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Part 8. Markov processes and renewal theory

QUASISTOCHASTIC MATRICES AND MARKOV RENEWAL THEORY

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BY GEROLD ALSMEYER

Abstract

Let $\mathcal{S}$ be a finite or countable set. Given a matrix $F = (F_{ij})_{i,j \in \mathcal{S}}$ of distribution functions on $\mathbb{R}$ and a quasistochastic matrix $Q = (q_{ij})_{i,j \in \mathcal{S}}$, i.e. an irreducible nonnegative matrix with maximal eigenvalue 1 and associated unique (modulo scaling) positive left and right eigenvectors u and v, the matrix renewal measure $\sum_{n \geq 0} Q^n \otimes F^{*n}$ associated with $Q \otimes F := (q_{ij} F_{ij})_{i,j \in \mathcal{S}}$ (see below for precise definitions) and a related Markov renewal equation are studied. This was done earlier by de Saporta (2003) and Sgibnev (2006, 2010) by drawing on potential theory, matrix-analytic methods, and Wiener–Hopf techniques. In this paper we describe a probabilistic approach which is quite different and starts from the observation that $Q \otimes F$ becomes an ordinary semi-Markov matrix after a harmonic transform. This allows us to relate $Q \otimes F$ to a Markov random walk $\{(M_n, S_n)\}_{n \geq 0}$ with discrete recurrent driving chain $\{M_n\}_{n \geq 0}$. It is then shown that renewal theorems including a Choquet–Deny-type lemma may be easily established by resorting to standard renewal theory for ordinary random walks. The paper concludes with two typical examples.

Keywords: Quasistochastic matrix; Markov random walk; Markov renewal equation; Markov renewal theorem; spread out; Stone-type decomposition; age-dependent multi-type branching process; random difference equation; perpetuity

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1. Introduction and main results

Quasistochastic matrices (see below for the formal definition) constitute a generalization of stochastic matrices and, thus, of transition matrices of Markov chains with countable state space. In applications, two of which may be found in the final section of this paper, such matrices appear when studying the limit behaviour of certain functionals of processes which are driven by discrete Markov chains. These processes, called *Markov random walks* or *Markov-additive processes*, are characterized by having increments (referring only to the additive part) which are conditionally independent given the driving chain. Moreover, the conditional distribution of the nth increment depends only on the state of the chain at times $n - 1$ and n. Aiming at limit results as just mentioned, our main purpose is to show that, by using a harmonic transform, quasistochasticity may easily be reduced to stochasticity and, thus, to ordinary transition matrices. This in turn allows the use of more intuitive probabilistic arguments instead of analytic ones. Further information follows below after a description of the basic setup.

First we define quasistochasticity. Let $\mathcal{S} = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$ or $\mathcal{S} = \mathbb{N}$. Suppose that we are given an irreducible nonnegative matrix $Q = (q_{ij})_{i,j \in \mathcal{S}}$ with maximal eigenvalue 1

for which there exist unique positive left and right eigenvectors $u = (u_i)_{i \in \mathcal{S}}$, $v = (v_i)_{i \in \mathcal{S}}$ modulo scaling,

$$u^\top Q = u^\top \quad \text{and} \quad Qv = v.$$

A matrix of this kind will be called *quasistochastic* hereafter. If $\mathcal{S}$ is finite or, more generally, $\sum_{i \in \mathcal{S}} u_i < \infty$ and $u^\top v < \infty$, strict uniqueness is rendered upon choosing the normalization

$$\sum_{i \in \mathcal{S}} u_i = 1 \quad \text{and} \quad u^\top v = \sum_{i \in \mathcal{S}} u_i v_i = 1. \tag{1.1}$$

Note that, under these assumptions, all powers $Q^n =: (q_{ij}^{(n)})_{i,j \in \mathcal{S}}$ are also nonnegative matrices with finite entries (plainly, a nontrivial statement only if $\mathcal{S}$ is infinite).

In the example that first comes to mind, Q is the transition matrix of a recurrent discrete Markov chain on $\mathcal{S}$ and, thus, a proper stochastic matrix for which the left eigenvector u is the essentially unique stationary measure of the chain. In the positive recurrent case, we can choose u to be the unique stationary distribution and $v = (1, 1, \ldots)^\top$.

Next, let F_{ij} for $i, j \in \mathcal{S}$ be proper distribution functions on $\mathbb{R}$, thus nondecreasing, and right continuous with limits 0 at $-\infty$ and 1 at $+\infty$. Define the matrix function

$$\mathbb{R} \ni t \ \mapsto \ Q \otimes F(t) = ((Q \otimes F)_{ij}(t))_{i,j \in \mathcal{S}} := (q_{ij} F_{ij}(t))_{i,j \in \mathcal{S}},$$

where $F(t) := (F_{ij}(t))_{i,j \in \mathcal{S}}$. If $B(t) = (B_{ij}(t))_{i,j \in \mathcal{S}}$ denotes another matrix of real-valued functions, the convolution $(Q \otimes F) * B$ of $Q \otimes F(t)$ and $B(t)$ is defined by

$$((Q \otimes F) * B)_{ij}(t) := \sum_{k \in \mathcal{S}} \int_{\mathbb{R}} B_{kj}(t - x) \, (Q \otimes F)_{ik}(\mathrm{d}x), \qquad i, j \in \mathcal{S}, \ t \in \mathbb{R},$$

provided that the integrals exist. Since, for all $i, j \in \mathcal{S}$,

$$((Q \otimes F) * (Q \otimes F))_{ij}(t) = \sum_{k \in \mathcal{S}} q_{ik} q_{kj} F_{ik} * F_{kj}(t) \ \leq \ \sum_{k \in \mathcal{S}} q_{ik} q_{kj} \ = \ q_{ij}^{(2)},$$

it follows that $(Q \otimes F)^{*2}$ exists (as a componentwise finite-valued function), and then, using induction over n, the same is true for $(Q \otimes F)^{*n}$ which is defined recursively by

$$(Q \otimes F)^{*n}(t) = (Q \otimes F) * (Q \otimes F)^{*(n-1)}(t), \qquad t \in \mathbb{R},$$

for $n \geq 1$, where $A^{*0}(t)$ equals the identity matrix for each $t \geq 0$ and any matrix function A. The same induction argument also shows that

$$(Q \otimes F)^{*n}(t) = (q_{ij}^{(n)} F_{ij}^{*n}(t))_{i,j \in \mathcal{S}} = Q^n \otimes F^{*n}(t), \qquad t \in \mathbb{R}, \ n \in \mathbb{N}_0.$$

Of particular interest in this work is the *matrix renewal measure associated with $Q \otimes F$*, namely the $\mathcal{S} \times \mathcal{S} \times \mathbb{R}$-valued measure for which

$$\mathbb{V}((t, t+h]) := \sum_{n \geq 0} ((Q \otimes F)^{*n}(t+h) - (Q \otimes F)^{*n}(t)), \qquad t \in \mathbb{R}, \ h > 0,$$

under conditions ensuring that the entries of $\mathbb{V} = (\mathbb{V}_{ij})_{i,j \in \mathcal{S}}$ are Radon measures. The matrix measure $\mathbb{V}$ arises in connection with the solution $Z(t) = (Z_i(t))_{i \in \mathcal{S}}$ of a system of renewal equations, namely,

$$Z_i(t) = z_i(t) + \sum_{j \in \mathcal{S}} q_{ij} \int_{\mathbb{R}} Z_j(t - x) \, F_{ij}(\mathrm{d}x), \qquad t \in \mathbb{R}, \ i \in \mathcal{S},$$

written shortly as $Z = z + (Q \otimes F) * Z$, where $z(t) = (z_i(t))_{i \in \mathcal{S}}$ is a vector of real-valued functions. Indeed, if

$$Z(t) = \mathbb{V} * z(t) = (\mathbb{V}_i * z(t))_{i \in \mathcal{S}} = \left(\sum_{j \in \mathcal{S}} q_{ij}^{(n)} \int_{\mathbb{R}} z_j(t - x) \, F_{ij}^{*n}(\mathrm{d}x) \right)_{i \in \mathcal{S}}$$

exists for all $t \in \mathbb{R}$, then it forms a solution which is unique under additional assumptions that we sketch around Theorem 1.4 below.

