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Theta Series, Eisenstein Series and Poincaré Series over Function Fields

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Abstract. We construct analogues of theta series, Eisenstein series and Poincaré series for function fields of one variable over finite fields, and prove their basic properties.

1 Introduction

1.1 Motivation

This paper is part of the project to prove analogues of the formulas of Gross-Zagier (see [8]) and Gross (see [7]) for function fields of one variable over finite fields. Both papers [7] and [8] have two main parts. The first part consists of the computation of certain pairings between Heegner points on modular curves, so it is geometric in nature. The second part is analytical: approximately it is the computation of Fourier coefficients of the unique holomorphic cuspidal newform of weight 2 and level N whose Petersson product with any other holomorphic cuspidal newform of weight 2 and the same level is equal to the central critical value (or derivative) of a certain L-function of the latter.

The aim of this paper is to construct the analogues of the three special types of automorphic forms appearing in these analytical computations and prove their basic properties which we will need later. Partially because of its technical nature we also tried to present this material in a simple and transparent way, therefore we might spend more time with certain parts than it is strictly necessary using the results available in the literature. The three special types of automorphic forms mentioned above are theta series, Eisenstein series and Poincaré series. The first two of these are necessary in the analogue of the Rankin-Selberg method. On the other hand Poincaré series can be used to construct linear generators of different linear sub-spaces of cusp forms, and so they can be used to compute the projection of automorphic forms into these sub-spaces. Our results are extensions of the results of [15], [16], and [17], where the special case of the rational function field is treated. In [18] these results were used to prove an analogue the Gross-Zagier formula in this special case with several other restrictions. Our methods are also similar, but it is necessary to treat all objects from the adélic point of view for the generalization.

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1.2 Contents

In Section 2 we collect some basic definitions and results on automorphic forms and their Fourier coefficients together from [11] and [22] for the convenience of the reader and to fix notation. The only new result is Proposition 2.8 which says that if the Fourier coefficients of a holomorphic family of automorphic forms can be continued analytically then the family also has analytical continuation.

The aim of Section 3 is to construct theta series and determine their conductor and central character. There are several constructions of theta series in the literature: in [5] it is constructed using the Weil representation, in [15] the analogue of the classical treatment is worked out in the special case of the rational function field and unramified central character. In both cases it is somewhat cumbersome to show that the series, as defined, is invariant under the congruence group of the required conductor. For example in [15] this is proved via a fairly complicated computation using the Poisson summation formula. We will choose a third way which is perhaps the simplest one: we apply Weil's converse theorem to induced representations to show the existence of attached automorphic forms. The determination of their conductor and central character follows easily from standard results of representation theory of finite groups.

Section 4 contains some preliminary results, some with proofs, mainly for the convenience of the reader. In Section 5 we compute the Fourier coefficients of certain Eisenstein series. Eisenstein series were studied in various generality in the papers [9], [10] and [14]. In particular the authors computed the constant terms in the Fourier expansion of these series, remarked that they are Hecke eigenforms, and gave a formula without proof for the eigenvalues (at least for primes not dividing the conductor) which could be used to reduce the problem above to computing just one Fourier coefficient. Because it is neither more complicated nor longer to compute all Fourier coefficients at once rather than to compute just one, and because we do not wish to burden our paper with references, we decided to prove our result independently of theirs.

The aim of Sections 6 and 7 is to study analogues of Poincaré series for function fields. In Section 6 after a preliminary result (Lemma 6.3) for which we could not find an adequate reference, we prove the convergence of Poincaré series and give a characterization using the Petersson product. In Section 7 we prove our main results which are two estimates of the Fourier coefficients of Poincaré series. The estimates are reduced to an estimate of local integrals which are reduced to estimates of Kloosterman sums. Again we gave a proof for the latter (Lemma 7.2) because of the lack of adequate reference. Here the adélic approach is essential, otherwise the computations would be too complicated.

1.3 Notation

In this paper, if not otherwise stated, we will use the following conventions, assumptions and notation. For any geometrically connected non-singular projective curve C over the finite field $\mathbf{f} = \mathbf{f}_q$ of characteristic p let $|C|$, $\mathrm{Div}(C)$ and $\mathrm{Pic}(C)$ denote the set of closed points, the divisor group and Picard group of an algebraic curve C,

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respectively. We will identify the set $|C|$ and the set of places of the function field of C. For any $x \in |C|$ let $\deg(x)$ denote its degree. For any $D \in \mathrm{Div}(C)$ let $[D]$ denote its linear equivalence class.

X will be a fixed geometrically connected smooth projective curve over the field $\mathbf{f}_q$. We denote the function field and the ring of adèles of X by F and $\mathbf{A}$, respectively. We denote by $|\cdot|$ the normalized absolute value of any adèle in $\mathbf{A}$. We denote by $\mathcal{O}$ the maximal compact sub-ring of $\mathbf{A}$. For any closed point v in $|X|$ we let F_v, $\mathbf{f}_v$ and $\mathcal{O}_v$ denote the corresponding completion of F, its constant field, and its discrete valuation ring, respectively. Let (a) denote the divisor of any idèle $a \in \mathbf{A}^*$. Sometimes we will drop the parentheses for the sake of convenience.

For any idèle, adèle, adèle-valued matrix or function defined on the above which decomposes as an infinite product of functions defined on the individual components the subscript v will denote the v-th component. For example $|\cdot|_v$ will denote the normalized absolute value of the field F_v. Similar convention will be applied to subsets of adèles and adèle-valued matrices. Sometimes, when it would be too inconvenient otherwise, we will drop the subscript v.

Let G denote the group scheme of invertible two by two matrices. Let B denote the group scheme of invertible upper triangular two by two matrices. Let P denote the group scheme of invertible upper triangular two by two matrices with 1 on the lower left corner. Let U denote the group scheme of invertible upper triangular two by two matrices with ones on the diagonal. Let Z denote the center of G.

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2 Automorphic Forms

2.1 Definition

By an automorphic form over F of central character ψ, (where ψ is a quasi-character of $\mathbf{A}^*$), we mean a continuous function $\phi\colon G(\mathbf{A}) \to \mathbf{C}$ satisfying the following properties:

(i) $\phi(\gamma g) = \phi(g)$ for all $\gamma \in G(F)$,
(ii) $\phi(gz) = \phi(g)\psi(z)$ for all $z \in Z(\mathbf{A})$,
(iii) ϕ is right $G(\mathcal{O})$-finite, *i.e.*, its right translates with respect to the elements of the maximal compact subgroup $G(\mathcal{O})$ of $G(\mathbf{A})$ generate a finite dimensional vector space over $\mathbf{C}$.

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Moreover, if for almost all $g \in G(\mathbf{A})$:

$$\int_{F\backslash\mathbf{A}} \phi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right)\, dx = 0,$$

where dx is a Haar measure on $\mathbf{A}$, we call ϕ a cusp form.

2.2 Definition

Let ψ be a quasi-character of $\mathbf{A}^*$ whose conductor divides $\mathfrak{n}$, where $\mathfrak{n}$ is an effective divisor in $\mathrm{Div}(X)$. By an automorphic form over F of level $\mathfrak{n}$, and central character ψ, we mean an automorphic form satisfying the property (iii*) instead of the weaker property (iii) above.

(iii*) $\phi(gk) = \phi(g)\psi(k)$ for all $k \in \mathbf{K}_0(\mathfrak{n})$, where

$$\mathbf{K}_0(\mathfrak{n}) = \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in G(\mathbf{A}) \;\middle|\; c \equiv 0 \bmod \mathfrak{n} \right\},$$

and the (well-defined) representation $\psi\colon \mathbf{K}_0(\mathfrak{n}) \to \mathbf{C}^*$ is given by:

$$\psi\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right) = \prod_{v|\mathfrak{n}} \psi_v(d_v) \quad \forall \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \mathbf{K}_0(\mathfrak{n}),$$

if $\mathfrak{n}$ is not the trivial divisor and 1, otherwise.

Let $\mathcal{A}(\mathfrak{n},\psi)$ (resp. $\mathcal{A}_0(\mathfrak{n},\psi)$) denote the space of automorphic forms (resp. cuspidal automorphic forms) of level $\mathfrak{n}$, and central character ψ.

Lemma 2.3 *Every $\phi \in \mathcal{A}(\mathfrak{n},\psi)$ is uniquely determined by its restriction to $P(\mathbf{A})$.*

Proof See [22, p. 19]. ∎

2.4

Let $\tau\colon F \setminus \mathbf{A} \to \mathbf{C}^*$ be a non-trivial additive character. Then every character of $F \setminus \mathbf{A}$ is of the form $x \mapsto \tau(\eta x)$ for some $\eta \in F$.

Proposition 2.5 *For every $\phi \in A(\mathfrak{n},\psi)$ there are functions $\phi^0\colon \mathrm{Pic}(X) \to \mathbf{C}$ and $\phi^*\colon \mathrm{Div}(X) \to \mathbf{C}$, the latter vanishing on non-effective divisors, such that*

$$\phi\left(\begin{pmatrix}y & x\\ 0 & 1\end{pmatrix}\right) = \phi^0(y) + \sum_{\eta\in F^*} \phi^*(\eta y \mathfrak{d}^{-1})\tau(\eta x),$$

for all $y \in \mathbf{A}^*$ and $x \in \mathbf{A}$, where $\mathfrak{d}$ is an idèle such that $\mathcal{D} = \mathfrak{d}\mathcal{O}$, where $\mathcal{D}$ is the $\mathcal{O}$-module defined as

$$\mathcal{D} = \{x \in \mathbf{A} \mid \tau(x\mathcal{O}) = 1\}.$$

Proof This is Proposition 1 of Chapter III in [22], p. 21. For proof see [22, pp. 19–20]. ■

Definition 2.6 The functions ϕ^0 and ϕ^* are called the Fourier coefficients of the automorphic form ϕ with respect to the additive character τ.

