscr{F}(i, j)$ satisfies $V^d(M^{-1})$.

Example 11.5. In the following example, (X_t) is a controlled amart satis-

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

fying Doob's condition, such that the net $(X_\tau)_{\tau\in T_c}$ does not converge stochastically, X_t does not converge essentially and $(|X_t|)$ is not a controlled amart. Let $\Omega = [0, 1[$, and let P be Lebesgue measure on $[0, 1[$. Set $J = \{(i, j) | 1 \leqq j \leqq 2^{i-1}\}$, ordered by $(i, j) \leqq (k, l)$ if $(i, j) = (k, l)$, or $i < k$. Let $\mathscr{F}(i, j)$ be the σ-algebra generated by the partition $[(k-1)2^{-i}, k2^{-i}[$, $1 \leqq k \leqq 2^i$. Set

$$X(i, j) = 1_{[(2j-2)2^{-i},(2j-1)2^{-i}[} - 1_{[(2j-1)2^{-i},2j2^{-i}[}$$

for $(i, j) \in J$. Let $\tau \in T_c$, $(i, 1) <_c \tau$; then

$$|E(X_\tau)| \leqq \textstyle\sum_{k\geqq i} 2^{-k} = 2^{-i+1}.$$

$(X(i, j))$ is a controlled amart, and an ordered potential. Set $\tau = (i + 1, k)$ on $[(k-1)2^{-i}, k2^{-i}]$, for $1 \leqq k \leqq 2^i$; $\tau \in T_c$ (it is controlled by $(i, 1)$), and $|X_\tau| = 1$.

We finally observe that in the present example the condition $V^d(\frac{1}{2})$ holds (cf. Example 11.4 above). Thus V^d is not sufficient for essential convergence of L^1-bounded controlled amarts.

12. Complements and remarks. In this section we discuss, rather briefly, some extensions of our results, in particular to directed sets filtering to the left, to σ-finite measure spaces, and to Banach-valued stochastic processes.

We at first observe that while we state in the paper the well-known Vitali conditions as they appear in the literature, a stopping times formulation would have been more compact. Thus e.g. the condition V is equivalent with the following: For each $\epsilon > 0$ and for each adapted family of sets (A_t), there exists $\tau \in T$ such that $P(e \limsup A_t \backslash A_\tau) < \epsilon$. Similarly W becomes: For each adapted family of sets (A_t) there exists a stopping time γ taking countably many values, such that $P(e \limsup A_t \backslash A_\gamma) = 0$. There are analogous shorthand versions of V' and W'.

Going in the opposite direction, it is possible to define amarts without stopping times. This was in fact done by Lamb [**24**], who also proved a result essentially equivalent with the amart convergence theorem. The stopping times approach initiated by J. Baxter [**3**], Austin-Edgar-Ionescu Tulcea [**2**], and Chacon [**6**], is more intuitive and transparent, and an amart theory [**12**] paralleling martingale theory could not have been developed without it.

A. *The descending case.* Let J be a directed set filtering to the right, and write $-J$ for J with the reversed ordering. Given a stochastic basis $(\mathscr{F}_t)_{t\in-J}$, the sets $(-T, \leqq)$, $(-T', < | <)$ and $(-T_c, <_c)$ are filtering to the left. A stochastic process $(X_t, \mathscr{F}_t, -J)$ is an *amart* (resp. an *ordered amart*) if the net $(E(X_\tau))_{\tau\in-T}$(resp. $(E(X_\tau))_{\tau\in-T'}$) converges. A stochastic process is a *subpramart* (resp. *ordered subpramart*) if $s \limsup_{\sigma,\tau\in-T} X(\sigma, \tau) \leqq 0$ (resp. $s \limsup_{\sigma,\tau\in-T'} X(\sigma, \tau) \leqq 0$). Unlike in the case of J filtering to the right (cf. Section 3) there seems to be no easy approximation of amarts by submartingales or supermartingales. Let $(X_t)_{t\in -J}$ be an amart (resp. an ordered amart); the net $(X_\tau)_{\tau\in -T}$ (resp. $(X_\tau)_{\tau\in -T'}$) converges in L^1 norm. (For the case $J = N$ see [**12**], Theorem 2.3 and 2.9; for the case of amarts indexed by $-J$, see [**1**] Proposition 4.1). We now show that asymptotic behavior of subpramarts is similar to that in the ascending case.