Apart from allowing $\mathcal{S}$ to be infinite, our setup is the same as in the papers by de Saporta [12] and Sgibnev [26], who derived a Blackwell-type renewal theorem for $\mathbb{V}$ and determined the asymptotic behaviour of $Z(t) = \mathbb{V} * z(t)$ under appropriate conditions. De Saporta's approach is based on potential theory and rather technical, while Sgibnev used a matrix-analytic approach in combination with a matrix Wiener–Hopf factorization as described in [3]. The main purpose of this paper is to provide a different, purely probabilistic approach within the framework of discrete Markov renewal theory; this allows us to interpret assumptions in a more natural context and is also considerably simpler. This simplification is due to the fact that the core results in discrete Markov renewal theory, which deals with random walks driven (or modulated) by a recurrent Markov chain with discrete state space, can be easily deduced from classical renewal theory dealing with ordinary random walks with positive drift. This is done by drawing on stopping times, occupation measures, and regeneration techniques, and is demonstrated in Section 3, for it has apparently never been carried out in the literature (though a similar approach may already be found in the classical paper by Athreya *et al.* [6]). For basic definitions and properties of Markov random walks and Markov renewal processes with discrete driving chain, we refer the reader to the textbooks by Asmussen [4, pp. 206ff.] and Çinlar [10, Chapter 10], or [9].

Besides quasistochasticity, the following two standing assumptions about Q are made throughout this work:

(A1) $\sum_{n \geq 1} q_{ii}^{(n)} = \infty$ for some $i \in \mathcal{S}$,

(A2) $\mu := \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} u_i q_{ij} v_j \int x \, F_{ij}(\mathrm{d}x) > 0$.

In terms of the stochastic matrix P that is associated with Q and is introduced in Section 2 below, condition (A1) means that P is recurrent, while (A2) ensures that the Markov random walk associated with $P \otimes F$ has positive stationary drift (see Lemma 2.1). Since Q (and, thus, P) is irreducible, it follows by solidarity that (A1) actually implies that $\sum_{i \in \mathcal{S}} q_{ii}^{(n)} = \infty$ *for all $i \in \mathcal{S}$*. Moreover, (A1) automatically holds if $\mathcal{S}$ is finite.

We also need the following lattice-type condition on $Q \otimes F$; it is due to Shurenkov [28]. We call $Q \otimes F$ the d-arithmetic if d is the maximal positive number such that

$$F_{ij}(\gamma(j) - \gamma(i) + d\,\mathbb{Z}) = F_{ij}(\infty) \tag{1.2}$$

for all $i, j \in \mathcal{S}$ with $u_i q_{ij} v_j > 0$ and some measurable $\gamma \colon \mathcal{S} \to [0, d)$ which we call the shift function. If no such d exists, $Q \otimes F$ is called nonarithmetic. Note that (1.2) for all i and j as stated implies that

$$F_{ij}^{*n}(\gamma(j) - \gamma(i) + d\,\mathbb{Z}) = F_{ij}^{*n}(\infty)$$

for all $i, j \in \mathcal{S}$ with $u_i q_{ij}^{(n)} v_j > 0$ and all $n \in \mathbb{N}$. Consequently, if F_{ij}^{*n} is nonsingular with respect to the Lebesgue measure λ for some $n \in \mathbb{N}$ and $i, j \in \mathcal{S}$ with $u_i q_{ij}^{(n)} v_j > 0$, then $Q \otimes F$ must be nonarithmetic and is called *spread out*. As in the classical renewal setup, this property

entails a Stone-type decomposition of the matrix renewal measure $\mathbb{V}$; this leads in turn to some improvements of the renewal results on $\mathbb{V}$ in the nonarithmetic case.

We proceed to the statement of our main results, all proofs of which are presented in Section 4. For the sake of brevity, we restrict attention to the case of nonarithmetic $Q \otimes F$; note that all given results have obvious arithmetic counterparts which are obtained in a similar manner.

If $\mathcal{S}$ is finite, the following result can be found in [12, Theorem 3] or [26, Theorem 1].

Theorem 1.1. *Let Q be a quasistochastic matrix satisfying* (A1) *and* (A2)*, and suppose that $Q \otimes F$ is nonarithmetic. Then the associated renewal measure $\mathbb{V}$ satisfies*

$$\lim_{t \to \infty} \mathbb{V}_{ij}((t, t+h]) = \frac{v_i u_j h}{\mu} \quad and \quad \lim_{t \to -\infty} \mathbb{V}_{ij}((t, t+h]) = 0$$

for all $h > 0$ and $i, j \in \mathcal{S}$.

The next result provides a Stone-type decomposition of $\mathbb{V}$. It was derived by other means for finite $\mathcal{S}$ in [25, Theorem 2] (one-sided case) and [26, Theorem 5].

Theorem 1.2. *Let Q be a quasistochastic matrix satisfying* (A1) *and* (A2)*, and suppose that $Q \otimes F$ is spread out. Then the associated renewal measure allows a Stone-type decomposition $\mathbb{V} = \mathbb{V}^1 + \mathbb{V}^2$, where*

(a) $\mathbb{V}^1 = (\mathbb{V}^1_{ij})_{i,j \in \mathcal{S}}$ *consists of finite measures $\mathbb{V}^1_{ij}$, and*

(b) $\mathbb{V}^2 = (\mathbb{V}^2_{ij})_{i,j \in \mathcal{S}}$ *consists of λ-continuous measures $\mathbb{V}^2_{ij}$ with densities h_{ij} that are bounded, continuous, and satisfy, for all i and $j \in \mathcal{S}$,*

$$\lim_{t \to \infty} h_{ij}(t) = \frac{v_i u_j}{\mu} \quad and \quad \lim_{t \to -\infty} h_{ij}(t) = 0.$$

Furthermore,

$$\lim_{t \to \infty} \sup_{\mathcal{B}(\mathbb{R}) \ni B \subset [0,h]} \left| \mathbb{V}_{ij}(t+B) - \frac{v_i u_j \lambda(B)}{\mu} \right| = 0$$

for all $h > 0$ and $i, j \in \mathcal{S}$.

Turning to the functional version of the two previous results, consider a positive sequence $\lambda = (\lambda_i)_{i \in \mathcal{S}}$ and a measurable function $g : \mathcal{S} \times \mathbb{R} \to \mathbb{R}$. The function g is called λ-*directly Riemann integrable* if

$$g_i \text{ is } \lambda\text{-almost everywhere continuous for all } i \in \mathcal{S}, \tag{1.3}$$

$$\sum_{i \in \mathcal{S}} \lambda_i \sum_{n \in \mathbb{Z}} \sup_{n\varepsilon < x \le (n+1)\varepsilon} |g_i(x)| < \infty \quad \text{for some } \varepsilon > 0, \tag{1.4}$$

where $g_i := g(i, \cdot)$. If $\mathcal{S}$ is finite then this reduces to the statement that g_i for each $i \in \mathcal{S}$ is directly Riemann integrable in the ordinary sense and Theorem 1.3 reduces to [12, Theorem 4] or [26, Theorem 4] for the general nonarithmetic case. For the spread-out case, see also [25, Theorem 3] and [26, Theorem 6]. (In Theorem 1.3 below the $\mathcal{S} \times \mathbb{R}$-valued function $\mathbb{V} * g$ has components $(\mathbb{V} * g)_i(t) := \sum_j \int g(j, t-x) \mathbb{V}_{ij}(\mathrm{d}x) =: (\mathbb{V}_i * g)(t)_{i \in \mathcal{S}}$.)

Theorem 1.3. *Under the same assumptions as in Theorem 1.1, let g be u-directly Riemann integrable. Then $\mathbb{V} * g = (\mathbb{V}_i * g(t))_{i \in \mathcal{S}}$ has bounded components, i.e.*

$$\sup_{t \in \mathbb{R}} |\mathbb{V}_i * g(t)| < \infty \quad \text{for all } i \in \mathcal{S},$$

and

$$\lim_{t \to \infty} (\mathbb{V} * g)_i(t) = \frac{v_i}{\mu} \sum_{j \in \mathcal{S}} u_j \int g_j(x)\, dx \quad and \quad \lim_{t \to -\infty} (\mathbb{V} * g)_i(t) = 0$$

for all $i \in \mathcal{S}$. If also $Q \otimes F$ is spread out then the assertions remain valid for all functions g satisfying

$$g_i \in L^\infty(\lambda) \quad and \quad \lim_{|x| \to \infty} g_i(x) = 0 \quad for\ all\ i \in \mathcal{S}, \tag{1.5}$$

$$\sum_{i \in \mathcal{S}} u_i \|g_i\|_\infty < \infty, \tag{1.6}$$

$$g \in L^1(u \otimes \lambda), \quad i.e. \sum_{i \in \mathcal{S}} u_i \|g_i\|_1 < \infty. \tag{1.7}$$

Turning finally to the Markov renewal equation $Z = z + (Q \otimes F) * Z$, it is now relatively easy to provide conditions such that $Z^* = \mathbb{V} * z$ is a solution. On the other hand, the question of uniqueness of Z^* within a reasonable class of functions is more difficult, especially when the state space $\mathcal{S}$ of the driving chain is infinite. Conditions that guarantee uniqueness are often hard to verify in concrete applications.