Lemma 2.7 *Assume that two functions $\phi^0\colon \operatorname{Pic}(X) \to \mathbf{C}$ and $\phi^*\colon \operatorname{Div}(X) \to \mathbf{C}$ are given, the latter vanishing on non-effective divisors. Then the sum*

$$\phi\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \phi^0(y) + \sum_{\eta \in F^*} \phi^*(\eta y \mathfrak{d}^{-1})\tau(\eta x)$$

has only finitely many non-zero terms over every compact set of $P(\mathbf{A})$, so it defines a continuous function on $P(\mathbf{A})$.

Proof This is Lemma 2 of Chapter V in [22], p. 35. For proof see [22, pp. 35–36]. ■

Proposition 2.8 *Let $U \subset V$ be two non-empty sub-domains of $\mathbf{C}$. Let $\psi\colon \mathbf{A}^* \to \mathbf{C}$, $\phi^0_s\colon \operatorname{Pic}(X) \to \mathbf{C}$ and $\phi^*_s\colon \operatorname{Div}(X) \to \mathbf{C}$ be a quasi-character of $\mathbf{A}^*$ and two functions, respectively, the latter two are holomorphic in s for all $s \in V$, and the last function vanishing on non-effective divisors such that*

$$\phi_s\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \phi^0_s(y) + \sum_{\eta \in F^*} \phi^*_s(\eta y \mathfrak{d})\tau(\eta x)$$

is the restriction of an automorphic form in $\mathcal{A}(\mathfrak{n}, \psi)$ for all $s \in U$. Then ϕ_s is the restriction of an automorphic form in $\mathcal{A}(\mathfrak{n}, \psi)$ for all $s \in V$, and for all $g \in G(\mathbf{A})$ the function $\Phi_s(g)$ is holomorphic in s.

Proof We first prove that $G(\mathbf{A}) = G(F)B(\mathbf{A})\mathbf{K}$ for every compact, open subgroup $\mathbf{K} = \prod_{v \in |X|} \mathbf{K}_v$. Take any element g of $G(\mathbf{A})$. There is a finite set S of places such that if $\mathbf{K}_v$ is not $G(\mathcal{O}_v)$ then $s \in S$. As the natural image of $G(F)$ in $\prod_{v \in S} G(F_v)$ is dense, there is a $\gamma \in G(F)$ such that the v-component of $\gamma^{-1}g$ is in $\mathbf{K}_v$ for all $v \in S$. But $\gamma^{-1}g$ is in $B(F_v)\mathbf{K}_v = B(F_v)G(\mathcal{O}_v)$ for all other v by the Iwasawa decomposition, so the claim follows. Let us start the proof of the proposition. The function ϕ_s is well defined for all s by Lemma 2.7. By what we have just proved it is sufficient to prove that for every $\gamma_1, \gamma_2 \in G(F)$, $b_1, b_2 \in P(\mathbf{A})$, $k_1, k_2 \in \mathbf{K}_0(\mathfrak{n})$ and $z_1, z_2 \in Z(\mathbf{A})$, if we have $\gamma_1 b_1 k_1 z_1 = \gamma_2 b_2 k_2 z_2$, then $\phi_s(b_1)\psi_s(k_1)\psi(z_1) = \phi_s(b_2)\psi_s(k_2)\psi(z_2)$ for all $s \in V$. But we know this to hold for all $s \in U$, so it holds for all $s \in V$ as well by unique continuation of holomorphic functions. The last assertion of the proposition is obvious. ■

3 Theta Series

Notation 3.1 For any field K let $\overline{K}$ denote its separable closure. For any Galois extension $F_2|F_1$ of global or local fields of positive characteristic let $W(F_2|F_1)$ denote the Weil group of the extension. For any global or local field K of positive characteristic let $W(K)$ simply denote $W(\overline{K}|K)$, the absolute Weil group of K. For any n-dimensional complex representation $\rho\colon W(K) \to \mathrm{GL}_n(\mathbf{C})$ of the absolute Weil group of the global or local field K of positive characteristic let $\det(\rho)$, $\mathrm{cond}(\rho)$ denote the determinant of ρ, and its Artin conductor, respectively. If the the field K is a global function field, let $L_x(\rho, t)$ denote the Euler factor of the Grothendieck L-series of ρ at x for each place x of K. If the representation above is 1-dimensional and the field K is a global function field, then let ρ also denote the corresponding grössencharacter of K by abuse of notation.

Definition 3.2 We will call the divisors $\mathfrak{m}$, $\mathfrak{n}$ on a non-singular projective curve C defined over a finite field relatively prime if their support is disjoint. We will call a function $f\colon \mathrm{Div}(C) \to \mathbf{C}$ multiplicative if it vanishes on non-effective divisors, and for every pair of relatively prime divisors $\mathfrak{n}$ and $\mathfrak{m}$ we have $f(\mathfrak{n}\mathfrak{m}) = f(\mathfrak{n})f(\mathfrak{n})$. Let ρ be an n-dimensional complex representation of the absolute Weil group of the function field of C. Then let ρ also denote by abuse of notation the unique multiplicative function $\rho\colon \mathrm{Div}(C) \to \mathbf{C}$ such that for each prime divisor $\mathfrak{p} \in |C|$ we have

$$\sum_{n=0}^{\infty} \rho(\mathfrak{p}^n) t^{\deg(\mathfrak{p})n} = L_{\mathfrak{p}}(\rho, t).$$

Theorem 3.3 *Let ρ be a 2-dimensional complex representation of the absolute Weil group of F. Then there is an automorphic form $\mathcal{L}(\rho)$ of level $\mathrm{cond}(\rho)$ and central character $\det(\rho)$ such that for each effective divisor $\mathfrak{m} \in \mathrm{Div}(X)$ we have*

$$\mathcal{L}(\rho)^*(\mathfrak{m}) = \rho(\mathfrak{m}).$$

If the representation ρ is irreducible, then $\mathcal{L}(\rho)$ can be chosen to be cuspidal.

Proof This is the theorem of Section 76 of Chapter XI in [22], and the remark immediately following it, p. 157. For proof see [22, pp. 145–157]. ■

Notation 3.4 Let Δ be a subgroup of finite index of a group Γ. If $\chi\colon \Delta \to \mathrm{Aut}(V)$ is a representation of Δ on a complex vector space V then we will denote the representation induced from χ to Γ by $\mathrm{Ind}_{\Gamma|\Delta}(\chi)$. If $\chi\colon \Gamma \to \mathrm{Aut}(V)$ is a representation of Γ on a complex vector space V then the restriction of χ to Δ will be denoted by $\mathrm{Res}_{\Delta}(\chi)$. For any element of $g \in \Gamma$ the conjugate subgroup $g\Delta g^{-1}$ will be denoted by Δ^g. For any representation $\chi\colon \Gamma \to \mathrm{Aut}(V)$ as above we will denote by χ^g the conjugate representation defined by $\chi^g(h) = \chi(ghg^{-1})$ for all $h \in \Gamma$.

Lemma 3.5 *If Δ is a subgroup of a finite group Γ and $\chi\colon \Delta \to \mathrm{Aut}(V)$ is a finite dimensional representation on a complex vector space V, then the determinant of the induced representation $\mathrm{Ind}_{\Gamma|\Delta}(\chi)$ is $s(\pi)^m t\big(\det(\chi)\big)$, where π is the regular permutation*

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representation of Γ on the left cosets of Δ, $s(\pi)$ is its sign, t the map induced on the dual groups by the transfer map $\Gamma^{ab} \to \Delta^{ab}$, and $m = \dim(V)$.

Proof See [2, Proposition 13.15, part (i), pp. 338–339]. ■

Lemma 3.6 *Let Δ, Ω be two subgroups of a finite group Γ and let $\chi\colon \Delta \to \mathrm{Aut}(V)$ be a finite dimensional representation on a complex vector space V. Let $S \subset \Gamma$ be a representative system of double cosets of the pair Δ, Ω. Then*

$$\mathrm{Res}_{\Omega}\big(\mathrm{Ind}_{\Gamma|\Delta}(\chi)\big) = \bigoplus_{g\in S} \mathrm{Ind}_{\Omega|\Omega\cap\Delta^{g^{-1}}}\big(\mathrm{Res}_{\Omega\cap\Delta^{g^{-1}}}(\chi^{g})\big).$$

Proof See [2, Theorem 10.13, pp. 237–238]. ■

Notation 3.7 Let Y be a separable connected double cover of X, and let K denote the the corresponding extension of F. Let $\epsilon\colon W(F) \to \mathbf{C}^*$ denote the unique non-trivial character with kernel $W(K) < W(F)$. Let $\mathfrak{f}$ denote the conductor of ϵ, the discriminant of the extension $K|F$. Let $\mathbf{N}(\mathfrak{m}) \in \mathrm{Div}(X)$ denote the norm of any effective divisor $\mathfrak{m} \in \mathrm{Div}(Y)$.

Proposition 3.8 *Let $\lambda\colon W(K) \to \mathbf{C}^*$ be a continuous one-dimensional representation of the Weil group of K. Then the following holds:*

(i) *$\mathrm{Ind}_{W(F)|W(K)}(\lambda)$ is irreducible if and only if λ is not a restriction of a representation of $W(F)$,*
(ii) *the determinant of $\mathrm{Ind}_{W(F)|W(K)}(\lambda)$ is $\epsilon t\big(\deg(\rho)\big)$, where t denotes the same homomorphism as in Lemma 3.5,*
(iii) *the Artin conductor of $\mathrm{Ind}_{W(F)|W(K)}(\lambda)$ is $\mathfrak{f}\mathbf{N}\big(\mathrm{cond}(\lambda)\big)$,*
(iv) *for all $x \in |X|$ we have*

$$L_x\big(\mathrm{Ind}_{W(F)|W(K)}(\lambda), t\big) = \prod_{y|x} L_y(\lambda, t^{\deg(y)/\deg(x)}),$$

where the product is over all y places of K above x.