THEOREM 12.1. *Let $(X_t)_{t\in -J}$ be a subpramart (resp. ordered subpramart). Then the net $(X_\tau)_{\tau\in -T}$ (resp. $(X_\tau)_{\tau\in -T'}$) converges stochastically to a limit X_∞, $-\infty \leqq X_\infty < +\infty$. If* $\liminf E(X_t^-) < \infty$ *or if (X_t) is a pramart, then X_∞ is finite* a.s. *If $J = N$, in the conclusion stochastic convergence can be replaced by* a.s. *convergence.*

Proof. Let $(X_t)_{t\in -J}$ be a subpramart. We at first prove that $(X_\tau)_{\tau\in -T}$ converges stochastically in $\overline{\mathbf{R}}$. (The approach consisting in first proving convergence in $\overline{\mathbf{R}}$ was initiated-for ascending amarts-by Dvoretzky [**11**].) Since stochastic convergence in $\overline{\mathbf{R}}$ is defined by the distance of a complete metric space, it is enough to show that given a fixed sequence (s_n) in $-J$, for every decreasing sequence (τ_n) in $-T$ such that $(\tau_n) \leqq (s_n)$, X_{τ_n} converges stochastically in $\overline{\mathbf{R}}$. Choose a decreasing sequence (s_n) of indices in $-J$ such that if $\sigma, \tau \in -T$, $\sigma \leqq \tau \leqq s_n$, then

$$P[\{X_\sigma - E^{\mathscr{F}_\sigma} X_\tau > 1/n\}] \leqq 1/n.$$

Let (τ_n) be a decreasing sequence of elements of $-T$ such that $(\tau_n) \leqq (s_n)$. Let $Y_{-n} = X_{\tau_n}$ and $\mathscr{G}_{-n} = \mathscr{F}_{\tau_n}$. For every stopping time σ taking values in $-N$, $\tau_{-\sigma} \in -T$, and $\mathscr{G}_\sigma = \mathscr{F}_{\tau_{-\sigma}}$. Hence $(Y_n, \mathscr{G}_n, -N)$ is a subpramart. Assume that there exists $\alpha < \beta$ such that $P(A) = a > 0$, where

$$A = \liminf_{n\in -N} Y_n < \alpha < \beta < \limsup_{n\in -N} Y_n\}.$$

Observe that A is in the tail σ-algebra $\bigcap_{-N} \mathscr{G}_n$. We choose $M_1 \leqq M_2 \leqq M$ in $-N$ such that $P(A\backslash B) < \delta$ where

$$B = A \cap \{\inf_{M_2 \leqq n \leqq M} Y_n < \alpha < \beta < \sup_{M_1 \leqq n \leqq M_2} Y_n\}.$$

Set $C = A \cap \{\sup_{M_1 \leqq n \leqq M_2} Y_n > \beta\}$, and define stopping times $\sigma \leqq \tau \leqq M$ by

$$\sigma = \begin{cases} \inf\{n | M_1 \leqq n \leqq M_2,\, Y_n > \beta\} & \text{on } C \\ M & \text{on } C^c, \end{cases}$$

$$\tau = \begin{cases} \inf\{n | M_2 \leqq n \leqq M,\, Y_n < \alpha\} & \text{on } B \\ M & \text{on } B^c. \end{cases}$$

Since $Y_\sigma - Y_\tau \geqq (\beta - \alpha)1_B + 1_{C\backslash B}(\beta - Y_M)$,

$$\begin{aligned} Y_\sigma - E^{\mathscr{G}_\sigma} Y_\tau &\geqq (\beta - \alpha) E^{\mathscr{G}_\sigma}(1_B) + \beta E^{\mathscr{G}_\sigma}(1_{C\backslash B}) - E^{\mathscr{G}_\sigma}(1_{C\backslash B} Y_M) \\ &\geqq (\beta - \alpha) 1_A - (|\alpha| + |\beta|) E^{\mathscr{G}_\sigma}(1_{A\backslash B}) \\ &\quad - |\beta| E^{\mathscr{G}_\sigma}(1_{C\backslash B}) - E^{\mathscr{G}_\sigma}(1_{C\backslash B} |Y_M|). \end{aligned}$$

Since for every positive random variable X and for every $\eta > 0$ we have

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

$P(\{E^{\mathscr{G}_\sigma}X \geqq \eta\}) \leqq E(X)/\eta$, the following inequalities hold:

$$P(\{(|\alpha| + |\beta|)E^{\mathscr{G}_\sigma}(1_{A\setminus B}) \geqq \epsilon\}) \leqq \delta(|\alpha| + |\beta|)/\epsilon,$$

$$P(\{|\beta|E^{\mathscr{G}_\sigma}(1_{C\setminus B}) > \epsilon\}) \leqq \delta|\beta|/\epsilon.$$