Given any $Z \colon \mathcal{S} \times \mathbb{R} \to \mathbb{R}$, let $\widehat{Z} := D^{-1} Z = (v_i^{-1} Z_i)_{i \in \mathcal{S}}$. Then define

$$\mathcal{L} := \left\{ Z \colon \|\widehat{Z}_i\|_\infty < \infty \text{ and } \lim_{t \to -\infty} \widehat{Z}_i(t) = 0 \text{ for all } i \in \mathcal{S} \right\},$$

$$\mathcal{L}_0 := \left\{ Z \in \mathcal{L} \colon \sup_{i \in \mathcal{S}} \|\widehat{Z}_i\|_\infty < \infty \right\},$$

$$\mathcal{L}_0(g) := \{ Z \colon \widehat{Z} - \widehat{g} \in \mathcal{L}_0 \},$$

$$\mathcal{C}_b := \left\{ Z \colon \sup_{i \in \mathcal{S}} \|\widehat{Z}_i\|_\infty < \infty \text{ and } Z_i \text{ is continuous for all } i \in \mathcal{S} \right\}.$$

Note that $\mathcal{L} = \mathcal{L}_0$ if $\mathcal{S}$ is finite.

Theorem 1.4. *Let Q be a quasistochastic matrix satisfying (A1) and (A2), and suppose that $Q \otimes F$ is nonarithmetic. Let $z \colon \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ be u-directly Riemann integrable, or satisfy conditions (1.5)–(1.7) if $Q \otimes F$ is spread out. Then $Z^* = \mathbb{V} * z$ is an element of $\mathcal{L}$ and the unique solution to $Z = z + (Q \otimes F) * Z$ in $\mathcal{L}_0(Z^*)$. It is also the unique solution in the larger class $\mathcal{L}_0$ if $\mathcal{S}$ is finite or, more generally, $Z^* \in \mathcal{L}_0$.*

Note that, within the class of componentwise bounded functions, there are in fact infinitely many solutions to $Z = z + (Q \otimes F) * Z$, namely all functions

$$Z^c(t) := \mathbb{V} * z(t) + cv = (\mathbb{V}_i * z(t) + cv_i)_{i \in \mathcal{S}}, \qquad t \in \mathbb{R},$$

for $c \in \mathbb{R}$. This means that the constant vectors $cv = (cv_i)_{i \in \mathcal{S}}$ are solutions to the homogeneous (Choquet–Deny-type) equation $Z = (Q \otimes F) * Z$. The following theorem further shows that they are in fact the only solutions within the class $\mathcal{C}_b$. If $\mathcal{S}$ is finite, this was established analytically by de Saporta [12, Subsection 3.2] extending earlier results by Crump [11] and Athreya and Rama Murthy [5] in the one-sided case when all z_i, Z_i, and/or F_{ij} are concentrated on $[0, \infty)$. Not necessarily continuous solutions in the one-sided case are also discussed in some detail by Çinlar [9, Sections 3 and 4] in his survey of Markov renewal theory. For yet another and quite recent extension of these results, see [27]. Here we give a simple probabilistic argument which essentially reduces the problem to the classical renewal setup where the answer is known (see [14, p. 382]).

Theorem 1.5. *Let Q be a quasistochastic matrix satisfying (A1) and (A2), and suppose that $Q \otimes F$ is nonarithmetic. Then any solution $Z \in \mathcal{C}_b$ to the equation $Z = (Q \otimes F) * Z$ equals cv for some $c \in \mathbb{R}$.*

2. The Markov renewal setup

Put $D := \mathrm{diag}(v_i, \ i \in \mathcal{S})$ and $\pi = (\pi_i)_{i \in \mathcal{S}}$ with $\pi_i := u_i v_i$ for $i \in \mathcal{S}$. By (1.1), π defines a probability distribution on $\mathcal{S}$ if both the u_i and $u_i v_i$ are summable. Put further

$$P := D^{-1} Q D = \left(\frac{q_{ij} v_j}{v_i} \right)_{i,j \in \mathcal{S}},$$

which is an irreducible *stochastic matrix* having an essentially unique left eigenvector $\pi = u^\top D = (u_i v_i)_{i \in \mathcal{S}}$ associated with its maximal eigenvalue 1. Then

$$\Lambda(t) := P \otimes F(t) = D^{-1}(Q \otimes F)(t) D = \left(\frac{q_{ij} F_{ij}(t) v_j}{v_i} \right)_{i,j \in \mathcal{S}} \tag{2.1}$$

defines a matrix transition function of a Markov-modulated sequence $\{(M_n, X_n)\}_{n \geq 0}$ with state space $\mathcal{S} \times \mathbb{R}$. This means that the latter sequence forms a temporally homogeneous Markov chain satisfying

$$\mathbb{P}\{M_{n+1} = j, \ X_{n+1} \leq t \mid M_n = i\} = p_{ij} F_{ij}(t)$$

for all $n \in \mathbb{N}_0$, $i, j \in \mathcal{S}$, and $t \in \mathbb{R}$. Equivalently, $M = \{M_n\}_{n \geq 0}$ forms a Markov chain on $\mathcal{S}$ with transition matrix P and the X_n are conditionally independent given M with

$$\mathbb{P}\{X_n \leq t \mid M\} = \mathbb{P}\{X_n \leq t \mid M_{n-1}, M_n\} = F_{M_{n-1} M_n}(t)$$

for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. The Markov-additive process associated with $\{(M_n, X_n)\}_{n \geq 0}$, called the *Markov random walk* (MRW) hereafter, is defined to be $\{(M_n, S_n)\}_{n \geq 0}$, where $S_n = X_0 + \cdots + X_n$ for $n \in \mathbb{N}_0$. Its occupation measure on $\mathcal{S} \times \mathbb{R}$ under $\mathbb{P}_i := \mathbb{P}\{\cdot \mid M_0 = i\}$ is given by

$$\mathbb{U}_i(C) := \mathbb{E}_i\left[\sum_{n \geq 0} \mathbb{I}_C(M_n, S_n) \right] = \sum_{n \geq 0} \mathbb{P}_i\{(M_n, S_n) \in C\}$$

for measurable subsets C of $\mathcal{S} \times \mathbb{R}$; call $(\mathbb{U}_i)_{i \in \mathcal{S}}$ the *Markov renewal measure*. Since $\mathcal{S}$ is countable, there is a one-to-one correspondence between the vector measure $(\mathbb{U}_i)_{i \in \mathcal{S}}$ and the matrix renewal measure $\mathbb{U} = (\mathbb{U}_{ij})_{i,j \in \mathcal{S}}$, where

$$\mathbb{U}_{ij}(B) := \mathbb{E}_i\left[\sum_{n \geq 0} \mathbb{I}_{\{M_n = j, \ S_n \in B\}} \right] = \sum_{n \geq 0} \mathbb{P}_i\{M_n = j, \ S_n \in B\}, \qquad B \in \mathcal{B}(\mathbb{R}).$$

Lemma 2.1. *Let Q be a quasistochastic matrix satisfying (A1) and (A2). Then the associated MRW $\{(M_n, S_n)\}_{n \geq 0}$ has recurrent driving chain with stationary measure π and positive stationary drift μ defined in (A2), so $\mathbb{E}_\pi X_1 = \mu$.*

Proof. Obviously, (A1) is equivalent to

$$\sum_{n \geq 1} p_{ii}^{(n)} = \infty \quad \text{for some } i \in \mathcal{S},$$

which in turn is equivalent to the recurrence of $\{M_n\}_{n\geq 0}$ as claimed. The drift assertion follows from

$$\mathbb{E}_\pi X_1 = \sum_{i,j\in\mathscr{S}} \mathbb{P}_\pi\{M_0 = i,\ M_1 = j\}\,\mathbb{E}[X_1 \mid M_0 = i,\ M_1 = j] = \sum_{i,j\in\mathscr{S}} \pi_i\, p_{ij} \int x\, F_{ij}(\mathrm{d}x)$$

in combination with the definitions of the π_i and p_{ij}.