Proof (i) By Criterion 10.25 in [2], p. 245, this induced representation is irreducible if and only if λ is not equal to λ^g for any $g \in W(K) - W(F)$, since two 1-dimensional representations are orthogonal if and only if they are different. Clearly, the latter holds if and only if λ is not a restriction of a representation of $W(F)$.

(ii) Let L be a normal extension of K such that the representation λ factors through the finite quotient $W(L|K)$. This group is a two index subgroup of $W(L|F)$. Since the sign representation of the permutation of $W(L|F)$ on cosets of $W(L|K)$ is ϵ, the claim follows from Lemma 3.5 applied to λ.

(iii) This claim is a special case of Proposition 6 of chapter 6 in [19], p. 111.

(iv) Let L denote the same normal extension as above. Let z be a place of L above x. Let Γ, Δ and Ω denote the Weil groups $W(L|F)$, $W(L|K)$ and the decomposition group $W(L_z|F_x)$, respectively. Then by Lemma 3.6 we have

$$\begin{aligned}\operatorname{Res}_{\Omega}\big(\operatorname{Ind}_{\Gamma|\Delta}(\lambda)\big) &= \bigoplus_{g\in S}\operatorname{Ind}_{\Omega|\Omega\cap\Delta^{g^{-1}}}\big(\operatorname{Res}_{\Omega\cap\Delta^{g^{-1}}}(\lambda^{g})\big)\\ &= \bigoplus_{g\in S}\operatorname{Ind}_{\Omega^{g}|\Omega^{g}\cap\Delta}\big(\operatorname{Res}_{\Omega^{g}\cap\Delta}(\lambda)\big)^{g^{-1}},\end{aligned}$$

where S is a representative system of double cosets of the pair Δ, Ω. The group Γ acts on all places of L above x transitively such that the action on the corresponding decomposition groups is by conjugation. For every place y of K above x there is a unique $g \in S$ such that z^g is above y. These facts together imply that

$$\begin{aligned}L_x\big(\operatorname{Ind}_{W(F)|W(K)}(\lambda), t\big) &= \prod_{g\in S} L_x\Big(\operatorname{Ind}_{\Omega^{g}|\Omega^{g}\cap\Delta}\big(\operatorname{Res}_{\Omega^{g}\cap\Delta}(\lambda)\big)^{g^{-1}}, t\Big)\\ &= \prod_{y|x} L_y(\lambda, t^{\deg(y)/\deg(x)}),\end{aligned}$$

where in the second equation we used (ii) of Proposition 3.8 on p. 530 in [3]. ■

Remark 3.9 A slight modification of the argument in Section 3 of Chapter XII in [13], pp. 236–239, can also be used to give a proof for part (iv) of the proposition above.

Theorem 3.10 *Let $\lambda\colon W(K) \to \mathbf{C}^*$ be a continuous one-dimensional representation of the Weil group of K. Then there exists an automorphic form Θ_λ of conductor $\mathbf{N}\big(\operatorname{cond}(\lambda)\big)\mathfrak{f}$ and central character $\epsilon t(\lambda)$ such that*

$$\Theta_\lambda^*(\mathfrak{n}) = \sum_{\mathbf{N}(\mathfrak{m})=\mathfrak{n}} \lambda(\mathfrak{m}).$$

If λ is not a restriction of a representation of $W(F)$, then Θ_λ can be chosen to be cuspidal.

Proof Let Θ_λ denote the automorphic form $\mathcal{L}(\rho)$ attached to the 2-dimensional induced representation $\rho = \operatorname{Ind}_{W(F)|W(K)}(\lambda)$ in Theorem 3.3. It is clear that Θ_λ has the required conductor and central character, and it can be chosen to be cuspidal, by (iii), (ii) and (i) of Proposition 3.8, respectively. To conclude the proof of the theorem, we only have to show that

$$\operatorname{Ind}_{W(F)|W(K)}(\lambda)(\mathfrak{n}) = \sum_{\mathbf{N}(\mathfrak{m})=\mathfrak{n}} \lambda(\mathfrak{m})$$

for each effective divisor $\mathfrak{n}$ of X. Because both functions in the equality above are multiplicative, we only have to check this for prime divisors, but this case follows at once from (iv) of Proposition 3.8. ■

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4 Some Preliminary Results

Definition 4.1 Let ν_1, ν_2 be two continuous 1-dimensional complex representations of $\mathcal{O}^*$ which are trivial on $\mathbf{f}_q^*$. Let $\pi(\nu_1,\nu_2)$ denote the right-regular representation of $G(\mathcal{O})$ on $\mathcal{B}(\nu_1,\nu_2)$, where the latter is the set of all locally constant complex-valued functions f on $G(\mathcal{O})$ such that

$$f\left(\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} g\right) = \nu_1(a)\nu_2(c)\cdot f(g) \quad \forall \begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \in B(\mathcal{O}),\ \forall g \in G(\mathcal{O}).$$

Let ω be a grössencharacter of F such that composition of ω with the natural map $\mathcal{O}^* \to F^* \setminus \mathbf{A}^*$ is $\nu_1\nu_2$. Let $\Lambda(\nu_1,\nu_2,\omega)$ denote the set of all pairs (λ_1,λ_2) of quasi-characters of $\mathbf{A}^*$ such that the composition of λ_1, λ_2 with the natural map above is ν_1,ν_2, respectively and $\lambda_1\lambda_2 = \omega$. As a principal homogeneous set over the group of quasi-characters of $\mathrm{Pic}(X)$, the set $\Lambda(\nu_1,\nu_2,\omega)$ has a natural structure of a Riemann surface. For any pair $(\lambda_1,\lambda_2) \in \Lambda(\nu_1,\nu_2,\omega)$, let $\mathrm{Re}(\lambda_1\lambda_2^{-1})$ denote the real number such that for any idèle $i \in \mathbf{A}^*$ we have $|\lambda_1\lambda_2^{-1}(i)| = |i|^{2\,\mathrm{Re}(\lambda_1\lambda_2^{-1})}$.

Definition 4.2 Let ν_1, ν_2 and ω be as above and let $(\lambda_1,\lambda_2) \in \Lambda(\nu_1,\nu_2,\omega)$ be arbitrary. Let $\pi(\lambda_1,\lambda_2)$ denote the right-regular representation of $G(\mathbf{A})$ on $\mathcal{B}(\lambda_1,\lambda_2)$, where the latter is the set of all locally constant complex-valued functions f on $G(\mathbf{A})$ such that

$$f\left(\begin{pmatrix} a & b \\ 0 & c\end{pmatrix} g\right) = \lambda_1(a)\lambda_2(c)\cdot f(g) \quad \forall \begin{pmatrix} a & b \\ 0 & c\end{pmatrix} \in B(\mathbf{A}),\ \forall g \in G(\mathbf{A}).$$

Since by the Iwasawa decomposition the restriction map $f \mapsto f|_{G(\mathcal{O})}$ induces a $G(\mathcal{O})$-equivariant isomorphism between $\mathcal{B}(\lambda_1,\lambda_2)$ and $\mathcal{B}(\nu_1,\nu_2)$, we note that $\pi(\lambda_1,\lambda_2)$ as a representation of $G(\mathcal{O})$ is isomorphic to $\pi(\nu_1,\nu_2)$. For every $\phi \in \mathcal{B}(\nu_1,\nu_2)$ and $(\lambda_1,\lambda_2) \in \Lambda(\nu_1,\nu_2,\omega)$, let $\phi(\lambda_1,\lambda_2)$ denote the corresponding element in $\mathcal{B}(\lambda_1,\lambda_2)$.

Proposition 4.3 *Let ν_1,ν_2 and ω be as above.*

(i) *For all $\phi \in \mathcal{B}(\nu_1,\nu_2)$, the series*

$$E(\lambda_1,\lambda_2)(\phi)(g) = \sum_{\gamma \in B(F)\setminus G(F)} \phi(\lambda_1,\lambda_2)(\gamma g),$$

is locally uniformly absolute convergent on the open set $\mathrm{Re}(\lambda_1\lambda_2^{-1}) > 1$ of the Riemann surface $\Lambda(\nu_1,\nu_2,\omega)$ and defines an automorphic form of central character ω,

(ii) *The series $E(\lambda_1,\lambda_2)(\phi)(g)$ is holomorphic in (λ_1,λ_2) on the set above,*

(iii) *The map $\phi(\lambda_1,\lambda_2) \mapsto E(\lambda_1,\lambda_2)(\phi)$ is $G(\mathbf{A})$-equivariant.*

Proof Claims (ii) and (iii) are obvious, as well as the second part of claim (i). To prove the other part of (i), we first note that it is sufficient to prove it only for $E(|\cdot|^s, |\cdot|^{-s})(\chi)$, where χ is the function which is identically one on $G(\mathcal{O})$. To see

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this first note that for every $\phi \in \mathcal{B}(\nu_1, \nu_2)$ there is a constant M such that $|\phi(k)| \leq M$ for every $k \in G(\mathcal{O})$ as that group is compact and ϕ is continuous. Then we have the estimate

$$\left|\phi(\lambda_1, \lambda_2)\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} k\right)\right| \leq M|\lambda_1(a)\lambda_2(c)| = M\chi(|\cdot|^s, |\cdot|^{-s})\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} k\right),$$

where $\operatorname{Re}(\lambda_1\lambda_2^{-1}) = s$, for all $\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in B(\mathbf{A})$ and for all $k \in G(\mathcal{O})$ using the definition $\mathcal{B}(\lambda_1, \lambda_2)$. But $G(\mathbf{A}) = B(\mathbf{A})G(\mathcal{O})$ by the Iwasawa decomposition so the estimate holds for all $g \in G(\mathbf{A})$. On the other hand the series $E(|\cdot|^s, |\cdot|^{-s})(\chi)$ is locally uniformly absolute convergent for $\operatorname{Re}(s) > 1$ by Theorem 1.4.4. of [9], p. 264. ■

Definition 4.4 Let $\mathfrak{X}$ be a Hausdorff topological space. A Borel measure on $\mathfrak{X}$ is a positive measure on the Borel sets of $\mathfrak{X}$. The latter will be denoted by $\mathfrak{B}(\mathfrak{X})$. If Γ is a group which acts on $\mathfrak{X}$ continuously on the left, then we say that μ is invariant with respect to Γ, if for every Borel set $S \in \mathfrak{B}(\mathfrak{X})$ and every $\gamma \in \Gamma$ we have $\mu(S) = \mu(\gamma S)$. Let Δ be a subgroup of Γ. We say that the action of Γ on $\mathfrak{X}$ relative to Δ is discrete, if the restriction of the action to Δ is trivial and every $x \in \mathfrak{X}$ has an open neighborhood U such that if for some $\gamma \in \Gamma$ we have that $U \cap \gamma U \neq \varnothing$, then $\gamma \in \Delta$.