Furthermore, since $|Y_M|$ is integrable, we can choose $\delta > 0$ such that if $P(F) \leqq \delta$, then $E(1_F|Y_M|) \leqq \epsilon^2$. We can therefore choose $\delta > 0$ such that the inequality

$$Y_\sigma - E^{\mathscr{G}_\sigma}Y_\tau \geqq (\beta - \alpha)1_A - 3\epsilon$$

holds outside of a set of probability less than 3ϵ. Hence

$$s \lim \sup_{\sigma,\tau\in -T(N)} Y(\sigma, \tau) \geqq (\beta - \alpha)1_A,$$

which is a contradiction. We therefore deduce that $(X_\tau)_{\tau\in -T}$ converges stochastically in $\overline{\mathbf{R}}$. Set $D = \{s \lim X_\tau = +\infty\}$; $D \in \bigcap_{-J}\mathscr{F}_t$, and we want to show that $P(D) = 0$. Assume the contrary and let $0 < \epsilon < P(D)/2$. Choose t such that $\sigma \leqq t$ implies $P(\{X_\sigma - E^{\mathscr{F}_\sigma}X_t > \epsilon\}) \leqq \epsilon$. By the definition of D, we can choose a decreasing sequence (t_n) in $-J$ such that $t_1 \leqq t$, and $P(\{1_D X_{t_n} \leqq (n + 1)1_D\}) \leqq \epsilon$. Hence the inequality

$$P(\{E^{\mathscr{F}_{t_n}}(1_D X_t) \leqq n\}) \leqq 2\epsilon + P(D^c)$$

holds for each n. Since

$$P(D) - 2\epsilon \leqq P(\{E^{\mathscr{F}_{t_n}}(1_D X_t) > n\}) \leqq E(|X_t|)/n,$$

we get a contradiction, and it follows that $-\infty \leqq X_\infty < \infty$. Furthermore, if $\lim\inf E(X_t^-) < \infty$, Fatou's lemma together with the inequality $-\infty < s \lim\sup X_t \leqq s \lim\sup X_\tau$ yields that X_∞ is finite. The same argument is valid for ordered subpramarts. To obtain the conclusion that X_∞ is a.s. finite for (ordered) pramarts, observe that if (X_t) is a pramart, then (X_t) and $(-X_t)$ are subpramarts. Finally, the proof also establishes a.s. convergence in the case $-J = -N$.

We now state a result analogous to Theorems 5.1 and 7.4 with some conditions omitted for simplicity. Also the proof is omitted.

THEOREM 12.2. *Let $(\mathscr{F}_t)_{t\in -J}$ be a stochastic basis. The following assertions are equivalent:*

(1) *$(\mathscr{F}_t)$ satisfies the Vitali condition V (resp. V').*

(2) *Given any stochastic process $(X_t)_{t\in -J}$ such that the net $(X_\tau)_{\tau\in -T}$ (resp. $(X_\tau)_{\tau\in -T'}$) converges stochastically, X_t converges essentially.*

(3) *Every subpramart (resp. ordered subpramart) such that* $\lim\inf E(X_t^-) < \infty$ *converges essentially.*

(4) *Every amart (resp. ordered amart) $(1_{A_t})_{t\in -J}$ such that* $\lim \mathrm{P}(A_t) = 0$ *converges essentially (to zero).*

(5) *Let Y be any $\mathscr{F}_{-\infty} = \bigcap_{t\in -J}\mathscr{F}_t$ measurable random variable. Assume that for each ω, $Y(\omega)$ is a cluster point of the net $(X_t(\omega))_{t\in -J}$. Then given any sequence*

(s_k) *in* $-J$, *there exists a sequence* (τ_k) *in* $-T$ (*resp.* $-T'$), *such that* $\tau_k \leqq s_k$, *and* X_{τ_k} *converges* a.s. *to* Y.

(6) *Identical to* (5) *except that* $Y = e \limsup X_t$.