Lemma 2.2. *Let Q be a quasistochastic matrix satisfying (A1) and (A2). Then*

$$\mathbb{V} = D\,\mathbb{U}\,D^{-1} = \left(\frac{v_i\,\mathbb{U}_{ij}}{v_j}\right)_{i,j\in\mathscr{S}}. \tag{2.2}$$

Proof. For all $i,\,j \in \mathscr{S}$, $t \in \mathbb{R}$, and $h > 0$,

$$\mathbb{U}_{ij}((t, t+h]) = \sum_{n\geq 0} \mathbb{P}_i\{M_n = j,\ S_n \in (t, t+h]\} = \sum_{n\geq 0} p_{ij}^{(n)}(F_{ij}^{*n}(t+h) - F_{ij}^{*n}(t)),$$

and, therefore, using (2.1),

$$\begin{aligned}
\mathbb{U}((t, t+h]) &= \sum_{n\geq 0}((P\otimes F)^{*n}(t+h) - (P\otimes F)^{*n}(t))\\
&= \sum_{n\geq 0} D^{-1}((Q\otimes F)^{*n}(t+h) - (Q\otimes F)^{*n}(t))D.
\end{aligned}$$

But this equals the left-hand side below, and the rest proves the assertion:

$$D^{-1}\left(\sum_{n\geq 0}((Q\otimes F)^{*n}(t+h) - (Q\otimes F)^{*n}(t))\right)D = D^{-1}\mathbb{V}((t, t+h])\,D.$$

Equation (2.2) provides the crucial relation between the renewal measure $\mathbb{V}$ associated with $Q \otimes F$ and the matrix renewal measure $\mathbb{U}$ whose entries $\mathbb{U}_{ij}$ are actually ordinary renewal measures, as will be shown in Lemma 3.3 below. As a consequence, any result valid for $\mathbb{U}$ is now easily converted into a result for $\mathbb{V}$.

3. Discrete Markov renewal theory: a purely probabilistic approach

Throughout this section, let $\{(M_n, S_n)\}_{n\geq 0}$ be an arbitrary nonarithmetic MRW with discrete recurrent driving chain $M = \{M_n\}_{n\geq 0}$ having state space $\mathscr{S}$, transition matrix $P = (p_{ij})_{i,j\in\mathscr{S}}$, and stationary measure $\pi = (\pi_i)_{i\in\mathscr{S}}$, the latter being unique up to positive scalars. We denote by $X_1, X_2, \ldots$ the increments of $\{S_n\}_{n\geq 0}$ and by F_{ij} the conditional distribution of X_n given $M_{n-1} = i$ and $M_n = j$ for $i,\, j \in \mathscr{S}$. Put $\mathbb{P}_i(\cdot) := \mathbb{P}(\cdot \mid M_0 = i)$ with expectation operator $\mathbb{E}_i$, and let $S_0 = 0$ almost surely (a.s.) under $\mathbb{P}_i$ for each $i \in \mathscr{S}$. Finally, assume that the MRW has positive stationary drift μ, given by

$$\mu = \sum_{i\in\mathscr{S}} \sum_{j\in\mathscr{S}} \pi_i\, p_{ij}\, \mu_{ij} = \mathbb{E}_\pi X_1,$$

where $\mu_{ij} := \int x\, F_{ij}(\mathrm{d}x)$. Below, we call all these the *standard assumptions*. Note that μ, like π, is unique only up to positive scalars.

3.1. Auxiliary results

Let $i \in \mathcal{S}$ be arbitrary but fixed throughout this subsection. Then we can define π by

$$\pi_k := \pi_k^{(i)} := \mathbb{E}_i \left[\sum_{n=1}^{\sigma(i)} \mathbb{I}_{\{M_n = k\}} \right], \qquad k \in \mathcal{S}, \tag{3.1}$$

where $\sigma(i)$ denotes the first return time of M to i. With this choice, $\pi_i = 1$ and we can also easily deduce that

$$\mathbb{E}_i \left[\sum_{n=1}^{\sigma(i)} g(M_n, X_n) \right] = \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} g(M_n, X_n) \right] = \mathbb{E}_\pi g(M_1, X_1) \tag{3.2}$$

whenever $\mathbb{E}_\pi g(M_1, X_1)$ exists. Note that $\pi^{(j)} = c_j \pi^{(i)}$ for any $j \in \mathcal{S}$ together with $c_j \pi_j = c_j \pi_j^{(i)} = \pi_j^{(j)} = 1$ implies that $c_j = \pi_j^{-1}$.

If $\{\sigma_n(i)\}_{n \geq 1}$ denotes the renewal sequence of successive return times of M to i (thus, $\sigma(i) = \sigma_1(i)$) then $\{S_{\sigma_n(i)}\}_{n \geq 1}$ is an ordinary random walk under any $\mathbb{P}_j$ with increment distribution $\mathbb{P}_i\{S_{\sigma(i)} \in \cdot\}$ and drift

$$\mathbb{E}_i S_{\sigma(i)} = \mathbb{E}_i \left[\sum_{n=1}^{\sigma(i)} X_n \right] = \mathbb{E}_\pi X_1 = \mu,$$

where we have utilized (3.2). In particular, $\{S_{\sigma_n(i)}\}_{n \geq 0}$ with $\sigma_0(i) := 0$ forms a zero-delayed random walk under $\mathbb{P}_i$. The drift of any other $\{S_{\sigma_n(j)}\}_{n \geq 1}$ in terms of μ and π is given in the next lemma.

Lemma 3.1. *For each* $j \in \mathcal{S}$,

$$\mathbb{E}_j S_{\sigma(j)} = \frac{\mu}{\pi_j}.$$

Proof. This follows from

$$\mathbb{E}_j S_{\sigma(j)} = \mathbb{E}_{\pi^{(j)}} X_1 = \pi_j^{-1} \mathbb{E}_\pi X_1$$

valid for any $j \in \mathcal{S}$.

The following lemma on the lattice type of $\{S_{\sigma_n(j)}\}_{n \geq 1}$, $j \in \mathcal{S}$, is stated without proof, which can be given with the help of Fourier transforms.

Lemma 3.2. *Under the standard assumptions, for any* $j \in \mathcal{S}$, $S_{\sigma(j)}$ *is nonarithmetic under* $\mathbb{P}_j$.

The next lemma confirms that the Markov renewal measure $\mathbb{U}_i$ is directly related to the ordinary renewal measures of $\{S_{\sigma_n(j)}\}_{n \geq 1}$, $j \in \mathcal{S}$, under $\mathbb{P}_i$.

Lemma 3.3. *For all* $j \in \mathcal{S}$, $\mathbb{U}_i(\{j\} \times \cdot) = \mathbb{U}_{ij}$ *equals the (ordinary) renewal measure of* $\{S_{\sigma_n(j)}\}_{n \geq 1}$ *under* $\mathbb{P}_i$ *if* $j \neq i$, *and of* $\{S_{\sigma_n(i)}\}_{n \geq 0}$ *under* $\mathbb{P}_i$ *if* $j = i$.

Proof. The assertion follows directly from

$$\mathbb{U}_{ij}(B) = \mathbb{E}_i \left[\sum_{n \geq 0} \mathbb{I}_{\{M_n = j, \, S_n \in B\}} \right] = \begin{cases} \mathbb{E}_i \left[\sum_{n \geq 1} \mathbb{I}_{\{S_{\sigma_n(j)} \in B\}} \right] & \text{if } j \neq i, \\[2ex] \mathbb{E}_i \left[\sum_{n \geq 0} \mathbb{I}_{\{S_{\sigma_n(i)} \in B\}} \right] & \text{otherwise,} \end{cases}$$

for all $B \in \mathcal{B}(\mathbb{R})$.