Remarks 4.5 If a group Γ acts on a Hausdorff topological space $\mathfrak{X}$ discreetly and Δ is a subgroup of Γ, then the set $\Delta \setminus \mathfrak{X}$ inherits a natural Hausdorff topology such that the projection map $\mathfrak{X} \to \Delta \setminus \mathfrak{X}$ is a covering, and Γ acts on $\Delta \setminus \mathfrak{X}$ discreetly relative to Δ. Since $Z(F) \setminus G(F)$ acts on $Z(\mathbf{A}) \setminus G(\mathbf{A})$ discreetly, it acts on $Z(\mathbf{A})U(F) \setminus G(\mathbf{A})$ discreetly relative to $Z(F) \setminus U(F)$.

Proposition 4.6 *Let $\phi\colon Z(\mathbf{A})U(F)\setminus G(\mathbf{A})$ be a continuous function which has compact support. Then the series*

$$P(\phi)(g) = \sum_{\gamma \in Z(F)U(F)\setminus G(F)} \phi(\gamma g),$$

is uniformly absolute convergent and defines an automorphic form with trivial central character which is compactly supported as a function on $Z(\mathbf{A})G(F) \setminus G(\mathbf{A})$.

Proof Let $S \subset Z(\mathbf{A})U(F) \setminus G(\mathbf{A})$ be a compact subset containing the support of ϕ. It is clear that if $P(\phi)$ is convergent, then its support is contained in the image of S, hence it is compact. Therefore it is sufficient to prove that for each $g \in G(\mathbf{A})$ there is an open subgroup $\mathbf{U} < G(\mathbf{A})$ such that the number of $\gamma \in Z(F)U(F) \setminus G(F)$ such that $\phi(\gamma g u)$ is non-zero for some $u \in \mathbf{U}$ is finite. Since S is compact, it can be covered by finitely many open subsets V such that if for some $\gamma \in Z(F) \setminus G(F)$ we have that $V \cap \gamma V \neq \varnothing$, then $\gamma \in Z(F) \setminus U(F)$. We can assume that all sets V are right translates of an open subgroup $\mathbf{U}$ by refining the cover. This subgroup $\mathbf{U}$ will suffice. ■

The following proposition is an easy exercise in measure theory.

Proposition 4.7 *Let $\mathfrak{X}$ be a Hausdorff topological space which has a countable basis of compact open subsets. Let μ be a Borel measure on $\mathfrak{X}$ and let Γ be a group which acts on $\mathfrak{X}$ continuously on the right. Assume that μ is invariant respect to Γ. Let Δ be a subgroup of Γ such that the action of Γ on $\mathfrak{X}$ relative to Δ is discrete.*

Then there is a unique measure on $\Gamma \setminus \mathfrak{X}$, denoted μ (by abuse of notation) with the following property: if $f\colon \mathfrak{X} \to \mathbf{C}$ is a continuous function such that $\sum_{\gamma\in\Delta\setminus\Gamma} |f(\gamma x)|$ is locally uniformly convergent and

$$\int_{\Gamma\setminus\mathfrak{X}} \sum_{\gamma\in\Delta\setminus\Gamma} |f(\gamma x)|\, d\mu < \infty,$$

then

$$\sum_{\gamma\in\Delta\setminus\Gamma} f(\gamma x) \in L_1(\Gamma \setminus \mathfrak{X}, \mu), \quad f \in L_1(\mathfrak{X}, \mu),$$

and

$$\int_{\mathfrak{X}} f(x)\, d\mu = \int_{\Gamma\setminus\mathfrak{X}} \sum_{\gamma\in\Delta\setminus\Gamma} f(\gamma x)\, d\mu. \qquad \blacksquare$$

5 Fourier Coefficients of Eisenstein Series

Definition 5.1 Let K denote a separable ramified extension of F of degree two, and let ϵ, $\mathfrak{f}$ denote the corresponding grössencharacter and its conductor, respectively. Let H_s denote the unique complex-valued function on $G(\mathbf{A})$ which satisfies the following condition: for all $g \in G(\mathbf{A})$ which is of the form

$$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} k,$$

with

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B(\mathbf{A}), \quad k \in \mathbf{K}_0(\mathfrak{f}),$$

we have

$$H_s(g) = \left|\frac{a}{c}\right|^s \epsilon(c)\epsilon(k),$$

and $H_s(g) = 0$, otherwise.

By Proposition 4.3 the series

$$E_s(g) = \sum_{\gamma\in B(F)\setminus G(F)} H_s(\gamma g),$$

is locally uniformly absolute convergent if $\mathrm{Re}(s) > 1$, and defines an automorphic form with central character ϵ and level $\mathfrak{f}$.

Proposition 5.2 *Let $\tau\colon F\setminus \mathbf{A}\to \mathbf{C}^*$ be a non-trivial additive character such that the divisor $\mathfrak{d}$ defined in Proposition 2.3 has disjoint support from $\mathfrak{f}$. If E_s^0 and E_s^* denote the Fourier coefficients of the automorphic form E_s with respect to the additive character τ, then for* $\mathrm{Re}(s)>1$ *we have*

$$E_s^*(y)=\frac{\epsilon(y\mathfrak{d})|\mathfrak{d}|^{3/2-s}}{L(\epsilon,2s)|y|^{s-1}}\prod_{v|\mathfrak{f}}\kappa_v\epsilon_v(y_v)|y_v\pi_v|_v^{2s-1}\prod_{v\nmid f}\frac{1-\epsilon_v(y_v\pi_v)|y_v\pi_v|_v^{2s-1}}{1-\epsilon_v(\pi_v)|\pi_v|_v^{2s-1}},$$

where κ_v is defined by

$$\kappa_v=|\pi_v|\sum_{a\in\mathbf{f}_v^*}\epsilon_v(a/\pi_v)\tau_v(-a/\pi_v),$$

π_v is a local uniformizer for all $v\in|X|$, and $E_s^0(y)=|y|^s$.

Proof By the definition of Fourier coefficients $E_s^*(y\mathfrak{d}^{-1})$ is given by the formula

$$E_s^*(y\mathfrak{d}^{-1})=|\mathfrak{d}|^{1/2}\int_{F\setminus\mathbf{A}}E_s\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\tau(-x)\,dx,$$

where the product measure dx on $\mathbf{A}$ is chosen such that $\mathcal{O}_v$ has measure one for all place v. As $G(F)$ has the pair-wise disjoint decomposition

$$G(F)=B(F)\cup\bigcup_{u\in F}B(F)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&u\\0&1\end{pmatrix},$$

we get that

$$\int_{F\setminus\mathbf{A}}E_s\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\tau(-x)\,dx=\int_{F\setminus\mathbf{A}}H_s\left(\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\tau(-x)\,dx$$
$$+\int_{\mathbf{A}}H_s\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\tau(-x)\,dx.$$

By definition the first term is equal to

$$\int_{F\setminus\mathbf{A}}|y|^s\tau(-x)\,dx=0.$$

Using the identity

$$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}y&x\\0&1\end{pmatrix}=\begin{pmatrix}1&0\\0&y\end{pmatrix}\begin{pmatrix}0&1\\1&xy^{-1}\end{pmatrix}$$

and substituting x by xy^{-1}, the second integral can be written as

$$\int_{\mathbf{A}}H_s\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}y&x\\0&1\end{pmatrix}\right)\tau(-x)\,dx=\epsilon(y)|y|^{1-s}\prod_{v\in|X|}(V_s)_v(y_v)$$

where for $y \in F_v$ we define

$$(V_s)_v(y) = \int_{F_v} (H_s)_v\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right) \tau_v(-xy)\, dx.$$

For any $v \in |X|$ we have the formula

$$(H_s)_v\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right) = \begin{cases} \epsilon_v(x)|x|_v^{-2s} & \text{if } x \notin \mathcal{O}_v, \\ 1 & \text{if } v \nmid \mathfrak{f}, x \in \mathcal{O}_v, \\ 0 & \text{otherwise,} \end{cases}$$

upon using the identity

$$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} = \begin{pmatrix} x^{-1} & 1 \\ 0 & x \end{pmatrix} \begin{pmatrix} -1 & 0 \\ x^{-1} & 1 \end{pmatrix}$$

in the first case.