B. *σ-finite measure spaces.* Let $(\Omega, \mathscr{F}, \mu)$ be a σ-finite measure space. A stochastic process (X_t) is an amart (respectively an ordered amart) if the net $(\int X_\tau d\mu)_{\tau \in T}$ (respectively $(\int X_\tau d\mu)_{\tau \in T'}$) converges. One shows, as in the case when $\mu(\Omega) = 1$ (see Sections 1, 6) that (X_t) is an amart (respectively an ordered amart) if and only if the net $(\int |E_\mu{}^\sigma(X_\tau) - X_\sigma| d\mu)_{\sigma \leqq \tau}$ (respectively the net $(\int |E_\mu{}^\sigma(X_\tau) - X_\sigma| d\mu)_{\sigma < | < \tau})$ converges to zero. (For the definition of the conditional expectation with respect to a σ-finite measure, see e.g. [**26**] page 16.) The following generalization of Theorem 2.1 in [**1**] and Theorem 6.2 in [**19**] yields the Riesz decomposition of amarts (and ordered amarts).

Let $\mathscr{D}$ be the class of all stochastic processes (X_t) taking values modulo μ and satisfying the condition (D):

$$(D) \lim_{\sigma, \tau \in T} \|X_\sigma - E_\mu{}^{\mathscr{F}_\sigma} X_\tau\| = 0$$

where $\| \ \|$ is a complete norm defined on equivalence classes of random variables, such that the operator E_μ preserves the convergence in this norm. Then each stochastic process (X_t) in $\mathscr{D}$ can be written as a sum $X_t = Y_t + Z_t$, where (Y_t) is a martingale and $\lim_\tau \|Z_\tau\| = 0$. A similar statement can be obtained for the class $\mathscr{D}'$ of stochastic processes satisfying the condition (D'):

$$\lim_{\sigma, \tau \in T'} \|X_\sigma - E_\mu{}^{\mathscr{F}_\sigma} X_\tau\| = 0.$$

C. *Banach valued case.* Let J be a directed set filtering to the right, and let $(\mathscr{F}_t)$ be a stochastic basis of the probability space $(\Omega, \mathscr{F}, P)$. We denote by $\mathscr{E}$ a fixed Banach space with norm $| \ |$. A random variable will be a strongly measurable function $X: \Omega \to \mathscr{E}$, and a stochastic process will be a family (X_t) of $\mathscr{F}_t$ measurable random variables. Unless specified otherwise, the integral of a random variable X is defined in the Pettis sense, and we set $\|X\| = \sup_{A \in \mathscr{F}} |E(1_A X)|$. X is said *Bochner integrable* if $E(|X|) < \infty$, and a stochastic process is L^1 bounded if $\sup_t E(|X_t|) < \infty$. The Banach space $\mathscr{E}$ has the *Radon-Nikodym property* if for every probability space $(\Omega, \mathscr{F}, P)$ and every measure $\mu: \mathscr{F} \to \mathscr{E}$ such that μ is absolutely continuous with respect to P and μ has finite variation on Ω, there exists a Bochner integrable random variable $X: \Omega \to \mathscr{E}$ such that $\mu(A) = E(1_A X)$ for all $A \in \mathscr{F}$. A stochastic process (X_t) is an *amart* (respectively an *ordered amart*) if the net $(E(X_\tau))_{\tau \in T}$ (respectively $(E(X_\tau))_{\tau \in T'}$) converges in the strong topology of $\mathscr{E}$.

The Pettis norm characterization of amarts by the difference property, and the Riesz decomposition of amarts (proved in [**13**] in the case $J = N$, and in [**1**] in the general case) extend to ordered amarts. The definitions of pramart, ordered pramart and mil also extend, the norm in $\mathscr{E}$ replacing the absolute value. We notice that the proof of Proposition 4.1 extends without any modification to the case of the norm convergence in a Banach space $\mathscr{E}$. In the following theorem the adjective "strong" applies to the topology of the Banach space.

THEOREM 12.3. *The following statements are equivalent:*

(1) *The stochastic basis satisfies the Vitali condition* V.

(2) *For every Banach space* $\mathscr{E}$, *and for every stochastic process* (X_t), *the strong stochastic convergence of* $(X_\tau)_{\tau\in T}$ *implies the strong essential convergence of* X_t.

(3) *For every Banach space* $\mathscr{E}$, *every pramart is a mil.*

Also the conditions (1′) and (2′) similar to the conditions (1) and (2) above with V' and T' instead of V and T, are equivalent. This extends the equivalence (1) $\Leftrightarrow$ (3) in Theorem 7.4. It has been proved in [**15**] that every $\mathscr{E}$-valued amart $(X_n, \mathscr{F}_n, N)$ is a mil if and only if $\mathscr{E}$ is finite-dimensional. Hence an $\mathscr{E}$-valued amart is a pramart if and only if $\mathscr{E}$ is finite dimensional.