The next result concerns the *pre-$\sigma(i)$ occupation measure*

$$U_i(C) := \mathbb{E}_i\left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_C(M_n, S_n)\right]$$

defined on measurable subsets C of $\mathscr{S} \times \mathbb{R}$. Setting $C = \{j\} \times \mathbb{R}$, it follows that

$$U_i(\{j\} \times \mathbb{R}) = \mathbb{E}_i\left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_{\{M_n=j\}}\right] = \pi_j \quad \text{for all } j \in \mathscr{S}. \tag{3.3}$$

Lemma 3.4. *Under the standard assumptions,*

$$\mathbb{U}_i(C) = \sum_{j \in \mathscr{S}} \iint \mathbb{I}_C(j, x+y)\, U_i(\{j\} \times \mathrm{d}y)\, \mathbb{U}_{ii}(\mathrm{d}x)$$

for any measurable $C \subset \mathscr{S} \times \mathbb{R}$; in particular, for all $j \in \mathscr{S}$ and $B \in \mathcal{B}(\mathbb{R})$,

$$\mathbb{U}_{ij}(B) = \int U_i(\{j\} \times (B-x))\, \mathbb{U}_{ii}(\mathrm{d}x) = \int \mathbb{U}_{ii}(B-x)\, U_i(\{j\} \times \mathrm{d}x).$$

Proof. Apply a standard conditioning argument to the expression

$$\mathbb{U}_i(C) = \mathbb{E}_i\left[\sum_{n \geq 0}\sum_{k=0}^{\sigma_{n+1}(i)-\sigma_n(i)-1} \mathbb{I}_C(M_{\sigma_n(i)+k},\, S_{\sigma_n(i)+k})\right].$$

Lemma 3.5. *Under the standard assumptions, for all $j \in \mathscr{S}$ and $h > 0$,*

$$\sup_{t \in \mathbb{R}} \mathbb{U}_{ij}([t, t+h]) \leq \pi_j\, \mathbb{U}_{ii}([-h, h]).$$

Proof. It is well known from ordinary renewal theory that, for any $h > 0$,

$$\sup_{t \in \mathbb{R}} \mathbb{U}_{ii}([t, t+h]) \leq \mathbb{U}_{ii}([-h, h]).$$

Using this and (3.3) with Lemma 3.4, we obtain, for all $j \in \mathscr{S}$, $t \in \mathbb{R}$, and $h > 0$,

$$\begin{aligned}
\mathbb{U}_{ij}([t, t+h]) &= \int \mathbb{U}_{ii}([t-x, t+h-x])\, U_i(\{j\} \times \mathrm{d}x) \\
&\leq U_i(\{j\} \times \mathbb{R})\, \mathbb{U}_{ii}([-h, h]) \\
&= \pi_j\, \mathbb{U}_{ii}([-h, h]).
\end{aligned}$$

3.2. Markov renewal theorems

It is now fairly straightforward to derive the Markov renewal theorem in the present setup by drawing on Blackwell's renewal theorem and the key renewal theorem from standard renewal theory. Since π is generally unique only up to positive scalars, it should be observed that $\pi(\cdot)/\mu$ with μ defined by (A2) does not depend on the particular choice of π.

Theorem 3.1. (Markov renewal theorem I.) *Under the standard assumptions,*

$$\lim_{t \to \infty} \mathbb{U}_i(A \times [t, t+h]) = \frac{\pi(A)\, h}{\mu} \quad \text{and} \quad \lim_{t \to -\infty} \mathbb{U}_i(A \times [t, t+h]) = 0$$

for all $i \in \mathscr{S}$, π-finite $A \subset \mathscr{S}$, and $h > 0$.

Proof. This is now a direct consequence of Blackwell's renewal theorem (applied to the $\mathbb{U}_{ij}$) and the dominated convergence theorem, when using the facts that

$$\mathbb{U}_i(A \times [t, t+h]) = \sum_{j \in A} \mathbb{U}_{ij}([t, t+h]) = \sum_{j \in A} \mathbb{U}_{ij}([t, t+h])$$

by Lemma 3.3 and $\sum_{j \in A} \mathbb{U}_{ij}([t, t+h]) \leq \pi(A)\, \mathbb{U}_{ii}([-h, h])$ (Lemma 3.5), and, finally, by Lemma 3.1,

$$\lim_{t \to \infty} \mathbb{U}_{ij}([t, t+h]) = \frac{1}{\mathbb{E}_j S_{\sigma(j)}} = \frac{\pi_j}{\mu} \quad \text{for any } j \in \mathcal{S}.$$

Turning to the functional version of the previous result, recall from (1.3) and (1.4) the definition of a π-directly Riemann integrable function g. The asymptotic behaviour of

$$\mathbb{U}_i * g(t) = \sum_{j \in \mathcal{S}} \int g_j(t-x)\, \mathbb{U}_{ij}(\mathrm{d}x)$$

for any such g and $i, j \in \mathcal{S}$ is described by the second Markov renewal theorem.

Theorem 3.2. (Markov renewal theorem II.) *Under the standard assumptions, for any $i \in \mathcal{S}$ and π-directly Riemann integrable function g, $\mathbb{U}_i * g$ is a bounded function satisfying*

$$\lim_{t \to \infty} \mathbb{U}_i * g(t) = \frac{1}{\mu} \sum_{j \in \mathcal{S}} \pi_j \int g_j(x)\, \mathrm{d}x \quad and \quad \lim_{t \to -\infty} \mathbb{U}_i * g(t) = 0.$$

Proof. Without loss of generality, let g be nonnegative. Define, for any $\rho > 0$,

$$\overline{g}_k^\rho(t) := \sum_{n \in \mathbb{Z}} \Big(\sup_{n\rho < x \leq (n+1)\rho} g_k(x) \Big) \mathbb{I}_{(n\rho, (n+1)\rho]}(t), \qquad (k, t) \in \mathcal{S} \times \mathbb{R}.$$

By construction, for each $i \in \mathcal{S}$, g_i^ε majorizes g_i, and since all the π_j are positive, condition (1.4) ensures that g_i^ε is directly Riemann integrable, which when combined with (1.3) implies that g_i itself is directly Riemann integrable. Then, by the key renewal theorem,

$$\lim_{t \to \infty} \mathbb{U}_{ij} * g_j(t) = \frac{\pi_j}{\mu} \quad \text{and} \quad \lim_{t \to -\infty} \mathbb{U}_{ij} * g_j(t) = 0$$

for all $i, j \in \mathcal{S}$. Now fix any $i \in \mathcal{S}$ and choose $\pi = \pi^{(i)}$; thus, $\pi_i = 1$. Use Lemma 3.5 together with (1.4) to infer that

$$\begin{aligned}
\sum_{j \in \mathcal{S}} \mathbb{U}_{ij} * g(t) &\leq \sum_{j \in \mathcal{S}} \mathbb{U}_{ij} * \overline{g}^\varepsilon(t) \\
&= \sum_{j \in \mathcal{S}} \sum_{n \in \mathbb{Z}} \Big(\sup_{n\varepsilon < x \leq (n+1)\varepsilon} g_j(x) \Big) \mathbb{U}_{ij}([t-(n+1)\varepsilon, t-n\varepsilon)) \\
&\leq \mathbb{U}_{ii}([-\varepsilon, \varepsilon]) \sum_{j \in \mathcal{S}} \pi_j \sum_{n \in \mathbb{Z}} \Big(\sup_{n\varepsilon < x \leq (n+1)\varepsilon} g_j(x) \Big) \\
&< \infty. \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.4)
\end{aligned}$$

Now by Lemma 3.3 we also have

$$\mathbb{U}_i * g(t) = \sum_{j \in \mathcal{S}} \int g_j(t-x)\, \mathbb{U}_{ij}(\mathrm{d}x) = \sum_{j \in \mathcal{S}} \mathbb{U}_{ij} * g(t), \qquad (3.5)$$

so the convergence assertions follow on appealing to the dominated convergence theorem. Combining (3.4) and (3.5) also shows the boundedness of $\mathbb{U}_i * g$ for each $i \in \mathcal{S}$.

3.3. The spread-out case: a Stone-type decomposition

The final subsection deals with the situation when $\{M_n, S_n\}_{n \geq 0}$ is *spread out* which means that some convolution power of $\mathbb{P}_\pi\{X_1 \in \cdot\}$ is nonsingular with respect to λ or, equivalently, that F_{rs}^{*n} is nonsingular with respect to λ for some $r, s \in \mathcal{S}$ with $p_{rs}^{(n)} > 0$ and some $n \in \mathbb{N}$ (for a definition in a more general setup, see [21, Definition 2.1]). In this case, Lemma 3.3 further allows us to derive a Stone-type decomposition of the Markov renewal measure in a very straightforward manner. We begin with a preliminary result on the ordinary renewal measures $\mathbb{U}_{ij}$.

Proposition 3.1. *Let* $\{(M_n, S_n)\}_{n \geq 0}$ *be spread out. Then there exist finite measures* $\mathbb{U}_{ij}^1$ *and* λ-*continuous measures* $\mathbb{U}_{ij}^2 = \upsilon_{ij}\lambda$ *such that the following assertions hold for all* $i, j \in \mathcal{S}$.

(a) $\mathbb{U}_{ij} = \mathbb{U}_{ij}^1 + \mathbb{U}_{ij}^2$.