From this it follows that

$$(V_s)_v(y_v) = \int_{x \notin \mathcal{O}_v} \epsilon_v(x)|x|_v^{-2s}\tau_v(-xy_v)\, dx + \begin{cases} \int_{\mathcal{O}_v} \tau_v(-xy_v)\, dx & \text{if } v \nmid \mathfrak{f}, \\ 0 & \text{otherwise.} \end{cases}$$

We compute

$$\begin{aligned} \int_{x \notin \mathcal{O}_v} \epsilon_v(x)|x|_v^{-2s}\tau_v(-xy_v)\, dx &= \sum_{n \geq 1} \int_{\mathcal{O}_v^*} \epsilon_v(x\pi_v^{-n})|x\pi_v^{-n}|_v^{-2s}\tau_v(-xy_v\pi_v^{-n})\, d\pi_v^{-n}x \\ &= \sum_{n \geq 1} \epsilon_v(\pi_v)^n|\pi_v|_v^{(2s-1)n} \int_{\mathcal{O}_v^*} \epsilon_v(x)\tau(-xy_v\pi_v^{-n})\, dx. \end{aligned}$$

If $v \nmid \mathfrak{f}$, then since

$$\int_{\mathcal{O}_v} \tau_v(-xy_v)\, dx = \begin{cases} 1 & \text{if } v(y) \geq v(\mathfrak{d}), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\int_{\mathcal{O}_v^*} \tau_v(-xy_v\pi_v^{-n})\, dx = \begin{cases} 1 - |\pi_v|_v & \text{if } v(y) \geq v(\mathfrak{d}) + n, \\ -|\pi_v|_v & \text{if } v(y) = v(\mathfrak{d}) + n - 1, \\ 0 & \text{otherwise,} \end{cases}$$

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we obtain:

$$\begin{aligned}(V_s)_v(y) &= 1 + \sum_{1\le n\le v(y\mathfrak{d}^{-1})} \epsilon_v(\pi_v)^n |\pi_v|_v^{(2s-1)n}(1-|\pi_v|_v)\\ &\qquad - |\pi_v|_v\left(\epsilon_v(\pi_v)|\pi_v|_v^{(2s-1)}\right)^{(v(y\mathfrak{d}^{-1})+1)}\\ &= \left(1-\epsilon_v(\pi_v)|\pi_v|_v^{2s}\right)\sum_{n=0}^{v(y\mathfrak{d}^{-1})} \epsilon_v(\pi_v)^n|\pi_v|_v^{(2s-1)n}\\ &= \left(1-\epsilon_v(\pi_v)|\pi_v|_v^{2s}\right)\frac{1-\left(\epsilon_v(\pi_v)|\pi_v|_v^{(2s-1)}\right)^{(v(y\mathfrak{d}^{-1})+1)}}{1-\epsilon_v(\pi_v)|\pi_v|_v^{2s-1}},\end{aligned}$$

if $v(y)\ge v(\mathfrak{d})$ and 0, otherwise.

If $v|\mathfrak{f}$, then

$$\int_{\mathcal{O}_v^*}\epsilon_v(x)\tau_v(-xy_v\pi_v^{-n})\,dx = \begin{cases}\epsilon_v(y_v\pi_v^{-n})\kappa_v & \text{if } v(y)=n-1,\\ 0 & \text{otherwise.}\end{cases}$$

From this we obtain:

$$V_s(y) = \begin{cases}\epsilon_v(y_v)|y_v\pi_v|_v^{2s-1}\kappa_v & \text{if } v(y)\ge 0,\\ 0 & \text{otherwise.}\end{cases}$$

Putting this together, we have (for $y\in\mathfrak{d}\mathcal{O}$):

$$E_s^*(y\mathfrak{d}^{-1}) = \frac{\epsilon(y)|\mathfrak{d}|^{1/2}}{L(\epsilon,2s)|y|^{s-1}}\prod_{v|\mathfrak{f}}\kappa_v\epsilon_v(y_v)|y_v\pi_v|_v^{2s-1}\prod_{v\nmid f}\frac{1-\epsilon_v(y_v\mathfrak{d}_v^{-1}\pi_v)|y_v\mathfrak{d}_v^{-1}\pi_v|_v^{2s-1}}{1-\epsilon_v(\pi_v)|\pi_v|_v^{2s-1}}.$$

The Fourier coefficient $E_s^0(y)$ is given by the formula

$$E_s^0(y) = |\mathfrak{d}|^{1/2}\,|y|\int_{F\backslash\mathbf{A}} E_s\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)\,dx.$$

Using the Bruhat-Tits decomposition of $G(F)$ again we can write this integral as

$$\begin{aligned}E_s^0(y)|\mathfrak{d}|^{-1/2} &= \int_{\mathbf{A}/F} H_s\left(\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)\,dx\\ &\qquad + \int_{\mathbf{A}} H_s\left(\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} y & x\\ 0 & 1\end{pmatrix}\right)\,dx.\end{aligned}$$

The first term is equal to

$$\int_{F\backslash\mathbf{A}} |y|^s\,dx = |\mathfrak{d}|^{-1/2}\,|y|^s.$$

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Computing as above we can write the second term as

$$\int_{\mathbf{A}} H_s\left(\begin{pmatrix} 0 & 1 \\ y & x \end{pmatrix}\right) dx = \int_{\mathbf{A}} H_s\left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & xy^{-1} \end{pmatrix}\right) dx$$
$$= \epsilon(y)|y|^{1-s} \int_{\mathbf{A}} H_s\left(\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix}\right) dx,$$

where in the last equation we substituted x by xy^{-1}. This last expression only depends on the divisor class of y. But it is also the difference of two functions, the Fourier coefficient $|\mathfrak{d}|^{-1/2} E_s^0(y)$ and $|\mathfrak{d}|^{-1/2}|y|^s$. Since ϵ is ramified this can hold for this integral only if it is zero. We conclude that $E_s^0(y) = |y|^s$. ■

6 Poincaré Series

Lemma 6.1 *For any Borel measure μ on $Z(F_v) \setminus G(F_v)$ the following two properties are equivalent:*

(i) *μ is a left-invariant Haar-measure,*
(ii) *$\mu(Sk) = \mu(S)$ for all $k \in G(\mathcal{O}_v)$ and $S \in \mathfrak{B}\big(Z(F_v) \setminus G(F_v)\big)$, and $\mu(bS) = \mu(S)$ for all $b \in Z(F_v) \setminus B(F_v)$ and $S \in \mathfrak{B}\big(Z(F_v) \setminus G(F_v)\big)$.*

Proof We will first prove that (i) implies (ii). It is clear that the second property of (ii) holds. For every $k \in G(\mathcal{O}_v)$ the measure $\mu(\cdot\, k)$ is also a left-invariant Haar measure, therefore there is a positive real number $m(k)$ such that $\mu(Sk) = m(k)\mu(S)$ for all $S \in \mathfrak{B}\big(Z(F_v) \setminus G(F_v)\big)$. Since the function m is a continuous character on the compact group $G(\mathcal{O}_v)$, it must be constant, hence the first property in (ii) also holds.

In order to prove that (ii) implies (i) we first remark that the restriction of any μ satisfying (ii) to $\mathfrak{B}\big(Z(F_v) \setminus G(\mathcal{O}_v)\big)$ is a left-invariant Haar-measure of the group $Z(\mathcal{O}_v) \setminus G(\mathcal{O}_v)$, because it is right-invariant and the group is compact. This fact and the Iwasawa decomposition implies that $\mu(gS) = \mu(S)$ for all $g \in Z(F_v) \setminus G(F_v)$ and for all S which is a translate of a compact open group. But the latter generate the σ-algebra of Borel sets, hence this property holds for each Borel set as well. ■

Definition 6.2 Let $dk = \bigotimes dk_v$ be the Haar measure on $\mathbf{K}_0(1)$ with volume one on each component, let $dx = \bigotimes dx_v$ be the Haar measure on $\mathbf{A}$ such that $\mathcal{O}_v$ has volume one, and let $d^*x = \bigotimes d^*x_v$ be the Haar measure such that $\mathcal{O}_v^*$ has volume one. Now define the measure dg on $Z(\mathbf{A}) \setminus G(\mathbf{A})$ by the formula

$$\int_{Z(\mathbf{A})\setminus G(\mathbf{A})} f(g)\, dg = \int_{\mathbf{A}^*} \int_{\mathbf{A}} \int_{\mathbf{K}_0(1)} f\left(\begin{pmatrix} y & yx \\ 0 & 1 \end{pmatrix} k\right) dk\, dx\, d^*y.$$

Lemma 6.3 *The measure dg is a left-invariant Haar measure on $Z(\mathbf{A}) \setminus G(\mathbf{A})$.*

Proof The measure dg is a product of local measures which all satisfy (ii) of the lemma above, since dg satisfies it. Hence they are left-invariant, and so does dg. ■

Definition 6.4 Because dg is a left-invariant Haar measure, it induces a measure on $Z(\mathbf{A})G(F) \setminus G(\mathbf{A})$, which will be denoted by the same letter by abuse of notation. Let ω be a grössencharacter of F. For any pair of automorphic forms ϕ, ψ with central character ω, the integral

$$\int_{Z(\mathbf{A})G(F)\setminus G(\mathbf{A})} \phi(g)\overline{\psi}(g)\, dg,$$

if it is absolutely convergent, will be denoted by $\langle \phi, \psi \rangle$ and will be called the Petersson product of ϕ and ψ.

Definition 6.5 Let $\mathfrak{n}$, $\mathfrak{m}$ be two effective divisors on X. Let $\tau\colon F \setminus \mathbf{A} \to \mathbf{C}$ be a nontrivial additive character. Let $H^{\mathfrak{n},\mathfrak{m}}$ denote the unique complex-valued function on $G(\mathbf{A})$ which satisfies the following condition: for all $g \in G(\mathbf{A})$ which is of the form

$$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} k,$$

with

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B(\mathbf{A}), \quad k \in \mathbf{K}_0(\mathfrak{n}),$$

then

$$H^{\mathfrak{n},\mathfrak{m}}(g) = \tau(b/c),$$

if $(a/c) = \mathfrak{m}\mathfrak{d}$, and $H^{\mathfrak{n},\mathfrak{m}}(g) = 0$, otherwise.

Proposition 6.6

(i) *The series*

$$P^{\mathfrak{n},\mathfrak{m}}(g) = \sum_{\gamma \in Z(F)U(F)\setminus G(F)} H^{\mathfrak{n},\mathfrak{m}}(\gamma g),$$

is uniformly absolute convergent and defines an automorphic form with trivial central character and level $\mathfrak{n}$ *which is compactly supported as a function on the topological space* $Z(\mathbf{A})G(F) \setminus G(\mathbf{A})$.