The methods of the present paper allow the following generalization of Chatterji's important theorem [**8**].

THEOREM 12.4. *Let* $\mathscr{E}$ *be a Banach space with the Radon-Nikodym property, and let* $(\mathscr{F}_t)$ *be a stochastic basis satisfying the Vitali condition* V. *Every* L^1 *bounded martingale* (X_t) *converges essentially in the strong topology of* $\mathscr{E}$.

Proof. By the implication (1) $\Rightarrow$ (2) in Theorem 12.3, we only need to prove that there exists a random variable X_∞ (necessarily Bochner integrable) such that for every $\epsilon > 0$, the net $(P(\{|X_\tau - X_\infty| > \epsilon\}))_{\tau\in T}$ converges to 0. Since the strong convergence in probability is defined by the distance of a complete metric space, it suffices to prove that for every increasing sequence (τ_n) in T, (X_{τ_n}) strongly converges in probability. Since (X_{τ_n}) is an L^1-bounded martingale for $(\mathscr{F}_{\tau_n})$, this follows from Chatterji's Theorem [**8**].

D. *Dichotomy of behavior of pramarts*. We notice that Theorems 2.6 and 2.7 [**12**] extend with the same proof to classes $\mathscr{C}$ of stochastic processes $(X_n, \mathscr{F}_n, N)$ having the following properties:

(i) If $\lim E(X_n^+) < \infty$, (resp. $\lim E(X_n^-) < \infty$), then X_n converges almost surely to X_∞, $-\infty \leqq X_\infty < +\infty$ (resp. $-\infty < X_\infty \leqq +\infty$).

(ii) Given an element $(X_n, \mathscr{F}_n)$ in $\mathscr{C}$, and given an increasing sequence (τ_k) of simple stopping times for $(\mathscr{F}_n)$, the sequence $(X_{\tau_k}, \mathscr{F}_{\tau_k})$ belongs to $\mathscr{C}$.

The class of pramarts satisfies (i), and the classes of pramarts, subpramarts and superpramarts each satisfy (ii) (cf. Section 2).

THEOREM 12.5. *Let* $\mathscr{C}$ *be a class of stochastic processes satisfying the properties* (i) *and* (ii) *above, and let* (X_n) *be a predictable element* (i.e., X_n *is* $\mathscr{F}_{n-1}$ *measurable for all* n) *in* $\mathscr{C}$. *Then there exists a set* $G \subset \Omega$ *such that* X_n *converges* a.s. *on* G *and* $\limsup X_n = +\infty$, $\liminf X_n = -\infty$ *on* G^c.

THEOREM 12.6. *Let* $\mathscr{C}$ *be a class of stochastic processes satisfying* (i) *and* (ii) *above, and let* (X_n) *be an element of* $\mathscr{C}$. *Let* (τ_k) *be an increasing sequence of bounded stopping times with* $\tau_k \geqq k$, $k \in N$. *Suppose that*

https://doi.org/10.4153/CJM-1980-009-1 Published online by Cambridge University Press

$E[\sup_k |X_{\tau_k} - X_{k-1}|] < \infty$. *Then there exists a set* $G \subset \Omega$ *such that* X_n *converges* a.s. *on* G, *and* $\limsup X_n = +\infty$, $\liminf X_n = -\infty$ *on* G^c.

E. *Strong law of large numbers.* Theorem 1.11 [**19**] extends to pramarts and mils. More generally, let (X_n) be an adapted sequence such that for some constant $\alpha \geqq 1$, $\sum_{i=1}^{\infty} E|X_i - X_{i-1}|^2/i^{1+\alpha} < \infty$, and such that $E^n X_{n+1} - X_n \to 0$ a.s.; then $X_n/n \to 0$ a.s. The proof is the same as in [**19**].

F. *Semiamarts.* Semiamarts are defined for $J = N$ by the property $\sup |EX_\tau| < \infty$. The proper generalization to directed sets is: There exists $s \in J$ such that $\sup_{\tau \geqq s} |EX_\tau| < \infty$. An amart is a semiamart, and a considerable part of the semiamart theory on integers (cf. [**12**] Section 4, and [**19**]) extends to directed sets.

