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SPECTRA OF LINEAR FRACTIONAL COMPOSITION OPERATORS ON THE GROWTH SPACE AND BLOCH SPACE OF THE UPPER HALF-PLANE

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Abstract

In this article, we provide a complete description of the spectra of linear fractional composition operators acting on the growth space and Bloch space over the upper half-plane. In addition, we also prove that the norm, essential norm, spectral radius and essential spectral radius of a composition operator acting on the growth space are all equal.

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1. Introduction

Let Ω be a domain of the complex plane. Given a holomorphic self-map $\varphi : \Omega \to \Omega$ and a holomorphic function $u \in H(\Omega)$ over Ω, the associated *weighted composition operator* is defined by

$$uC_\varphi(f) = u \cdot f \circ \varphi \quad \text{for all } f \in H(\Omega).$$

In particular, when $u \equiv 1$, C_φ is called a *composition operator*.

Over the past four decades, the research on composition operators and weighted composition operators has undergone a fast and fruitful development. The convention is to relate the theoretical operator properties of uC_φ, that is, boundedness, compactness, spectra, normality and so on, to the function properties of its symbol. The work is usually carried over the unit disk. For general information, we refer readers to the excellent monographs by Sharpiro [21] and Cowen and Macluer [7].

In recent years, there has been a great deal of interest in holomorphic function spaces of the upper half-plane $\Pi^+ = \{z : \operatorname{Im} z > 0\}$. Matache, in his series of papers [17–19], made a systematic study of composition operators on the Hardy space $H^2(\Pi^+)$

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of the upper half-plane. He found [18] that, unlike in the case of the unit disk, not every holomorphic self-map of the upper half-plane induces a bounded composition operator on $H^2(\Pi^+)$. In fact, he obtained a very simple criterion for the boundedness of a composition operator acting on $H^2(\Pi^+)$: C_φ is bounded if and only if the angular derivative of φ at infinity exists. Matache's result was strengthened by Elliott and Jury [9] in the following precise form.

Theorem A. *Acting on the Hardy space $H^2(\Pi^+)$, C_φ is bounded if and only if φ has a finite angular derivative $0 < \lambda < \infty$ at infinity. In that case, the norm, essential norm, spectral radius and essential spectral radius are all equal to $\sqrt{\lambda}$.*

As a corollary of Theorem A, we see immediately that no compact composition operator exists on $H^2(\Pi^+)$. Motivated by this fact, Shapiro and Smith [22] investigated the existence of compact composition operators on Hardy spaces of various domains.

It should be noted that an analogue of Theorem A also holds on weighted Bergman spaces $\mathcal{A}^2_\alpha(\Pi^+)$ [10] of the upper half-plane.

Sharma *et al.* [23] introduced upper half-plane versions of the growth space $\mathcal{A}(\Pi^+)$ and the Bloch space $\mathcal{B}(\Pi^+)$,

$$\mathcal{A}(\Pi^+) = \Big\{f \in H(\Pi^+) : \|f\|_{\mathcal{A}(\Pi^+)} = \sup_{w\in\Pi^+} \operatorname{Im} w|f(w)| < \infty\Big\}$$

and

$$\mathcal{B}(\Pi^+) = \Big\{f \in H(\Pi^+) : \|f\|_{\mathcal{B}(\Pi^+)} = \sup_{w\in\Pi^+} \operatorname{Im} w|f'(w)| < \infty\Big\}.$$

Both spaces are natural counterparts of the respective ones over the unit disk. The authors then studied the boundedness of composition operators acting on these spaces.

Recall that the classical Bloch space of the unit disk is given by

$$\mathcal{B} = \Big\{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty\Big\}.$$

It is well known that the Cayley transform $\sigma(z) = i((1 + z)/(1 - z))$ is a conformal map from the unit disk onto the upper half-plane. If we define the 'composition operator'

$$\begin{aligned} C_\sigma : \mathcal{B}(\Pi^+) &\to \mathcal{B} \\ f &\mapsto f \circ \sigma, \end{aligned}$$

then calculation shows that $\|f \circ \sigma\|_{\mathcal{B}} = 2\|f\|_{\mathcal{B}(\Pi^+)}$. So $\mathcal{B}(\Pi^+)$ is isomorphic to $\mathcal{B}$ and a composition operator C_φ on $\mathcal{B}(\Pi^+)$ is similar to another composition operator C_ψ on $\mathcal{B}$ via C_σ.

The story for a growth space is somewhat different. The authors [23] found that a composition operator $C_\varphi : \mathcal{A}(\Pi^+) \to \mathcal{A}(\Pi^+)$ is bounded if and only if

$$\sup_{w\in\Pi^+} \frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)} < \infty,$$

which is equivalent to the existence of the finite angular derivative $\varphi'(\infty)$ by the Julia–Carathéodory theorem of the upper half-plane (see [9] or Lemma 2.5 below). Recently, the authors of [24] proved that there is also no compact composition operator on the growth space. Both facts indicate that the growth space behaves like the Hardy space and the weighted Bergman space over the upper half-plane. So one may naturally ask the following question.

Is there an analogue of Theorem A on the growth space?

We will answer this question affirmatively in the second section.

Among composition operators, those induced by linear fractional transformations are well understood in several backgrounds, including the unit disk [3, 5, 6, 12–14, 16] and the upper half-plane [11, 20]. Recall that a *linear fractional transformation* (LFT) is a meromorphic bijection of the extended complex plane $\mathbb{C} \cup \{\infty\}$ onto itself, which can be expressed in the form

$$\varphi(w) = \frac{aw + b}{cw + d}, \quad ad - bc \neq 0.$$

It is clear that each LFT has exactly two fixed points, counting multiplicities. According to the behavior at the fixed point, LFTs are classified into four classes: elliptic, hyperbolic, parabolic and loxodromic. For details, see Sharpiro's book [21] or the article [12].