(ii) *Let* f *be a cuspidal automorphic form of level* $\mathfrak{n}$ *and of trivial central character. Then we have:*

$$\langle f, P^{\mathfrak{n},\mathfrak{m}} \rangle = \frac{|\mathfrak{m}|^{-1}|\mathfrak{d}|^{-3/2} f^*(\mathfrak{m})}{[\mathbf{K}_0(1) : \mathbf{K}_0(\mathfrak{n})]},$$

where $[\mathbf{K}_0(1) : \mathbf{K}_0(\mathfrak{n})]$ *is the index of* $\mathbf{K}_0(\mathfrak{n})$ *in* $\mathbf{K}_0(1)$.

Proof By Proposition 4.6 it is sufficient to prove that $H^{\mathfrak{n},\mathfrak{m}}$ is compactly supported as a function on $Z(F)U(F) \setminus G(\mathbf{A})$ in order to prove that (i) holds, since $P^{\mathfrak{n},\mathfrak{m}}$ is $\mathbf{K}_0(\mathfrak{n})$-invariant on the right by definition. The support of this function is the image of the product $U(\mathbf{A})\mathbf{S}(\mathfrak{m})\mathbf{K}_0(\mathfrak{n})$, where the set $\mathbf{S}(\mathfrak{m})$ is defined as

$$\mathbf{S}(\mathfrak{m}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in B(\mathbf{A}) \;\middle|\; (a) = \mathfrak{m} \right\}.$$

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The images of all these sets are compact modulo $Z(\mathbf{A})U(F)$, since $\mathbf{S}(\mathfrak{m})$ and $\mathbf{K}_0(\mathfrak{n})$ are themselves compact, and because the image of $U(\mathbf{A})$ factors through the compact set $U(F)\backslash U(\mathbf{A})$. In order to show (ii), we first note that the function $f\cdot H^{\mathfrak{n},\mathfrak{m}}\colon Z(\mathbf{A})U(F)\backslash G(\mathbf{A}) \to \mathbf{C}$ has compact support, so the series

$$\sum_{\gamma\in Z(F)U(F)\backslash G(F)} |f(g)H^{\mathfrak{m},\mathfrak{n}}(\gamma g)|$$

is uniformly convergent, has compact support, hence defines a function which is Lebesgue-integrable by Proposition 4.6. Therefore by the definition of $P^{\mathfrak{n},\mathfrak{m}}$ and Proposition 4.7 we have that

$$\begin{aligned}
\langle f, P^{\mathfrak{n},\mathfrak{m}}\rangle &= \int_{Z(\mathbf{A})G(F)\backslash G(\mathbf{A})} f(g)\overline{P^{\mathfrak{m},\mathfrak{n}}}(g)\,dg\\
&= \int_{Z(\mathbf{A})U(F)\backslash G(\mathbf{A})} f(g)\overline{H^{\mathfrak{m},\mathfrak{n}}}(g)\,dg\\
&= \int_{\mathbf{A}^*}\int_{\mathbf{A}/F}\int_{\mathbf{K}_0(1)} (f\overline{H^{\mathfrak{m},\mathfrak{n}}})\left(\begin{pmatrix} y & yx\\ 0 & 1\end{pmatrix}k\right)dk\,dx\,d^*y\\
&= \int_{(y)=\mathfrak{m}\mathfrak{d}}\int_{yF\backslash\mathbf{A}}\int_{\mathbf{K}_0(\mathfrak{n})} f\left(\begin{pmatrix} y & yx\\ 0 & 1\end{pmatrix}\right)\tau(-x)\,dk\,dx\,d^*y\\
&= \frac{|\mathfrak{m}\mathfrak{d}|^{-1}}{[\mathbf{K}_0(1):\mathbf{K}_0(\mathfrak{n})]}\int_{\mathbf{A}/F} f\left(\begin{pmatrix} \mathfrak{m}\mathfrak{d} & x\\ 0 & 1\end{pmatrix}\right)\tau(-x)\,dk\,dx\,d^*y\\
&= \frac{|\mathfrak{m}|^{-1}|\mathfrak{d}|^{-3/2}f^*(\mathfrak{m})}{[\mathbf{K}_0(1):\mathbf{K}_0(\mathfrak{n})]},
\end{aligned}$$

where $\mathfrak{m}$ also denotes an idèle whose divisor is $\mathfrak{m}$, by abuse of notation. ■

7 Estimates of the Fourier Coefficients of Poincaré Series

Lemma 7.1 *There is a constant C depending only on X such that for any line bundle $\mathcal{L}\in\operatorname{Pic}(X)$ we have:*

$$r(\mathcal{L}) = \left|\{s\in H^0(\mathcal{L},X) \mid \text{the divisor } (s) \text{ has only even coefficients}\}\right| \le Cq^{\frac{\deg(\mathcal{L})}{2}}.$$

Proof A section $s\in H^0(\mathcal{L},X)$ is as above if and only if there is a unique effective divisor $D\in\operatorname{Div}(X)$ such that $2D=(s)$. This implies that

$$r(\mathcal{L}) \le q\left|\{D\in\operatorname{Div}(X) \mid D \text{ is effective, } 2[D]=\mathcal{L}\}\right|.$$

But we can estimate as follows:

$$\begin{aligned}
\left|\{D\in\operatorname{Div}(X) \mid D \text{ is effective, } 2[D]=\mathcal{L}\}\right| &\le \sum_{\substack{\mathcal{M}\in\operatorname{Pic}(X)\\ 2\mathcal{M}=\mathcal{C}}} q^{\dim(H^0(\mathcal{M},X))}\\
&\le |\operatorname{Pic}^0(X)|q^{1+\frac{\deg(\mathcal{L})}{2}}.
\end{aligned}$$

■

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Lemma 7.2 *Let $L = \mathbf{f}_{q^n}((t))$ be a field of Laurent series, let $\mathbf{D} = \mathbf{f}_{q^n}[[t]]$ be its discrete valuation ring and let $\chi\colon L \to \mathbf{C}$ be the additive character defined by the formula $\chi(\sum_{i=-N}^{\infty} a_i t^i) = \chi'(a_{-1})$, where $\chi'\colon \mathbf{f}_{q^n} \to \mathbf{C}$ is a nontrivial additive character. Then for any a, b, and $c \in \mathbf{D} - \{0\}$, for the sum*

$$S(a,b,c) = \sum_{x \in (\mathbf{D}/c\mathbf{D})^*} \chi\Big(\frac{ax^{-1} + bx}{c}\Big)$$

we have the estimate

$$|S(a,b,c)| \leq 2q^{\frac{n}{2}\nu(c) + n\min(\nu(a),\nu(b),\nu(c))},$$

where $\nu\colon L^* \to \mathbf{Z}$ is the valuation.

Proof We can write c as a product $c = t^m u$, where $u \in \mathbf{D}^*$. As we have that $S(a,b,c) = S(u^{-1}a, u^{-1}b, t^n)$, we can assume that $c = t^m$. Similarly, if $a = a't^r$, $b = b't^r$ and $c = c't^r$ for some $a', b', c' \in \mathbf{D}$ and positive r, then we have $S(a,b,c) = q^{nr}S(a',b',c')$, so we can assume that one of a and b, for example a is in $\mathbf{D}^*$. Then there are three cases:

(i) $c = t^{2m}$ for some positive m. Choose a representative system S in the set $(\mathbf{D}/t^{2m}\mathbf{D})^*$ for the factor set $(\mathbf{D}/t^m\mathbf{D})^*$. As every element in $(\mathbf{D}/t^{2m}\mathbf{D})^*$ can be written uniquely as a product $y(1+xt^m)$ for some $y \in S$ and $x \in \mathbf{D}/t^m\mathbf{D}$, we can compute as follows:

$$\begin{aligned}\sum_{x \in (\mathbf{D}/t^{2m}\mathbf{D})^*} \chi\Big(\frac{ax^{-1} + bx}{t^{2m}}\Big) &= \sum_{y \in S}\sum_{x \in \mathbf{D}/t^m\mathbf{D}} \chi\left(\frac{ay^{-1}(1 - xt^m) + by(1 + xt^m)}{t^{2m}}\right)\\ &= \sum_{y \in S} \chi\left(\frac{ay^{-1} + by}{t^{2m}}\right) \sum_{x \in \mathbf{D}/t^m\mathbf{D}} \chi\left(\frac{(by - ay^{-1})x}{t^m}\right),\end{aligned}$$

where we also used that the inverse of $1 + xt^m$ is $1 - xt^m$. The sum

$$\sum_{x \in \mathbf{D}/t^m\mathbf{D}} \chi\left(\frac{(by - ay^{-1})x}{t^m}\right)$$

can be nonzero only if $by \equiv ay^{-1} \bmod t^m\mathbf{D}$, in which case it is equal to q^{nm}. Since a is in $\mathbf{D}^*$ there at most two $y \in S$ which satisfies this congruence, hence the estimate.