(b) *If* $i \neq j$ *then* $\mathbb{U}_{ij}^1 = G_{ij} * \mathbb{U}_{jj}^1$, $\mathbb{U}_{ij}^2 = G_{ij} * \mathbb{U}_{jj}^2$, *and* $\upsilon_{ij} = G_{ij} * \upsilon_{jj}$, *where* $G_{ij}(\cdot) := \mathbb{P}_i\{S_{\sigma_1(j)} \in \cdot\}$.

(c) υ_{ij} *is continuous and bounded (uniformly in* $i \in \mathcal{S}$*) with*

$$\lim_{t \to \infty} \upsilon_{ij}(t) = \frac{\pi_j}{\mu} \quad and \quad \lim_{t \to -\infty} \upsilon_{ij}(t) = 0.$$

Proof. Choose any $i \in \mathcal{S}$. If $r, s \in \mathcal{S}$ are such that F_{rs}^{*n} has a convolution power that is nonsingular with respect to λ for some $n \in \mathbb{N}$, then choose a cyclic path $(i, r_1, \ldots, r_m, i)$ of positive probability p that passes through r and s at consecutive times. This is possible because $\{M_n\}_{n \geq 0}$ is irreducible and $p_{rs} > 0$. It follows that

$$
\begin{aligned}
G_{ii}(\cdot) &:= \mathbb{P}_i\{S_{\sigma_1(i)} \in \cdot\} \\
&= p_{ii} F_{ii} + \sum_{n \geq 2} \sum_{i_1, \ldots, i_{n-1} \in \mathcal{S} \setminus \{i\}} p_{ii_1} \cdots p_{i_{n-1}i} \, F_{ii_1} * \cdots * F_{i_{n-1}i} \\
&\geq p \, F_{ir_1} * \cdots * F_{rs} * \cdots * F_{r_{n-1}i}
\end{aligned}
$$

and, hence, that G_{ii} is spread out. Consequently, Stone's decomposition for ordinary renewal measures provides us with $\mathbb{U}_{ii} = \mathbb{U}_{ii}^1 + \mathbb{U}_{ii}^2$ for some finite measure $\mathbb{U}_{ii}^1$ and some λ-continuous measure $\mathbb{U}_{ii}^2 = \upsilon_{ii}\lambda$ such that υ_{ii} is bounded and continuous with limit 0 at $-\infty$ and

$$\lim_{t \to \infty} \upsilon_{ii}(t) = \frac{1}{\mathbb{E}_i S_{\sigma(i)}} = \frac{\pi_i}{\mu}.$$

All remaining assertions are now easily derived by using $\mathbb{U}_{ij} = F_{ij} * \mathbb{U}_{jj}$ for $i, j \in \mathcal{S}$ with $i \neq j$. Further details are therefore omitted.

It is now easy to obtain a Stone-type decomposition of the Markov renewal measures $\mathbb{U}_i$, $i \in \mathcal{S}$.

Theorem 3.3. (Stone-type decomposition.) *Let* $\{(M_n, S_n)\}_{n \geq 0}$ *be spread out. Then, for each* $i \in \mathcal{S}$*, there exists a finite measure* $\mathbb{U}_i^1$ *and a* $(\pi \otimes \lambda)$-*continuous measure* $\mathbb{U}_i^2$ *with density* υ_i *such that*

(a) $\mathbb{U}_i = \mathbb{U}_i^1 + \mathbb{U}_i^2$,

(b) υ_i *is bounded on any* $\mathcal{S}_0 \times \mathbb{R}$ *with* $\sup_{i \in \mathcal{S}_0} \pi_i < \infty$, *and*

(c) $\upsilon_{ij}(\cdot) := \upsilon_i(j, \cdot)$ *is continuous for any* $j \in \mathcal{S}$ *and satisfies*

$$\lim_{t \to \infty} \upsilon_{ij}(t) = \frac{1}{\mu} \quad and \quad \lim_{t \to -\infty} \upsilon_{ij}(t) = 0.$$

Proof. Fix any $i \in \mathcal{S}$, and again let π be defined by (3.1) so that $\pi_i = 1$. Using

$$\mathbb{U}_i(C) = \int_{\mathbb{R}} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_C(M_n, x + S_n) \right] \mathbb{U}_{ii}(\mathrm{d}x)$$

and Stone's decomposition for $\mathbb{U}_{ii}$ from the previous result, we arrive at the decompositon $\mathbb{U}_i = \mathbb{U}_i^1 + \mathbb{U}_i^2$ into the finite measure

$$\mathbb{U}_i^1(C) := \int_{\mathbb{R}} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_C(M_n, x + S_n) \right] \mathbb{U}_{ii}^1(\mathrm{d}x)$$

with total mass $\mathbb{U}_i^1(\mathcal{S} \times \mathbb{R}) = \pi_i^{-1} \mathbb{U}_{ii}^1(\mathbb{R})$ and the σ-finite measure

$$\mathbb{U}_i^2(C) := \int_{\mathbb{R}} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_C(M_n, x + S_n) \right] \upsilon_{ii}(x) \, \lambda(\mathrm{d}x)$$

$$= \int_{\mathbb{R}} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_C(M_n, x) \upsilon_{ii}(x - S_n) \right] \lambda(\mathrm{d}x).$$

Choosing $C = \{j\} \times B$ for arbitrary $j \in \mathcal{S}$ and $B \in \mathcal{B}(\mathbb{R})$, it follows that

$$\mathbb{U}_i^2(\{j\} \times B) = \int_B \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_{\{M_n = j\}} \upsilon_{ii}(x - S_n) \right] \lambda(\mathrm{d}x)$$

and thereby that $\mathbb{U}_i^2$ has $(\pi \otimes \lambda)$-density

$$\upsilon_{ij}(t) = \pi_j^{-1} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \mathbb{I}_{\{M_n = j\}} \upsilon_{ii}(t - S_n) \right], \qquad t \in \mathbb{R},$$

which satisfies (with $\| \cdot \|_\infty$ denoting the sup norm)

$$\upsilon_{ij}(t) \leq \|\upsilon_{ii}\|_\infty$$

for all $j \in \mathcal{S}$ and $t \in \mathbb{R}$, and, thus, $\|\upsilon_{ij}\|_\infty \leq \|\upsilon_{ii}\|_\infty$, and is continuous in the second argument. The remaining asymptotic assertions are now derived by using the asymptotic properties of υ_{ii} stated in Proposition 3.1 and the dominated convergence theorem.

In the spread-out case the class of functions g satisfying the assertions of Theorem 3.2 can be relaxed.

Theorem 3.4. (Markov renewal theorem II: spread-out case.) *Let $\{(M_n, S_n)\}_{n \geq 0}$ be spread out, and let $g \colon \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying (compare (1.3) and (1.4))*

$$g_i \in L^\infty(\lambda) \quad and \quad \lim_{|x| \to \infty} g_i(x) = 0 \quad for\ all\ i \in \mathcal{S}, \tag{3.6}$$

$$\sum_{i \in \mathcal{S}} \pi_i \|g_i\|_\infty < \infty,$$

$$g \in L^1(\pi \otimes \lambda), \quad i.e.\ \sum_{i \in \mathcal{S}} \pi_i \|g_i\|_1 < \infty. \tag{3.7}$$

Then all assertions of Theorem 3.2 about the $\mathbb{U}_i * g$ remain valid.

Proof. Again, without loss of generality, let g be nonnegative. Fix any $i \in \mathcal{S}$, choose $\pi = \pi^{(i)}$ and use Stone's decomposition of $\mathbb{U}_{ii}$ from Proposition 3.1(a) to infer that

$$\mathbb{U}_i * g(t) = \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} [\mathbb{U}_{ii}^1 * g_{M_n}(t - S_n) + \mathbb{U}_{ii}^2 * g_{M_n}(t - S_n)] \right] =: J_1(t) + J_2(t)$$

for all $t \in \mathbb{R}$. Put $G(i) := \|g_i\|_\infty$, and recall that $\|\mathbb{U}_{ii}^1\| := \mathbb{U}_{ii}^1(\mathbb{R}) < \infty$. It follows that $\sum_{n=0}^{\sigma(i)-1} \mathbb{U}_{ii}^1 * g_{M_n}(t - S_n) \le \|\mathbb{U}_{ii}^1\| \sum_{n=0}^{\sigma(i)-1} G(M_n)$, $\mathbb{P}_i$-a.s. and

$$J_1(t) \le \|\mathbb{U}_{ii}^1\| \, \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} G(M_n) \right] = \|\mathbb{U}_{ii}^1\| \, \mathbb{E}_\pi G(M_0) < \infty \quad \text{(use (3.2))},$$

implying the boundedness of J_1, and then, by the dominated convergence theorem, we have $\lim_{|t| \to \infty} J_1(t) = 0$ when $\lim_{|t| \to \infty} g_i(t) = 0$.