Recently, Schroderus [20] considered the spectrum problem of a linear fractional composition operator on $H^2(\Pi^+)$ and $\mathcal{A}^2_\alpha(\Pi^+)$ and got a complete solution. Schroderus's result extended some earlier work of Gallardo-Gutiérrez and Montes-Rodríguez [11]. We will consider the spectra problem of a LFT composition operator on the growth space in Section 2 and on the Bloch space in Section 3. We believe that these results will be beneficial for further research on the spectra of weighted composition operators with LFT symbols acting on the growth space over the unit disk.

2. Composition operators on the growth space

2.1. The growth space. Before our discussion, we will make a slight generalization of the definition of a growth space.

For $\alpha > 0$, the *growth space* $\mathcal{A}_\alpha(\Pi^+)$ over the upper half-plane is defined by

$$\mathcal{A}_\alpha(\Pi^+) = \Big\{ f \in H(\Pi^+) : \|f\|_{\mathcal{A}_\alpha} = \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha |f(w)| < \infty \Big\}.$$

Taking the notation from [20] for the weighted Bergman space $\mathcal{A}^2_\gamma(\Pi^+)$, $\gamma > -1$, we associate $\mathcal{A}^2_{-1}(\Pi^+) = H^2(\Pi^+)$ and let K^γ_w be the reproducing kernel for $\mathcal{A}^2_\gamma(\Pi^+), \gamma \geq -1$. According to [20, Lemma 2],

$$|f(w)| \leq \|f\|_{\mathcal{A}^2_\gamma} \|K^\gamma_w\|_{\mathcal{A}^2_\gamma} = \|f\|_{\mathcal{A}^2_\gamma} \frac{c_\gamma}{(\operatorname{Im} w)^{\gamma/2+1}}$$

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for all $f \in \mathcal{A}^2_\gamma(\Pi^+)$, where c_γ is a constant depending only on γ. Therefore $\mathcal{A}^2_\gamma(\Pi^+)$ is contained in the growth space $\mathcal{A}_{\gamma/2+1}(\Pi^+)$.

Denote by δ_w the linear functional of evaluation at $w \in \Pi^+$,

$$\delta_w(f) = f(w), \quad f \in \mathcal{A}_\alpha(\Pi^+).$$

It is clear that δ_w is continuous, that is, $\mathcal{A}_\alpha(\Pi^+)$ is a functional Banach space. In fact, we have the following result on the norm of the evaluation functional.

Lemma 2.1. *As a linear functional on $\mathcal{A}_\alpha(\Pi^+)$, $\|\delta_{w_0}\| = 1/(\operatorname{Im} w_0)^\alpha$.*

Proof. By the definition of $\mathcal{A}_\alpha(\Pi^+)$, it is obvious that $\|\delta_{w_0}\| \le 1/(\operatorname{Im} w_0)^\alpha$. To prove the equality, take the test function

$$f_{w_0}(w) = \frac{1}{(w - \operatorname{Re} w_0)^\alpha}.$$

Then

$$\|f_{w_0}\|_{\mathcal{A}_\alpha} = \sup_{w\in\Pi^+} \left| \frac{\operatorname{Im} w}{(\operatorname{Re} w - \operatorname{Re} w_0) + i \operatorname{Im} w} \right|^\alpha = 1$$

and

$$|\delta_{w_0}(f_{w_0})| = \frac{1}{(\operatorname{Im} w_0)^\alpha},$$

which completes the proof. □

Recall the classical growth space $H^\infty_\alpha(\mathbb{D})$ over the unit disk

$$H^\infty_\alpha(\mathbb{D}) = \Big\{ f \in H(\mathbb{D}) : \|f\|_{H^\infty_\alpha} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f(z)| < \infty \Big\}.$$

Composition operators and weighted composition operators on $H^\infty_\alpha(\mathbb{D})$ have been widely studied, and also more generally on the spaces $H^\infty_v(\mathbb{D})$, where v is a positive weight function on $\mathbb{D}$ (see [1, 2, 4, 16] and the references therein).

Unlike the Bloch space, the Cayley transform no longer induces an isomorphic composition operator from $\mathcal{A}_\alpha(\Pi^+)$ to $H^\infty_\alpha(\mathbb{D})$. However, as the following lemma shows, we can perform an isomorphism via the 'weighted composition operator'

$$J : \mathcal{A}_\alpha(\Pi^+) \to H^\infty_\alpha(\mathbb{D})$$
$$f \mapsto Jf(z) = \frac{f(\sigma(z))}{(1 - z)^{2\alpha}}, \tag{2.1}$$

where $\sigma(z) = i((1 + z)/(1 - z))$ is the Cayley transform.

Lemma 2.2. *Let J be as defined in (2.1). Then J is an isometric isomorphism from $\mathcal{A}_\alpha(\Pi^+)$ onto $H^\infty_\alpha(\mathbb{D})$.*

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PROOF. For any function $f \in \mathcal{A}_\alpha(\Pi^+)$,

$$\begin{aligned}
\|Jf\|_{H^\infty_\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\left|\frac{f(\sigma(z))}{(1-z)^{2\alpha}}\right| \\
&= \sup_{w\in\Pi^+}(1-|\sigma^{-1}(w)|^2)^\alpha\frac{|f(w)|}{|1-\sigma^{-1}(w)|^{2\alpha}} \\
&= \sup_{w\in\Pi^+}\left(1-\left|\frac{w-i}{w+i}\right|^2\right)^\alpha\left|1-\frac{w-i}{w+i}\right|^{-2\alpha}|f(w)| \\
&= \sup_{w\in\Pi^+}\left(\frac{4\operatorname{Im} w}{|w+i|^2}\cdot\frac{|w+i|^2}{4}\right)^\alpha|f(w)| \\
&= \sup_{w\in\Pi^+}(\operatorname{Im} w)^\alpha|f(w)| \\
&= \|f\|_{\mathcal{A}_\alpha}.
\end{aligned}$$

The proof is complete. □

According to Lemma 2.2, the properties of a composition operator on $\mathcal{A}_\alpha(\Pi^+)$ can be directly deduced from the corresponding weighted composition operator on $H^\infty_\alpha(\mathbb{D})$. We collect, in the following, some known results on weighted composition operators on $H^\infty_\alpha(\mathbb{D})$, which will be used later.