(ii) $c = t^{2m+1}$ for some positive m. Choose a representative system S in the set $(\mathbf{D}/t^{2m+1}\mathbf{D})^*$ for the factor set $(\mathbf{D}/t^{m+1}\mathbf{D})^*$. Using the same argument as above we have:

$$\sum_{x \in (\mathbf{D}/t^{2m+1}\mathbf{D})^*} \chi\left(\frac{ax^{-1} + bx}{t^{2m+1}}\right) = q^{nm} \sum_{\substack{y \in S\\ by - ay^{-1} \in t^m\mathbf{D}}} \chi\left(\frac{ay^{-1} + by}{t^{2m+1}}\right).$$

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Since our choice of S was arbitrary, we can assume that set of elements $y \in S$ which satisfy the congruence $by \equiv ay^{-1} \bmod t^m\mathbf{D}$, if it is not empty, is of the form $e_i(1 + xt^m)$, where e_1, e_2 are two fixed solutions of this congruence and $x \in \mathbf{f}_{q^n}$ arbitrary. As for such elements $(1 + xt^m)^{-1} = 1 - xt^m + x^2t^{2m}$, we get

$$\sum_{\substack{y\in S\\ by-ay^{-1}\in t^m\mathbf{D}}} \chi\left(\frac{ay^{-1}+by}{t^{2m+1}}\right) = \sum_{i=1,2}\sum_{x\in\mathbf{f}_{q^n}} \chi\left(\frac{ae_i^{-1}(1-xt^m+x^2t^{2m})+be_i(1+xt^m)}{t^{2m+1}}\right)$$

$$= \sum_{i=1,2}\chi\left(\frac{ae_i^{-1}+be_i}{t^{2m+1}}\right)\sum_{x\in\mathbf{f}_{q^n}}\chi\left(\frac{ae_i^{-1}x^2}{t}\right),$$

where terms involving xt^m can be eliminated because of the congruence satisfied by e_i. The two sums we got are well known to have absolute value $q^{\frac{n}{2}}$, which implies our claim.

(iii) $c = t$. In this case the sum is identical to the one considered in [11] and our estimate follows from the main result of that paper. ■

Corollary 7.3 *For any $v \in |X|$, $a, b \in \mathcal{O}_v$ and $c \in \mathcal{O}_v - \{0\}$ we have the estimate:*

$$\left|\int_{\mathcal{O}_v^*}\tau_v\left(\frac{ax^{-1}+bx}{c}\right)dx\right| \leq 2\max(|\mathfrak{d}_v|_v, |\mathfrak{d}_v|_v^{-1})|c|_v^{1/2}\min(|a|_v, |b|_v, |c|_v)^{-1}.$$

Proof Define the character $\widehat{\tau}_v\colon F_v \to \mathbf{C}$ by the formula $\widehat{\tau}_v(x) = \tau_v(\mathfrak{d}_v x)$. Then we can write

$$\int_{\mathcal{O}_v^*}\tau_v\left(\frac{ax^{-1}+bx}{c}\right)dx = \begin{cases}\int_{\mathcal{O}_v^*}\widehat{\tau}_v\left(\frac{ax^{-1}+bx}{\mathfrak{d}_v c}\right)dx, & \text{if } |\mathfrak{d}_v| \leq 1,\\ \int_{\mathcal{O}_v^*}\widehat{\tau}_v\left(\frac{a\mathfrak{d}_v^{-1}x^{-1}+b\mathfrak{d}_v^{-1}x}{c}\right)dx, & \text{if } |\mathfrak{d}_v| \geq 1.\end{cases}$$

Since for any $\alpha, \beta \in \mathcal{O}_v$, $\gamma \in \mathcal{O}_v - \{0\}$ we have

$$\int_{\mathcal{O}_v^*}\widehat{\tau}_v\left(\frac{\alpha x^{-1}+\beta x}{\gamma}\right)dx = |\gamma|_v\sum_{x\in(\mathcal{O}_v/\gamma\mathcal{O}_v)^*}\widehat{\tau}_v\left(\frac{\alpha x^{-1}+\beta x}{\gamma}\right)$$

and the sum on the right is of the type considered in Lemma 7.2, we have the claim above as a corollary. ■

Proposition 7.4 *For any $y \in \mathbf{A}^*$ and positive ϵ there is a constant $C(y,\epsilon)$ depending only on y and ϵ such that for any effective divisor $\mathfrak{n}$ and $\mathfrak{m}$ we have the estimates:*

$$|(P^{\mathfrak{n},\mathfrak{m}})^*(y)| \leq C(y,\epsilon)|\mathfrak{m}|^{-3/4-\epsilon}.$$

Proof We will denote by $\mathfrak{n}$, $\mathfrak{m}$ an idèle as well, whose divisor is $\mathfrak{n}$, $\mathfrak{m}$, respectively, by abuse of notation. We will really estimate $(P^{\mathfrak{n},\mathfrak{m}})^*(y\mathfrak{d})$. We can assume that $(y\mathfrak{d})$ is effective. The Fourier coefficient is given by the formula

$$(P^{\mathfrak{n},\mathfrak{m}})^*(y\mathfrak{d}) = \int_{F\backslash \mathbf{A}} P^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \tau(-x)\, dx,$$

where the measure dx on $\mathbf{A}$ is the usual one.

As $G(F)$ has the pair-wise disjoint decomposition

$$G(F) = \bigcup_{u\in F^*} Z(F)U(F)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \cup \bigcup_{u\in F^*, w\in F} Z(F)U(F)\begin{pmatrix} 0 & u \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix},$$

we have

$$\begin{aligned}(P^{\mathfrak{n},\mathfrak{m}})^*(y\mathfrak{d}) &= \sum_{u\in F^*} \int_{F\backslash \mathbf{A}} H^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} uy & ux \\ 0 & 1 \end{pmatrix}\right) \tau(-x)\, dx \\ &\quad + \sum_{u\in F^*} \prod_{v\in |X|} \int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u_v \\ y_v & x \end{pmatrix}\right) \tau_v(-x)\, dx.\end{aligned}$$

For the first term we have:

$$\begin{aligned}\left|\sum_{u\in F^*} \int_{F\backslash \mathbf{A}} H^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} uy & ux \\ 0 & 1 \end{pmatrix}\right) \tau(-x)\, dx\right| &= \left|\sum_{(uy)=(\mathfrak{m}\mathfrak{d})} \int_{F\backslash \mathbf{A}} \tau\big((u-1)x\big)\, dx\right| \\ &\leq q \int_{F\backslash \mathbf{A}} dx.\end{aligned}$$

To estimate the second term we first prove a lemma.

Lemma 7.5 *For any $u, y \in F_v^*$ we have*

$$\int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u \\ y & x \end{pmatrix}\right) \tau_v(-x)\, dx = \begin{cases} \int_{x\in y\mathcal{O}_v} \tau_v(x)\, dx, \\ \int_{|x|_v^2 = |\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v} \tau_v\left(\frac{u}{x} - x\right)\, dx, \\ 0, \end{cases}$$

if $v(u) = v(y) + v(\mathfrak{m}\mathfrak{d})$ and $v(\mathfrak{n}) = 0$, $v(y) + v(\mathfrak{m}\mathfrak{d}) - v(u)$ is even and $v(u) \leq v(y) + v(\mathfrak{m}\mathfrak{d}) - \max\big(1, v(\mathfrak{n})\big)$, and otherwise, respectively.

Proof Write the integral as a sum of

$$I_v(u, y, \mathfrak{m}, \mathfrak{n}) = \int_{xy^{-1}\in \mathcal{O}_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u \\ y & x \end{pmatrix}\right) \tau_v(-x)\, dx$$

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and

$$J_v(u,y,\mathfrak{m},\mathfrak{n}) = \int_{yx^{-1}\in\pi_v\mathcal{O}_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix}0 & u\\ y & x\end{pmatrix}\right)\tau_v(-x)\,dx.$$

For any u, x and y as above we have the product decompositions:

$$\begin{pmatrix}0 & u\\ y & x\end{pmatrix} = \begin{pmatrix}u & 0\\ 0 & y\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & xy^{-1}\end{pmatrix}$$

and

$$\begin{pmatrix}0 & u\\ y & x\end{pmatrix} = \begin{pmatrix}uyx^{-1} & u\\ 0 & x\end{pmatrix}\begin{pmatrix}-1 & 0\\ yx^{-1} & 1\end{pmatrix}.$$

Since in the first product the first matrix is in $B(F_v)$ and the second is in $G(\mathcal{O}_v)$ when $xy^{-1}\in\mathcal{O}_v$, we can write the first integral above as follows:

$$\begin{aligned} I_v(u,y,\mathfrak{m},\mathfrak{n}) &= \int_{xy^{-1}\in\mathcal{O}_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix}u & 0\\ 0 & y\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & xy^{-1}\end{pmatrix}\right)\tau_v(-x)\,dx\\ &= \int_{x\in y\mathcal{O}_v}\tau_v(x)\,dx, \end{aligned}$$

if $v(u)=v(y)+v(\mathfrak{m}\mathfrak{d})$, and $v(\mathfrak{n})=0$ and $I_v(u,y,\mathfrak{m},\mathfrak{n})=0$ otherwise. The second condition for $I_v(u,y,\mathfrak{m},\mathfrak{n})$ to be non-zero is necessary to have the second matrix in the decomposition above to be in $K_0(\mathfrak{n})_v$.

Similarly in the second product the first matrix is in $B(F_v)$ and the second is in $G(\mathcal{O}_v)$ when $yx^{-1}\in\pi_v\mathcal{O}_v$, so we have:

$$\begin{aligned} J_v(u,y,\mathfrak{m},\mathfrak{n}) &= \int_{yx^{-1}\in\pi_v\mathcal{O}_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix}\frac{uy}{x} & u\\ 0 & x\end{pmatrix}\begin{pmatrix}-1 & 0\\ yx^{-1} & 1\end{pmatrix}\right)\tau_v(-x)\,dx\\ &= \int_{|x|_v^2=|\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v}\tau_v\Big(\frac{u}{x}-x\Big)\,dx, \end{aligned}$$

if $v(\mathfrak{m}\mathfrak{d})-v(u)-v(y)$ is even and $\max\big(1,v(\mathfrak{n})\big)\le 1/2\big(v(\mathfrak{m}\mathfrak{d})+v(y)-v(u)\big)$, and $J_v(u,y,\mathfrak{m},\mathfrak{n})=0$ otherwise.