It remains to consider $J_2(t)$. Write

$$J_2(t) = \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} \int_\mathbb{R} g_{M_n}(t - x - S_n) \upsilon_{ii}(x) \, \lambda(dx) \right]$$

$$= \int_\mathbb{R} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} g_{M_n}(x) \upsilon_{ii}(t - x - S_n) \right] \lambda(dx)$$

$$= \int_\mathbb{R} \mathbb{E}_\pi g_{M_0}(x) \, \upsilon_{ii}(t - x) \, \lambda(dx)$$

$$+ \int_\mathbb{R} \mathbb{E}_i \left[\sum_{n=0}^{\sigma(i)-1} g_{M_n}(x) [\upsilon_{ii}(t - x - S_n) - \upsilon_{ii}(t - x)] \right] \lambda(dx).$$

By combining the assumptions on g with the properties of υ_{ii}, it is now straightforward to conclude that J_2 is bounded and that the first term of the last two lines converges to the asserted respective limit as $t \to \pm\infty$, while the second term converges to 0. We omit further details.

4. Proofs of the main results

In view of the results of the two previous sections, it is now straightforward to deduce our main theorems.

Proof of Theorem 1.1. As noted at the beginning of Section 2, $P = D^{-1}QD$ has the essentially unique left eigenvector $\pi = u^\top D = (u_i v_i)_{i \in \mathcal{S}}$ associated with eigenvalue 1, so that π is the essentially unique stationary measure of the Markov chain $(M_n)_{n \ge 0}$ with transition matrix P. Moreover, the MRW $\{(M_n, S_n)\}_{n \ge 0}$ has stationary drift μ as defined in (A2) under π and is nonarithmetic if $Q \otimes F$ has this property. Hence, a combination of Lemma 2.2 and Markov renewal theorem I (Theorem 3.1) yields

$$\lim_{t \to \infty} \mathbb{V}_{ij}([t, t+h]) = \frac{v_i}{v_j} \lim_{t \to \infty} \mathbb{U}_{ij}([t, t+h]) = \frac{v_i \pi_j h}{\mu v_j} = \frac{v_i u_j h}{\mu}$$

as well as $\lim_{t \to -\infty} \mathbb{V}_{ij}([t, t+h]) = 0$ for all $h > 0$ and $i, j \in \mathcal{S}$.

Proof of Theorem 1.2. If $Q \otimes F$ is spread out then so is $\{(M_n, S_n)\}_{n \geq 0}$. Therefore, by another use of Lemma 2.2 in combination with Theorem 3.3, the assertions of the theorem follow directly from observing that $\mathbb{V} = D\,\mathbb{U}^1 D^{-1} + D\,\mathbb{U}^2 D^{-1}$ provides a Stone-type decomposition of $\mathbb{V}$. Further details can be omitted.

Proof of Theorem 1.3. Recall that $\widehat{g}(t) := D^{-1} g(t) = (v_i^{-1} g_i(t))_{i \in \mathcal{S}}$. Then it is easily seen that $\widehat{g}$ is π-directly Riemann integrable if and only if g is u-directly integrable, and that $\widehat{g}$ satisfies (3.6) and (3.7) if and only if g itself satisfies (1.5) and (1.7). Furthermore, observing that

$$\mathbb{V} * g = (D\,\mathbb{U}\,D^{-1}) * D\,\widehat{g} = D\,\mathbb{U} * \widehat{g},$$

all assertions are directly inferred from Theorem 3.2 or 3.4 when applied to $\mathbb{U} * \widehat{g}$.

Proof of Theorem 1.4. The fact that $Z^* \in \mathcal{L}$ follows directly from Theorem 1.3, so we may immediately turn to the uniqueness assertions regarding the Markov renewal equation

$$Z = z + (Q \otimes F) * Z. \tag{4.1}$$

Note that if Z is in $\mathcal{L}$ and a solution to (4.1) then $\widehat{Z} = D^{-1} Z$ is in the same class (with respect to P, thus replacing v by $(1, 1, \ldots)^\top$ in the definition of $\mathcal{L}$) and is a solution to the probabilistic counterpart of (4.1), namely,

$$\widehat{Z} = \widehat{z} + (P \otimes F) * \widehat{Z}.$$

Hence, we may assume without loss of generality that $Q = P$, $v = (1, 1, \ldots)^\top$, and, thus, $\widehat{Z} = Z$. Given any further solution $Z' \in \mathcal{L}_0(Z^*)$, the difference $\Delta := Z' - Z^*$ is an element of $\mathcal{L}_0$ and a solution to the homogeneous equation $\Delta = (P \otimes F) * \Delta$; thus, $\Delta_i(t) = \mathbb{E}_i[\Delta(M_1, t - S_1)]$ and then upon iteration

$$\Delta_i(t) = \mathbb{E}_i[\Delta(M_n, t - S_n)]$$

for all $t \in \mathbb{R}$, $n \in \mathbb{N}$, and $i \in \mathcal{S}$. This shows that, for all $i \in \mathcal{S}$, $\{\Delta(M_n, t - S_n)\}_{n \geq 0}$ forms a bounded $\mathbb{P}_i$-martingale which thus converges $\mathbb{P}_i$-a.s. to a limit. But the latter equals 0 because

$$\lim_{n \to \infty} \Delta(M_n, t - S_n) = \lim_{n \to \infty} \Delta(i, t - S_{\sigma_n(i)}) = 0,$$

where, as before, the $\sigma_n(i)$ denote the almost surely finite return times to i of the chain $\{M_n\}_{n \geq 0}$. If $\mathcal{S}$ is finite or $Z^* \in \mathcal{L}_0$, then $\mathcal{L}_0(Z^*) = \mathcal{L}_0$ and the previous argument extends to all solutions $Z' \in \mathcal{L}_0$.

Proof of Theorem 1.5. Given a solution $Z \in \mathcal{C}_b$ of the homogeneous Markov renewal equation $Z = (Q \otimes F) * Z$, the function $\widehat{Z}$ is a bounded, componentwise continuous solution to $\widehat{Z} = (P \otimes F) * \widehat{Z}$ and, therefore, $\{\widehat{Z}(M_n, t - S_n)\}_{n \geq 0}$ is a bounded $\mathbb{P}_i$-martingale for all $i \in \mathcal{S}$. Using the optional sampling theorem, it follows that

$$\widehat{Z}_i(t) = \mathbb{E}_i[\widehat{Z}(M_{\sigma(i)}, t - S_{\sigma(i)})] = \mathbb{E}_i[\widehat{Z}_i(t - S_{\sigma(i)})] \quad \text{for all } i \in \mathcal{S}.$$

In other words, $\widehat{Z}_i$ forms a bounded, continuous solution to the ordinary Choquet–Deny equation $\widehat{Z}_i = \widehat{F}_i * \widehat{Z}_i$ for each $i \in \mathcal{S}$, where $\widehat{F}_i$ denotes the law of $S_{\sigma(i)}$ under $\mathbb{P}_i$. Since $\widehat{F}_i$ is nonarithmetic (Lemma 3.2) and $\widehat{Z}_i$ is continuous, the latter function must equal a constant c_i (see [14, p. 382]). By another appeal to the optional sampling theorem, now for distinct $i, j \in \mathcal{S}$, we find that

$$c_i = \widehat{Z}_i(t) = \mathbb{E}_i[\widehat{Z}_j(t - S_{\sigma(j)})] = c_j,$$

where $\mathbb{P}_i\{\sigma(j) < \infty\} = 1$ is guaranteed by the recurrence of $\{M_n\}_{n \geq 0}$. Consequently, $\widehat{Z}_i \equiv c$ for all $i \in \mathcal{S}$ and some $c \in \mathbb{R}$ as asserted.

5. Two examples

Quasistochastic matrices arise in various areas of applied probability, typically in connection with an exponential change of measure. We present two illustrative examples but make no attempt to completely elaborate all technical details.

5.1. Age-dependent multitype branching processes

This is an example from the class of multitype Crump–Mode–Jagers processes. We refer the reader to Mode's book [20, Chapter 3] for more detailed information and further mention a paper by the same author about a related model used for cell-cycle analysis [19].