The first result concerns the essential norm of a weighted composition operator, which is a special case of [4, Theorem 4.2]. The second gives the spectrum of a weighted composition operator in the case where φ has an interior fixed point.

LEMMA 2.3 [4, Theorem 4.2]. *For φ an analytic self-map of the unit disk and $u \in H(\mathbb{D})$ such that the weighted composition operator uC_φ is bounded acting on $H^\infty_\alpha(\mathbb{D})$, $\alpha > 0$, the essential norm is*

$$\|uC_\varphi\|_e = \lim_{r\to 1}\sup_{|\varphi(z)|>r}|u(z)|\left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^\alpha.$$

LEMMA 2.4 [1, Theorem 7]. *Let $\alpha > 0$ and suppose that φ, which is not an automorphism, has fixed point $a \in \mathbb{D}$ and that $uC_\varphi : H^\infty_\alpha(\mathbb{D}) \to H^\infty_\alpha(\mathbb{D})$ is bounded. Then*

$$\sigma(uC_\varphi) = \{\lambda : |\lambda| \le r_e(uC_\varphi)\} \cup \{u(a)\varphi'(a)^n\}_{n=0}^\infty.$$

We will use the Julia–Carathéodory theorem frequently. The unit disk version is well known; see, for example [7, Theorem 2.44]. The following lemma gives the upper half-plane version of the Julia–Carathéodory theorem, which is easily deduced from the unit disk version using the Cayley transform. Here, if $\varphi : \Pi^+ \to \Pi^+$, then we write $\varphi(\infty) = \infty$ if $\varphi(z_n) \to \infty$ whenever $z_n \to \infty$ nontangentially. In this case, we write

$$\varphi'(\infty) = \lim_{z\to\infty}\frac{z}{\varphi(z)}$$

if this nontangential limit exists and is finite. For more details, see [9].

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Lemma 2.5 (Julia–Carathéodory). *For an analytic self-map φ of the upper half-plane, the following conditions are equivalent:*

(1) $\varphi(\infty) = \infty$ *and* $\varphi'(\infty)$ *exists;*
(2) $\sup_{\operatorname{Im} z>0} \operatorname{Im} z/\operatorname{Im} \varphi(z) < \infty$*; and*
(3) $\limsup_{z\to\infty} \operatorname{Im} z/\operatorname{Im} \varphi(z) < \infty$.

Moreover, the quantities in (2) *and* (3) *are both equal to the angular derivative* $\varphi'(\infty)$.

2.2. Norm and spectral radius. The following result answers the first question raised in the introduction.

Theorem 2.6. *Suppose that φ is a holomorphic self-map of the upper half plane. Then acting on $\mathcal{A}_\alpha(\Pi^+)$, C_φ is bounded if and only if $\sup_{w\in\Pi^+}(\operatorname{Im} w/\operatorname{Im} \varphi(w))^\alpha < \infty$, that is, φ has a finite angular derivative at infinity. Moreover, the norm, essential norm, spectral radius and essential spectral radius are all equal to $\sup_{w\in\Pi^+}(\operatorname{Im} w/\operatorname{Im} \varphi(w))^\alpha = \varphi'(\infty)^\alpha$.*

Proof. On the one hand, if C_φ is bounded, then

$$\|C_\varphi^*(\delta_w)\| \le \|C_\varphi\|\|\delta_w\|.$$

Since, for any $f \in \mathcal{A}_\alpha(\Pi^+)$,

$$C_\varphi^*(\delta_w)(f) = \delta_w(C_\varphi f) = f(\varphi(w)) = \delta_{\varphi(w)}(f),$$

we get $C_\varphi^*(\delta_w) = \delta_{\varphi(w)}$. Then by Lemma 2.1,

$$\sup_{w\in\Pi^+}\left(\frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)}\right)^\alpha = \sup_{w\in\Pi^+}\frac{\|C_\varphi^*(\delta_w)\|}{\|\delta_w\|} \le \|C_\varphi\|.$$

On the other hand, if $\sup_{w\in\Pi^+}(\operatorname{Im} w/\operatorname{Im} \varphi(w))^\alpha < \infty$, then, for any $f \in \mathcal{A}_\alpha(\Pi^+)$,

$$\begin{aligned}
\|C_\varphi f\| &= \sup_{w\in\Pi^+}(\operatorname{Im} w)^\alpha|f(\varphi(w))| \\
&= \sup_{w\in\Pi^+}\left(\frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)}\right)^\alpha(\operatorname{Im} \varphi(w))^\alpha|f(\varphi(w))| \\
&\le \sup_{w\in\Pi^+}\left(\frac{\operatorname{Im} w}{\operatorname{Im} \varphi(w)}\right)^\alpha\|f\|.
\end{aligned}$$

Therefore, C_φ is bounded if and only if $\sup_{w\in\Pi^+}(\operatorname{Im} w/\operatorname{Im} \varphi(w))^\alpha < \infty$ and $\|C_\varphi\| = \sup_{w\in\Pi^+}(\operatorname{Im} w/\operatorname{Im} \varphi(w))^\alpha$.

It remains to prove the formula concerning the essential norm, spectral radius and essential spectral radius of C_φ. For this, we will pull back to the unit disk via the isomorphism J constructed above.