The conditions above make it impossible that $I_v(u,y,\mathfrak{m},\mathfrak{n})$ and $J_v(u,y,\mathfrak{m},\mathfrak{n})$ are both non-zero, hence the lemma follows. ■

Now we will estimate the infinite sum:

$$\sum_{u\in F^*}\prod_{v\in|X|}\int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix}0 & u_v\\ y_v & x\end{pmatrix}\right)\tau_v(-x)\,dx.$$

By Lemma 7.5 the term for u can only be non-zero, if the divisor $(\frac{y\mathfrak{m}\mathfrak{d}}{u})$ is effective and has only even coefficients. By Lemma 7.1 this implies that the number of those u for which the term is non-zero is at most $C|y\mathfrak{m}\mathfrak{d}|^{-1/2}$ for some constant C depending only on X.

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Now we will estimate the individual terms. Fix u and let $\pi_1, \pi_2 \in F_v$ such that $|\pi_1|_v = |\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v^{1/2}$ and $|\pi_2|_v = \min(|\mathfrak{d}|_v, |\mathfrak{d}|_v^{-1})|\frac{y\mathfrak{m}\mathfrak{d}}{u}|_v^{1/2}$. It is clear that $|\pi_2|_v \leq 1$. Then the integral

$$\int_{|x|_v^2=|\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v} \tau_v\Big(\frac{u}{x} - x\Big)\, dx = \Big|\frac{uy}{\mathfrak{m}\mathfrak{d}}\Big|_v^{1/2} \int_{\mathcal{O}_v^*} \tau_v\left(\frac{u\pi_2\pi_1^{-1}x^{-1} - \pi_1\pi_2 x}{\pi_2}\right) dx,$$

can be estimated using Corollary 7.3, since

$$|u\pi_2\pi_1^{-1}|_v = \left|\frac{u\mathfrak{m}\mathfrak{d}}{y}\right|_v^{1/2} \left|\frac{y\mathfrak{m}\mathfrak{d}}{u}\right|_v^{1/2} \min(|\mathfrak{d}|_v, |\mathfrak{d}|_v^{-1}) \leq |\mathfrak{m}|_v \leq 1$$

and

$$|\pi_1\pi_2|_v = \left|\frac{uy}{\mathfrak{m}\mathfrak{d}}\right|_v^{1/2} \left|\frac{y\mathfrak{m}\mathfrak{d}}{u}\right|_v^{1/2} \min(|\mathfrak{d}|_v, |\mathfrak{d}|_v^{-1}) \leq |y\mathfrak{d}|_v \leq 1,$$

as follows:

$$\left|\int_{|x|_v^2=|\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v} \tau_v\Big(\frac{u}{x} - x\Big)\, dx\right| \leq 2\left|\frac{uy^3}{\mathfrak{m}\mathfrak{d}}\right|_v^{1/4} |y\mathfrak{d}|_v.$$

Using this estimate for the factor of the term for those $v \in |X|$ for which we have $|\frac{y\mathfrak{m}\mathfrak{d}}{u}|_v < 1$ and the trivial estimate by $|y|_v\, |\frac{y\mathfrak{m}\mathfrak{d}}{u}|_v^{-1/4}$ otherwise, we get the estimate

$$\begin{aligned}\left|\prod_{v\in|X|} \int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u_v \\ y_v & x\end{pmatrix}\right) \tau_v(-x)\, dx\right| &\leq C'(y) 2^{t((\frac{y\mathfrak{m}\mathfrak{d}}{u}))} \left|\frac{y\mathfrak{m}\mathfrak{d}}{u}\right|^{-1/4} \\ &\leq C(y) 2^{t((\frac{y\mathfrak{m}\mathfrak{d}}{u}))} |\mathfrak{m}|^{-1/4},\end{aligned}$$

where we used that $|u| = 1$, moreover $C(y)$ and $C'(y)$ are constants depending on y, and for any divisor $\mathfrak{g}$ the number $t(\mathfrak{g})$ is defined as the number of prime divisors of $\mathfrak{g}$. Since for every positive ϵ there is a constant $C(\epsilon)$ such that for any effective divisor $\mathfrak{g}$ we have the estimate $2^{t(\mathfrak{g})} \leq C(\epsilon)|\mathfrak{g}|^{-\epsilon}$, using our estimates on the individual terms and the number of non-zero terms we get the inequality claimed in Proposition 7.4. ■

Proposition 7.6 *For any $y \in \mathbf{A}^*$, closed point $\infty \in |X|$, effective divisor $\mathfrak{m}_0$ having disjoint support from ∞, and positive ϵ there is a constant $C(y, \infty, \mathfrak{m}_0, \epsilon)$ depending only on y, ∞, $\mathfrak{m}_0$ and ϵ such that for any effective divisor $\mathfrak{n}$ and natural number n we have the estimate:*

$$|(P^{\mathfrak{n},\mathfrak{m}})^0(y)| \leq C(y, \infty, \mathfrak{m}_0, \epsilon)|\mathfrak{m}|^{-1/2-\epsilon},$$

where $\mathfrak{m} = \mathfrak{m}_0\infty^n$.

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Proof For the Fourier coefficient $(P^{\mathfrak{n},\mathfrak{m}})^0(y)$ we have

$$(P^{\mathfrak{n},\mathfrak{m}})^0(y) = \sum_{u\in F^*}\int_{F\backslash\mathbf{A}} H^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} uy & ux \\ 0 & 1\end{pmatrix}\right)dx + \sum_{u\in F^*}\prod_{v\in|X|}\int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u_v \\ y_v & x\end{pmatrix}\right)dx.$$

The first term is zero. To estimate the second term we need the following lemma whose proof is same as the proof of Lemma 7.5.

Lemma 7.7 *For any $u, y \in F_v^*$ we have*

$$\int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u \\ y & x\end{pmatrix}\right)dx = \begin{cases} |y|_v, \\ \int_{|x|_v^2=|\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v}\tau_v\left(\frac{u}{x}\right)dx, \\ 0, \end{cases}$$

if $v(u) = v(y) + v(\mathfrak{m}\mathfrak{d})$ and $v(\mathfrak{n}) = 0$, $v(y) + v(\mathfrak{m}\mathfrak{d}) - v(u)$ is even and $v(u) \leq v(y) + v(\mathfrak{m}\mathfrak{d}) - \max\bigl(1, v(\mathfrak{n})\bigr)$, and otherwise, respectively. ■

Now we will estimate for all $u \in F^*$ the infinite product:

$$\prod_{v\in|X|}\int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u_v \\ y_v & x\end{pmatrix}\right)dx.$$

The product can be non-zero only if the divisor $(\frac{y\mathfrak{m}\mathfrak{d}}{u})$ is effective and has only even coefficients, which we will assume about u. For every $v \in |X|$ choose a $\pi_v \in F_v$ such that $|\pi_v|_v = |\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v^{1/2}$. Then we have the estimate

$$\left|\int_{|x|_v^2=|\frac{uy}{\mathfrak{m}\mathfrak{d}}|_v}\tau_v\left(\frac{u}{x}\right)dx\right| = \left|\frac{uy}{\mathfrak{m}\mathfrak{d}}\right|_v^{1/2}\left|\int_{\mathcal{O}_v^*}\tau_v(u\pi_v^{-1}x)\,dx\right| \leq \left|\frac{uy}{\mathfrak{m}\mathfrak{d}}\right|_v^{1/2}\cdot\min\left(\left|\frac{y\mathfrak{d}}{u\mathfrak{m}}\right|_v^{1/2}, 1\right),$$

where we used the explicit expression for the second integral in the estimate. Using this estimate for those $v \in |X|$ for which we have $|\frac{y\mathfrak{m}\mathfrak{d}}{u}|_v < 1$ and the trivial estimate by $|y|_v\,|\frac{y\mathfrak{m}}{u\mathfrak{d}}|_v^{1/2}$ otherwise, we get the estimate

$$\left|\prod_{v\in|X|}\int_{F_v} H_v^{\mathfrak{n},\mathfrak{m}}\left(\begin{pmatrix} 0 & u_v \\ y_v & x\end{pmatrix}\right)dx\right| \leq C(y)|\mathfrak{m}|^{-1/2}\prod_{v\in|X|}\min\left(\left|\frac{y\mathfrak{d}}{u\mathfrak{m}}\right|_v^{1/2}, 1\right),$$

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where $C(y)$ is a constant depending on y. But

$$\prod_{v\in|X|}\min\left(\left|\frac{y\mathfrak{d}}{u\mathfrak{m}}\right|_v^{1/2},1\right)\leq\prod_{v\in|X|-\{\infty\}}\min\left(\left|\frac{y\mathfrak{d}}{u\mathfrak{m}_0}\right|_v^{1/2},1\right)$$

$$\leq\prod_{v\in|X|-\{\infty\}}\left|\frac{y\mathfrak{d}}{u\mathfrak{m}_0}\right|_v^{1/2}=C(y,\mathfrak{m}_0)|u|_\infty^{1/2},$$

where $C(y,\mathfrak{m}_0)$ is a constant depending on y and $\mathfrak{m}_0$. By Lemma 7.1 for all natural number l the number of such u which satisfies the assumptions above and $|u|_\infty^{1/2}=q^{-\deg(\infty)l}=q^{-ml}$, is at most $C'(y,\mathfrak{m}_0)q^{ml}$, where $C'(y,\mathfrak{m}_0)$ is a constant depending on y and $\mathfrak{m}_0$. Also this set is empty if $2l\geq n+C''(y,\mathfrak{m}_0)$ or $2l\leq -n-C''(y,\mathfrak{m}_0)$, where $C''(y,\mathfrak{m}_0)$ is a constant depending on y and $\mathfrak{m}_0$, so we get

$$|(P^{\mathfrak{n},\mathfrak{m}})^0(y)|\leq|\mathfrak{m}|^{-1/2}C(y)C(y,\mathfrak{m}_0)C'(y,\mathfrak{m}_0)\big(n+C''(y,\mathfrak{m}_0)\big)\,. \qquad\blacksquare$$

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