Added in proof. The condition V is now known not to be necessary for convergence of L_1-bounded martingales (cf. C. R. Acad. Sc. Paris, Série A, *288* (1979), 595–598). For the Banach-valued case, see also Can. J. Math. *31* (1979), 1033–1046.

References

1. K. Astbury, *On Amarts and other topics*, Ph.D. Dissertation, Ohio State University, (1976). Also *Amarts indexed by directed sets*, Ann. Prob. *6* (1978), 267–278.

2. D. G. Austin, G. A. Edgar and A. Ionescu Tulcea, *Pointwise convergence in terms of expectations*, Zeit. Wahrscheinlichkeitstheorie verw. Gebiete *30* (1974), 17–26.

3. J. R. Baxter, *Pointwise in terms of weak convergence*, Proc. Amer. Math. Soc. *46* (1974), 395–398.

4. A. Bellow, *Les amarts uniformes*, C.R. Acad. Sci. Paris *284*, Série A, 1295–1298.

5. A. Brunel and L. Sucheston, *Sur les amarts à valeurs vectorielles*, C.R. Acad. Sci. Paris, Serie A, *283*, 1037–1039.

6. R. V. Chacon, *A stopped proof of convergence*, Adv. in Math. *14* (1974) 365–368.

7. R. V. Chacon and L. Sucheston, *On convergence of vector-valued asymptotic martingales*, Zeit. Wahrscheinlichkeitstheorie verw. Gebiete *33* (1975), 55–59.

8. S. D. Chatterji, *Martingale convergence and the Radon-Nikodym theorem*, Math. Scand. *22* (1968), 21–41.

9. J. Dieudonné, *Sur un théorème de Jessen*, Fund. Math. *37* (1950), 242–248.

10. J. L. Doob, *Stochastic processes* (Wiley, New York, 1953).

11. A. Dvoretzky, *On stopping times directed convergence*, Bull. Amer. Math. Soc. *82*, No. 2 (1976), 347–349.

12. G. A. Edgar and L. Sucheston, *Amarts: A class of asymptotic martingales*, J. Multivariate Anal. *6* (1976), 193–221; 572–591.

13. ——— *The Riesz decomposition for vector-valued amarts*, Zeit. Wahrscheinlichkeitstheorie verw. Gebiete *36* (1976), 85–92.

14. ——— *On vector-valued amarts and dimension of Banach spaces*, Zeit. Wahrscheinlichkeitstheorie verw. Gebiete *39* (1977), 213–216.

15. ——— *Martingales in the limit and amarts*, Proc. Amer. Math. Soc. *67* (1977), 315–320.

16. N. Ghoussoub and L. Sucheston, *A Refinement of the Riesz decomposition for amarts and semi amarts*, J. Multivariate Analysis *8* (1978), 146–150.

17. C. A. Hayes and C. Y. Pauc, *Derivations and martingales* (Springer-Verlag, New York, 1970).

18. L. L. Helms, *Mean convergence of martingales*, Trans. Amer. Math. Soc. *87* (1958), 439–446.
19. U. Krengel and L. Sucheston, *On semiamarts, amarts, and processes with finite value*, Advances in Probability *4* (1978), 197–266.
20. K. Krickeberg, *Convergence of martingales with a directed index set*, Trans. Amer. Math. Soc. *83* (1956), 313–337.
21. ——— *Stochastische Konvergenz von Semimartingalen*, Math. Z. *66* (1957), 470–486.
22. ——— *Notwendige Konvergenzbedingungen bei Martingalen und verwandten Prozessen*, Transactions of the Second Prague conference on information theory, statistical decision functions, random processes [1959 Prague], (1960), 279–305 (Prague, Publishing House of the Czechoslovak Academy of Sciences).
23. K. Krickeberg and C. Pauc, *Martingales et dérivation*, Bull. Soc. Math. France *91* (1963), 455–544.
24. Ch. Lamb, *A ratio limit theorem for approximate martingales*, Can. J. Math. *25* (1973), 772–779.
25. A. G. Mucci, *Another Martingale convergence theorem*, Pacific J. Math. *64* (1976), 539–541.
26. J. Neveu, *Discrete parameter martingales* (North Holland, Amsterdam, 1975).
27. H. L. Royden, *Real analysis* (Macmillan Company, N.Y., 1968).
28. W. Rudin, *Real and complex analysis* (McGraw-Hill, 1970).
29. L. Sucheston, *On existence of finite invariant measures*, Math. Z. *86* (1964), 327–336.
30. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. *72* (1952), 46–66.

Ohio State University
Columbus, Ohio