By Lemma 2.2, the composition operator

$$C_\varphi : \mathcal{A}_\alpha(\Pi^+) \to \mathcal{A}_\alpha(\Pi^+)$$

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is isometrically isomorphic to the weighted composition operator

$$JC_\varphi J^{-1} : H_\alpha^\infty(\mathbb{D}) \to H_\alpha^\infty(\mathbb{D}).$$

Specifically, we have $\|C_\varphi\| = \|JC_\varphi J^{-1}\|$ and $\|C_\varphi\|_e = \|JC_\varphi J^{-1}\|_e$. We need to calculate the explicit expression of $JC_\varphi J^{-1}$. For any $f \in H_\alpha^\infty(\mathbb{D})$,

$$\begin{aligned} JC_\varphi J^{-1} f(z) &= JC_\varphi((1-\sigma^{-1})^{2\alpha} f \circ \sigma^{-1})(z) \\ &= J((1-\sigma^{-1}\circ\varphi)^{2\alpha} f \circ \sigma^{-1} \circ \varphi)(z) \\ &= \Big(\frac{1-\sigma^{-1}\circ\varphi\circ\sigma(z)}{1-z}\Big)^{2\alpha} f \circ \sigma^{-1}\circ\varphi\circ\sigma(z) \\ &= u(z) f(\psi(z)), \end{aligned}$$

where $\psi = \sigma^{-1}\circ\varphi\circ\sigma$ and $u(z) = ((1-\psi(z))/(1-z))^{2\alpha}$.

Since $\sup_{w\in\Pi^+} \operatorname{Im} w/\operatorname{Im}\varphi(w) < \infty$, Lemma 2.5 implies that $\varphi(\infty) = \infty$ and the angular derivative $\varphi'(\infty)$ exists, which, translating to the unit disk via the Cayley transform, is equivalent to $\psi(1) = 1$ and the angular derivative $\psi'(1)$ exists. Now, by Lemma 2.3 and the Julia–Carathéodory theorem of the unit disk [7, Theorem 2.44],

$$\begin{aligned} \|C_\varphi\|_e = \|JC_\varphi J^{-1}\|_e &= \lim_{r\to 1} \sup_{|\psi(z)|>r} |u(z)| \Big(\frac{1-|z|^2}{1-|\psi(z)|^2}\Big)^\alpha \\ &\ge \lim_{r\to 1} |u(r)| \Big(\frac{1-r^2}{1-|\psi(r)|^2}\Big)^\alpha \\ &= \lim_{r\to 1} \Big(\frac{|1-\psi(r)|}{1-r}\Big)^{2\alpha} \Big(\frac{1-r^2}{1-|\psi(r)|^2}\Big)^\alpha \\ &\ge \lim_{r\to 1} \Big(\frac{1-|\psi(r)|}{1-r}\Big)^{2\alpha} \Big(\frac{1-r^2}{1-|\psi(r)|^2}\Big)^\alpha \\ &= \lim_{r\to 1} \Big(\frac{1-|\psi(r)|}{1-r}\Big)^{\alpha} \Big(\frac{1+r}{1+|\psi(r)|}\Big)^\alpha \\ &= \psi'(1)^\alpha = \varphi'(\infty)^\alpha. \end{aligned}$$

Therefore,

$$\|C_\varphi\| \ge \|C_\varphi\|_e \ge \varphi'(\infty)^\alpha = \sup_{w\in\Pi^+} \Big(\frac{\operatorname{Im} w}{\operatorname{Im}\varphi(w)}\Big)^\alpha = \|C_\varphi\|,$$

which implies that

$$\|C_\varphi\| = \|C_\varphi\|_e = \varphi'(\infty)^\alpha = \sup_{w\in\Pi^+} \Big(\frac{\operatorname{Im} w}{\operatorname{Im}\varphi(w)}\Big)^\alpha.$$

For any $n \ge 1$, we denote by φ_n the nth iterate of φ. Since φ has finite angular derivative $\varphi'(\infty)$ at ∞, by the definition of $\varphi'(\infty)$, we know that $\varphi(z_j)$ converges to ∞ nontangentially whenever z_j converges to ∞ nontangentially. Therefore,

$$\varphi_n'(\infty) = \lim_{z\to\infty} \frac{z}{\varphi_n(z)} = \lim_{z\to\infty} \frac{z}{\varphi(z)} \frac{\varphi(z)}{\varphi_2(z)} \cdots \frac{\varphi_{n-1}(z)}{\varphi_n(z)} = \varphi'(\infty)^n.$$

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By similar arguments, we can prove that $\|C_{\varphi_n}\| = \|C_{\varphi_n}\|_e = \varphi'(\infty)^{n\alpha}$. Therefore, the spectral radius and essential spectral radius of C_φ are both equal to $\varphi'(\infty)^\alpha$. The proof is complete. □

Note that Theorem 2.6 strengthens the results of [23, Theorem 4.1] and [24, Theorem 2.6]. In particular, the following corollary is immediate.

Corollary 2.7 [24, Theorem 2.16]. *There exists no compact composition operator on* $\mathcal{A}_\alpha(\Pi^+)$.

2.3. Spectra. According to Theorem 2.6, if a linear fractional transformation

$$\varphi(w) = \frac{aw + b}{cw + d}$$

induces a bounded composition operator on the growth space $\mathcal{A}_\alpha(\Pi^+)$, then we must have $\varphi(\infty) = \infty$, that is, $a \neq 0$ and $c = 0$. So we can write φ in the form

$$\varphi(z) = \mu w + w_0,$$

where $\mu > 0$ and $\operatorname{Im} w_0 \geq 0$ since φ maps the upper half-plane into itself. Therefore, only two kinds of LFTs can induce bounded composition operators on the growth space $\mathcal{A}_\alpha(\Pi^+)$:

- *parabolic* if $\varphi(w) = w + w_0$ with $\operatorname{Im} w_0 \geq 0$; and
- *hyperbolic* if $\varphi(w) = \mu w + w_0$ with $\mu > 0, \mu \neq 1$ and $\operatorname{Im} w_0 \geq 0$.