Consider a population stemming from one ancestor born at time 0 which may be of any type $s \in \mathcal{S} = \{1, \ldots, m\}$. At the end of its life, each individual of type i gives birth to a random number of offspring of type j with finite mean μ_{ij} for any $j \in \mathcal{S}$ and has a nonarithmetic lifetime distribution G_i on $(0, \infty)$. Moreover, all individuals behave independently. We are interested in the asymptotic behaviour of $S(t) = (S_{ij}(t))_{i,j \in \mathcal{S}}$, where $S_{ij}(t)$ denotes the mean number of type-j individuals alive at time $t \geq 0$ when starting from one individual of type i. For simplicity, let the numbers of offspring be independent of the lifetime of an individual. Put $\overline{G}_i := 1 - G_i$. Then a standard renewal argument leads to

$$ S_{ij}(t) = \delta_{ij} \overline{F}_i(t) + \sum_{k=1}^{m} \mu_{ik} \int_{(0,t]} S_{kj}(t - x) \, G_k(dx), \qquad t \geq 0, $$

for all $1 \leq i, j \leq m$, that is, $S = g + (M \otimes G) * S$ with $M := (\mu_{ij})_{1 \leq i, j \leq m}$,

$$ g(t) := \begin{pmatrix} \overline{G}_1(t) & & 0 \\ & \ddots & \\ 0 & & \overline{G}_m(t) \end{pmatrix}, \quad \text{and} \quad G(t) := \begin{pmatrix} G_1(t) & \ldots & G_1(t) \\ & \ddots & \\ G_m(t) & \ldots & G_m(t) \end{pmatrix}. $$

Here $S(t)$ and $z(t)$ are matrices instead of vectors, but we may of course consider their column vectors $S_{.j}(t) = (S_{ij}(t))_{1 \leq i \leq m}$ and $g_{.j}(t) = (\delta_{ij} \overline{G}_j(t))_{1 \leq i \leq m}$ separately, or any linear combination $v^\top S(t) = \sum_{j=1}^{m} v_j S_{.j}(t)$.

Now consider $\alpha \in \mathbb{R}$ such that $\phi_i(\alpha) := \int e^{-\alpha t} G_i(dt) < \infty$ for each $i = 1, \ldots, m$. Defining $Z(t) := e^{-\alpha t} S(t)$, we then find that $Z = z + (Q \otimes F) * Z$ with $z(t) := e^{-\alpha t} g(t)$, $Q := (m_{ij} \phi_i(\alpha))_{1 \leq i, j \leq m}$, and

$$ F(t) := \begin{pmatrix} F_1(t) & \ldots & F_1(t) \\ & \ddots & \\ F_m(t) & \ldots & F_m(t) \end{pmatrix}, \quad \text{where} \quad F_i(t) := \phi_i(\alpha)^{-1} \int_{[0,t]} e^{-\alpha x} G_i(dx). $$

If α, called the Malthusian parameter of the population, can be chosen such that Q has maximal eigenvalue 1 and is primitive (thus, Q^n is a strictly positive matrix for some $n \in \mathbb{N}$—see [24]), then the results of Section 1 can be used to determine the limit of $e^{-\alpha t} S(t)$ as $t \to \infty$. In principle, these considerations may be extended to the case of infinite type space ($\mathcal{S} = \mathbb{N}$) in the sense that the Markov renewal equations as above remain valid. On the other hand, the quasistochasticity of Q, including therefore the necessary existence of the Malthusian parameter α, is a more delicate matter.

Note finally that other functionals of the population described here may be studied in a similar manner. For example, if $A_{ij}(t)$ denotes the average total age of all type-j individuals

alive at time t when the ancestor of the population is of type i, then it is readily verified that $A(t) = (A_{ij}(t))_{1 \le i, j \le m}$ satisfies the Markov renewal equation $A = f + (M \otimes G) * A$ with M and G as before and

$$f(t) := (\delta_{ij} \, t \, \overline{G}_i(t))_{1 \le i, j \le m}.$$

5.2. Random difference equations in a Markovian environment

Let $\{(A_n, B_n)\}_{n \in \mathbb{Z}}$ be a doubly infinite stationary ergodic sequence and consider the random difference equation

$$Y_n = A_n Y_{n-1} + B_n \qquad (5.1)$$

for $n \ge 0$. It was shown by Brandt [7] that, if

$$\mathbb{E}[\log |A_0|] < 0 \quad \text{and} \quad \mathbb{E}[\log^+ |B_0|] < \infty,$$

then a stationary solution of $\{Y_n\}_{n \ge 0}$ exists and may be realized by defining

$$Y_0 \;=\; B_0 + \sum_{n \ge 0} A_{-n} A_{-(n-1)} \cdots A_{-1} A_0 \, B_{-n-1}.$$

Regarding the existence and properties of the stationary law of Y_0 (often called *perpetuity*), many papers have dealt with the situation when the (A_n, B_n) are independent and identically distributed (i.i.d.) and possibly multivariate; see [1, 2, 8, 15, 16, 17, 18, 29]. The case where $\{A_n\}_{n \in \mathbb{Z}}$ forms an irreducible stationary Markov chain taking values in a finite subset $\mathcal{S}$ of $\mathbb{R}$, and the B_n are i.i.d. and independent of the A_n was treated by de Saporta [13]; see also [22, 23] for the more general case of continuous state space $\mathcal{S}$.

Here we look more closely at the situation treated in [13], for simplicity confining ourselves to the case when $\mathcal{S} \subset (0, \infty)$, but allowing $\mathcal{S}$ to be a countably infinite set. Denote by $P = (p_{ss'})_{s, s' \in \mathcal{S}}$ the transition matrix of $\{A_n\}_{n \ge 0}$ and by $\pi = (\pi_s)_{s \in \mathcal{S}}$ its unique stationary distribution. Note that the dual backward chain $(A_{-n})_{n \ge 0}$ has transition probabilities $\widehat{p}_{ss'} = \pi_{s'} p_{s's} / \pi_s$.

Being interested in $\mathbb{P}\{\pm Y_1 > t, \, A_0 = s\}$ for $(s, t) \in \mathcal{S} \times \mathbb{R}$, observe that, by (5.1),

$$\mathbb{P}\{\pm Y_1 > t, \, A_1 = s\} = \mathbb{P}\{\pm s Y_0 > t, \, A_1 = s\} + \psi_s^\pm(t),$$

where

$$\psi_s^\pm(t) := \mathbb{P}\{\pm s Y_0 + B_1 > t, \, A_1 = s\} - \mathbb{P}\{\pm s Y_0 > t, \, A_1 = s\}.$$

For α still to be specified, define the smoothed tail functions

$$Z_s^\pm(t) := \frac{1}{\pi_s e^t} \int_0^{e^t} u^\alpha \, \mathbb{P}\{\pm s Y_1 > u, \, A_1 = s\} \, \mathrm{d}u$$

and $z_s^\pm(t) := \pi_s^{-1} e^{-t} \int_0^{e^t} u^\alpha \, \psi_s^\pm(u) \, \mathrm{d}u$. Put $F_{ss'}(t) := \mathbb{I}_{[\log s, \infty)}(t)$. Then it is not difficult to show (see [13, Section 3] for details) that

$$Z_s^\pm(t) = z_s^\pm(t) + s^\alpha \sum_{s' \in \mathcal{S}} \widehat{p}_{ss'} F_{ss'} * Z_{s'}^\pm(t)$$

for all $(s, t) \in \mathcal{S} \times \mathbb{R}$. Consequently, $Z^+(t) = (Z_s^+(t))_{s \in \mathcal{S}}$ and $Z^-(t) = (Z_s^-(t))_{s \in \mathcal{S}}$ both satisfy the Markov renewal equation $Z = z + (Q \otimes F) * Z$ with $z(t) = z^+(t) = (z_s^+(t))_{s \in \mathcal{S}}$ and $z(t) = z^-(t) = (z_s^-(t))_{s \in \mathcal{S}}$, respectively, and

$$Q = (s^\alpha \widehat{p}_{ss'})_{s, s' \in \mathcal{S}} = \left(\frac{s^\alpha \pi_{s'} p_{s's}}{\pi_s} \right)_{s, s' \in \mathcal{S}}.$$

Therefore, the asymptotic behaviour of $Z^+(t)$ and $Z^-(t)$ as $t \to \infty$ can be determined with the help of the results in Section 1 if (besides further technical assumptions) we can choose $\alpha > 0$ such that Q is quasistochastic, which in particular requires Q to have spectral radius

$$\rho(Q) = \lim_{n \to \infty} (\mathbb{E}(A_0 \cdots A_{-n+1})^\alpha)^{1/n} = 1.$$

In the case of finite $\mathcal{S}$, the latter already implies quasistochasticity as a consequence of the Perron–Frobenius theorem, but, for infinite state space, this needs further discussion.

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