For the remainder of this section, we will try to determine the spectra of composition operators induced by LFTs. Since no general results exist in the literature involving the spectra of weighted composition operators with LFT symbols over $H^\infty_\alpha(\mathbb{D})$, even though some special cases [1, 2, 16] have been investigated, we cannot carry the results directly from the unit disk.

First, consider the case when φ is parabolic, that is,

$$\varphi(w) = w + w_0.$$

Obviously, φ is an automorphism of the upper half-plane when $\operatorname{Im} w_0 = 0$. In this case, the author in [20] obtained the spectrum of C_φ on the Hardy and weighted Bergman space of the upper half-plane by performing the Fourier transform to convert C_φ into some multiplier on a certain L^2 space (Paley–Wiener theorem). The method cannot be applied to the growth space. Fortunately, $\mathcal{A}_\alpha(\Pi^+)$ is large enough to admit abundant eigenvectors of C_φ. For the nonautomorphism case, we will embrace Cowen's use of a semigroup argument, the idea for which he attributes to Kaufman [5].

Theorem 2.8. *Suppose that* $\varphi(w) = w + w_0$ *with* $\operatorname{Im} w_0 \geq 0$. *Then the spectrum of* C_φ *on* $\mathcal{A}_\alpha(\Pi^+)$ *is:*

(i) $\sigma(C_\varphi) = \mathbb{T}$ *if* $\operatorname{Im} w_0 = 0$*; and*
(ii) $\sigma(C_\varphi) = \{e^{itw_0} : t \geq 0\} \cup \{0\}$ *if* $\operatorname{Im} w_0 > 0$.

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Proof. By Theorem 2.6, the spectral radius $\rho(C_\varphi)$ is $\varphi'(\infty)^\alpha = 1$.

(i) If $\operatorname{Im} w_0 = 0$, then φ is an automorphism of the upper half-plane and the associated composition operator is invertible. The spectrum mapping theorem implies that $\lambda \in \sigma(C_\varphi)$ if and only if $\lambda^{-1} \in \sigma(C_\varphi^{-1})$, where $C_\varphi^{-1} = C_{\varphi^{-1}}$ and the spectral radius $r(C_{\varphi^{-1}})$ also equals one by Theorem 2.6. Therefore, $\sigma(C_\varphi) \subset \mathbb{T}$.

For the converse inclusion, we will show that each $\lambda \in \mathbb{T}$ is an eigenvalue of infinite multiplicities. To see this, denote $g_t(w) = e^{itw}$ for any $t > 0$. Then

$$\|g_t\|_{\mathcal{A}_\alpha} = \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha |e^{itw}| = \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha e^{-t \operatorname{Im} w} = \left(\frac{\alpha}{te}\right)^\alpha < \infty$$

and

$$C_\varphi g_t = e^{itw_0} g_t.$$

For each $\lambda \in \mathbb{T}$, there exists a sequence of distinct positive numbers $\{t_k : k \in \mathbb{N}\}$ such that $\lambda = e^{it_k w_0}$, so

$$C_\varphi g_{t_k} = \lambda g_{t_k} \quad \text{for all } 0 \le k < \infty.$$

We need to show that these eigenfunctions g_{t_k} are linearly independent. Suppose that $n \in \mathbb{N}$ and $c_k \in \mathbb{C}$, $0 \le k \le n-1$ are such that

$$\sum_{k=0}^{n-1} c_k e^{it_k w} = 0 \quad \text{for all } w \in \Pi^+. \tag{2.2}$$

By the uniqueness theorem of holomorphic functions, (2.2) actually holds for all $w \in \mathbb{C}$. Now, for any $0 \le m \le n-1$, taking the mth derivative and evaluating at the origin in (2.2), we get the system of linear equations given by

$$\sum_{k=0}^{n-1} c_k t_k^m = 0 \quad \text{for all } 0 \le m \le n-1. \tag{2.3}$$

Note that the coefficients of (2.3) make up a Vandermonde matrix whose determinant is not zero, so $c_k = 0$, $0 \le k \le n-1$. By the arbitrariness of n, the family $\{g_{t_k} : k \in \mathbb{N}\}$ is linearly independent.

(ii) If $\operatorname{Im} w_0 > 0$, similarly to (i), we can show that each nonzero point of the spiral $\{e^{itw_0} : t \ge 0\} \cup \{0\}$ is an eigenvalue of C_φ, and thus $\{e^{itw_0} : t \ge 0\} \cup \{0\} \subset \sigma(C_\varphi)$.

For the converse inclusion, we will apply Cowen's semigroup argument. For any $t \in \Pi^+$, let

$$\varphi_t(w) = w + t|w_0|$$

and let C_t be the associated composition operator. We first show that:

(1) $C_{t_1+t_2} = C_{t_1} C_{t_2}$; and
(2) C_t is holomorphic with respect to t.

The first statement is obvious. For the second statement, denote

$$X_0 = \overline{\operatorname{span}}\{\delta_w : w \in \Pi^+\} \subset \mathcal{A}_\alpha(\Pi^+)^*.$$

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Lemma 2.1 implies that, for any $f \in \mathcal{A}_\alpha(\Pi^+)$,

$$\begin{aligned}\|f\|_{\mathcal{A}_\alpha} &= \sup\left\{\frac{|f(w)|}{\|\delta_w\|} : w \in \Pi^+\right\}\\ &= \sup\{|L(f)| : \|L\| = 1, L \in X_0\},\end{aligned}$$

that is, X_0 is a determining manifold [15, 2.8.2] of $\mathcal{A}_\alpha(\Pi^+)$. According to [15, 3.10.1], it suffices to illustrate that $L(C_t f)$ is a holomorphic function in t for any $L \in X_0$ and any $f \in \mathcal{A}_\alpha(\Pi^+)$. So let $f \in \mathcal{A}_\alpha(\Pi^+)$ be fixed for the moment. For any point $w \in \Pi^+$, it is obvious that $\delta_w(C_t f) = f(w + tu)$ is holomorphic in t. For any linear functional $L \in X_0$, there exists a sequence of functionals $\{L_n\}_n \subset \operatorname{span}\{\delta_w : w \in \Pi^+\}$ such that $L_n \to L$. Since $\|C_t\| = 1$ for all $t \in \Pi^+$,

$$|L_n(C_t f) - L(C_t f)| \le \|L_n - L\|\|f\|_{\mathcal{A}_\alpha} \to 0,$$

which means that $L(C_t f)$ is the uniform limit of the sequence $\{L_n(C_t f)\}_n$ of holomorphic functions, so $L(C_t f)$ is also holomorphic in t.

Now, denote by C the norm closed algebra generated by $\{I\} \cup \{C_t : t \in \Pi^+\}$. Note that C is a commutative unital Banach algebra. The Gelfand theory states that

$$\sigma_C(C_t) = \{L(C_t) \colon L \text{ is a multiplicative linear functional on } C\}.$$

For any given multiplicative linear functional L, denote $l(t) = L(C_t)$. Then $l(t)$ is a holomorphic function over the upper half-plane and $l(t_1 + t_2) = l(t_1)l(t_2)$. An elementary argument shows that either $l \equiv 0$ or $l(t) = e^{i\beta t}$ for some $\beta \in \mathbb{C}$. In the latter case,

$$|e^{i\beta t}| = |L(C_t)| \le 1$$

for all $t \in \Pi^+$, so $\beta \ge 0$ and thus

$$\sigma(C_\varphi) \subset \sigma_C(C_t) \subset \{e^{it\beta} : \beta \ge 0\} \cup \{0\}.$$

Taking $t = e^{i \arg w_0}$,

$$\sigma(C_\varphi) \subset \{e^{i\beta e^{i \arg w_0}} : \beta \ge 0\} \cup \{0\} = \{e^{isw_0} : s \ge 0\} \cup \{0\}.$$

The proof is complete. □

Now, for the hyperbolic case,

$$\varphi(w) = \mu w + w_0 \quad \text{for all } \mu > 0,\ \mu \ne 1.$$

When $\operatorname{Im} w_0 = 0$, φ is an automorphism. In this case, we will prove that C_φ is similar to a multiple of an isometry. When $\operatorname{Im} w_0 > 0$, φ is not an automorphism. In this case, the spectrum is a closed disk, but the proofs for the case $0 < \mu < 1$ and the case $\mu > 1$ are different, which is due to the location of their fixed points.

Theorem 2.9. *Suppose that $\varphi(w) = \mu w + w_0$ with $\mu > 0$, $\mu \ne 1$ and $\operatorname{Im} w_0 \ge 0$. Then the spectrum of C_φ on $\mathcal{A}_\alpha$ is:*

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(i) $\sigma(C_\varphi) = \{\lambda : |\lambda| = \mu^{-\alpha}\}$ *if* $\operatorname{Im} w_0 = 0$*; and*
(ii) $\sigma(C_\varphi) = \{\lambda : |\lambda| \leq \mu^{-\alpha}\}$ *if* $\operatorname{Im} w_0 > 0$.

Proof. By Theorem 2.6, it follows immediately that the spectral radius of C_φ is $\mu^{-\alpha}$.

(i) Without loss of generality, we may assume that $\varphi(w) = \mu w$. Indeed, for the case $w_0 \neq 0$, since φ has two fixed points, $a = w_0/(1-\mu) \in \mathbb{R}$ and ∞, we can perform a conformal transform by setting

$$\widetilde{\varphi}(w) = \tau_a^{-1} \circ \varphi \circ \tau_a(w) = \mu w,$$

where $\tau_a(w) = w + a$. Then $C_{\widetilde{\varphi}} = C_{\tau_a} \circ C_\varphi \circ C_{\tau_a}^{-1}$ and C_φ are similar and have the same spectrum.

Now, for any $f \in \mathcal{A}_\alpha$,

$$\|C_{\widetilde{\varphi}} f\|_{\mathcal{A}_\alpha} = \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha |f(\mu w)| = \mu^{-\alpha} \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha |f(w)| = \mu^{-\alpha}\|f\|_{\mathcal{A}^\alpha_\infty},$$

that is, $C_{\widetilde{\varphi}}$ is a multiple of an isometry. Since $C_{\widetilde{\varphi}}$ is invertible, $\sigma(C_{\widetilde{\varphi}}) \subset \{\lambda : |\lambda| = \mu^{-\alpha}\}$.

Now, for any λ, $|\lambda| = \mu^{-\alpha}$, write $\lambda = \mu^{-\alpha+it}$ for some $t \geq 0$ and denote

$$g_t(w) = e^{(-\alpha+it)\log w},$$

where log denotes the principle branch of the logarithm. Then

$$\|g_t\|_{\mathcal{A}_\alpha} = \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha e^{-\alpha \log|w| - t \arg w} \leq \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha |w|^{-\alpha} = 1 < \infty$$

and

$$C_{\widetilde{\varphi}} g_t(w) = e^{(-\alpha+it)(\log\mu + \log w)} = \lambda g_t(w).$$

Therefore, $\{\lambda : |\lambda| = \mu^{-\alpha}\} \subset \sigma(C_{\widetilde{\varphi}})$.

(ii) If $\operatorname{Im} w_0 > 0$, it remains to show that $\{\lambda : |\lambda| \leq \mu^{-\alpha}\} \subset \sigma(C_\varphi)$. The proof is divided into two cases.

Case 1. $\mu > 1$. Denote by $a = w_0/(1-\mu)$ ($\operatorname{Im} a < 0$) the fixed point of φ other than infinity. The fact that the imaginary part of a is negative will enable us to construct eigenfunctions. As above, write $\lambda = \mu^{-\alpha s+it}$ with $s \geq 1$, $t \geq 0$ and define the function

$$h_t(w) = e^{(-\alpha s+it)\log(w-a)},$$

where log is the principle branch of the logarithm. Then

$$\begin{aligned}\|h_t\|_{\mathcal{A}_\alpha} &= \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha e^{-\alpha \log|w-a| - t\arg(w-a)} \\ &\leq \sup_{w\in\Pi^+} (\operatorname{Im} w)^\alpha |\operatorname{Im} w - \operatorname{Im} a|^{-\alpha} < 1\end{aligned}$$

and

$$C_\varphi h_t = \lambda g_t.$$

Therefore, $\lambda \in \sigma(C_\varphi)$.

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Case 2. $\mu < 1$. In this case, φ has an interior fixed point $a = w_0/(1-\mu) \in \Pi^+$. By Lemma 2.2, the composition operator

$$C_\varphi : \mathcal{A}_\alpha(\Pi^+) \to \mathcal{A}_\alpha(\Pi^+)$$

is isometrically isomorphic to the weighted composition operator

$$uC_\psi : H_\alpha^\infty(\mathbb{D}) \to H_\alpha^\infty(\mathbb{D}),$$

where $\psi = \sigma^{-1} \circ \varphi \circ \sigma$ and $u(z) = ((1-\psi(z))/(1-z))^{2\alpha}$. By Theorem 2.6, the essential spectral radius is $r_e(uC_\psi) = r_e(C_\varphi) = \mu^{-\alpha}$. So according to Lemma 2.4,

$$\{\lambda : |\lambda| \le \mu^{-\alpha}\} \subset \sigma(C_\varphi).$$

Note that the set $\{u(\sigma^{-1}(a))\psi'(\sigma^{-1}(a))\}_{n=0}^\infty$ does not give any more points in the spectrum since the essential spectral radius equals the spectral radius by Theorem 2.6. □

REMARK. As shown above, the location of the fixed point $a = w_0/(1-\mu)$ enables us to construct eigenfunctions in the case $\mu > 1$. But this fails in the case $\mu < 1$. In fact, if $f \circ \varphi = \lambda f$, $|\lambda| \le \mu^{-\alpha}$, then

$$f(\varphi(w)) = \sum_{n=0}^\infty a_n \mu^n (w-a)^n = \lambda \sum_{n=0}^\infty a_n (w-a)^n = \lambda f(w)$$

in a neighborhood of $a \in \Pi^+$. So $\lambda = \mu^n$ and $f(w) = a_n(w-a)^n$ for some $n \in \mathbb{N}$. However, in this case, it is easy to see that $f \in \mathcal{A}_\alpha(\Pi^+)$ if and only if $a_n = 0$. Therefore, C_φ has no eigenvalues.

3. Regarding the Bloch space

Recall that the Bloch space of the half-plane is defined by

$$\mathcal{B}(\Pi^+) = \Big\{ f \in H(\Pi^+) : \|f\|_{\mathcal{B}(\Pi^+)} = \sup_{w \in \Pi^+} \operatorname{Im} w |f'(w)| < \infty \Big\}.$$

The quantity $\|\cdot\|_{\mathcal{B}(\Pi^+)}$ defines a seminorm on $\mathcal{B}(\Pi^+)$. A norm is given by adding the modulus $|f(i)|$ to the seminorm. As we mentioned in the introduction, the Cayley transform induces an isomorphic composition operator from $\mathcal{B}(\Pi^+)$ onto the classic Bloch space $\mathcal{B}$ of the unit disk

$$\mathcal{B} = \Big\{ f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}} = \sup_{z \in \mathbb{D}} (1-|z|^2)|f'(z)| < \infty \Big\}.$$

According to the Schwartz–Pick theorem [7, Theorem 2.39], every holomorphic self-map of the unit disk induces a bounded composition operator on $\mathcal{B}$.

In this section, we want to complete the characterization of the spectra of composition operators with LFT symbols acting on the Bloch space. First, recall that

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an automorphism $\varphi \in Aut(\mathbb{D})$ of the unit disk is a conformal map of the unit disk onto itself, which is of the form

$$\varphi(z) = \lambda \frac{a - z}{1 - \bar{a}z} \quad \text{for all } |\lambda| = 1, a \in \mathbb{D}.$$

The spectra of automorphic composition operators on the Bloch space were described, although not explicitly, in [16] as a corollary.

THEOREM 3.1 [16, Corollary 5.2]. *Let $\varphi \in Aut(\mathbb{D})$. Then the spectrum of C_φ acting on the Bloch space $\mathcal{B}$ is:*

(i) $\overline{\{\varphi'(a)^n : n \geq 0\}}$ *if φ has an interior fixed point $a \in \mathbb{D}$;*
(ii) *the unit circle $\mathbb{T}$ if φ has no interior fixed point.*

For the remainder of this paper, we will give the spectra when φ is not an automorphism. First, the spectral radius is already known.

LEMMA 3.2 [8, Corollary 3.9]. *Let φ be a holomorphic self-map of the unit disk. Then the spectral radius of C_φ acting on $\mathcal{B}$ is one.*

The spectrum of a composition operator on the Bloch space when the symbol admits an interior fixed point is shown in the following lemma.

LEMMA 3.3 [1, Corollary 9,10]. *Suppose that φ, which is not an automorphism, fixes a point in $\mathbb{D}$. Then, acting on the Bloch space,*

$$\sigma(C_\varphi) = \{\lambda : |\lambda| \leq r_e(C_\varphi)\} \cup \{u(a)\varphi'(a)^n\}_{n=0}^{\infty}.$$

Moreover, if φ is univalent and $r_{e,H^2}(C_\varphi) \neq 0$, where H^2 is the Hardy–Hilbert space of the unit disk, then $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

Now, we can complete the task of determining the spectra of nonautomorphism LFT composition operators acting on the Bloch space.

THEOREM 3.4. *Let φ, which is not an automorphism, be an LFT mapping the unit disk into itself. Then, acting on the Bloch space $\mathcal{B}$, the spectrum of C_φ is:*

(i) $\overline{\{\varphi'(a)^n : n \geq 0\}}$ *if φ has an interior fixed point a and an exterior fixed point;*
(ii) $\overline{\mathbb{D}}$ *if φ has two distinct fixed points with one on the boundary; and*
(iii) $\{e^{iw_0 t} : t \geq 0\} \cup \{0\}$*, where $w_0 = 2i\varphi(0)/(\zeta - \varphi(0))$, if φ has a unique fixed point $\zeta \in \mathbb{T}$ of 2-multiplicities.*

PROOF. (i) In this case, $\overline{\varphi(\varphi(\mathbb{D}))} \subset \mathbb{D}$ and thus $C_{\varphi\circ\varphi}$ is compact. It follows by Lemma 3.3 and the spectral mapping theorem that $\sigma(C_\varphi) = \overline{\{\varphi'(a)^n : n \geq 0\}}$.

(ii) The proof is divided into two cases.

Case 1. If φ has a boundary fixed point $\zeta \in \mathbb{T}$ and an interior fixed point, according to Lemma 3.3, it suffices to prove that the essential spectral radius $r_{e,H^2}(C_\varphi) > 0$. Since φ is a nonautomorphism LFT with a boundary fixed point, $\varphi(\mathbb{D})$ is a disk tangential to

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the unit circle at ζ. Applying [7, Theorem 7.31] and the Julia–Carathéodory theorem of the unit disk [7, Theorem 2.44],

$$\begin{aligned}
r_{e,H^2}(C_\varphi) &= \lim_{k\to\infty}\Big(\limsup_{|w|\to 1}\frac{\|K_{\varphi_k(w)}\|}{\|K_w\|}\Big)^{1/k}\\
&= \lim_{k\to\infty}\Big(\limsup_{|w|\to 1}\frac{1-|w|^2}{1-|\varphi_k(w)|^2}\Big)^{1/2k}\\
&= \lim_{k\to\infty}\Big(\limsup_{w\to \zeta}\frac{1-|w|^2}{1-|\varphi_k(w)|^2}\Big)^{1/2k}\\
&= \lim_{k\to\infty}\Big(\liminf_{w\to \zeta}\frac{1-|\varphi_k(w)|^2}{1-|w|^2}\Big)^{-1/2k}\\
&= \lim_{k\to\infty}(|\varphi_k'(\zeta)|)^{-1/2k} = \varphi'(\zeta)^{-1/2} > 0.
\end{aligned}$$

Case 2. If φ has a boundary fixed point $\zeta \in \mathbb{T}$ and an exterior fixed point, let $\widetilde{\varphi}(z) = \zeta^{-1}\varphi(\zeta z)$. Then $\widetilde{\varphi}(1) = 1$ and $C_{\widetilde{\varphi}}$ and C_φ are similar. Let $\sigma(z) = i((1+z)/(1-z))$ be the Cayley transform. Then $C_{\widetilde{\varphi}}$ acting on $\mathcal{B}$ is similar to C_ψ acting on $\mathcal{B}(\Pi^+)$, where $\psi = \sigma\circ\widetilde{\varphi}\circ\sigma^{-1}$. We need to prove that $\sigma(C_\psi) = \overline{\mathbb{D}}$.

Denote by w_0 ($\operatorname{Im} w_0 < 0$) the fixed point of ψ other than ∞. Then $\psi(w) = \mu(w-w_0)+w_0$, where $\mu > 1$ since ∞ is attractive. For $\lambda = \mu^{-s+it} \in \mathbb{D}$ with $s \geq 0$, $t \in \mathbb{R}$, take

$$f_{s,t}(w) = e^{(-s+it)\log(w-w_0)},$$

where log denotes the principle branch of the logarithm. Then

$$C_\psi f_{s,t} = \mu^{-s+it} f_{s,t}$$

and

$$\begin{aligned}
\|f_{s,t}\|_{\mathcal{B}(\Pi^+)} &= \sup_{w\in\Pi^+}\operatorname{Im} w|f_{s,t}'(w)|\\
&= |-s+it|\sup_{w\in\Pi^+}\operatorname{Im} w|e^{(-s-1+it)\log(w-w_0)}|\\
&= |-s+it|\sup_{w\in\Pi^+}\operatorname{Im} we^{-(s+1)\log|w-w_0|-t\arg(w-w_0)}\\
&\leq |-s+it|\sup_{w\in\Pi^+}\operatorname{Im} we^{-(s+1)\log|w-w_0|} \quad (\operatorname{Im} w_0 < 0)\\
&= |-s+it|\sup_{w\in\Pi^+}\frac{\operatorname{Im} w}{|w-w_0|^{s+1}} < \infty.
\end{aligned}$$

So $\overline{\mathbb{D}} \subset \sigma(C_\varphi)$, which, together with Lemma 3.2, gives the result $\sigma(C_\varphi) = \overline{\mathbb{D}}$.

(iii) Proceeding as above, C_φ acting on $\mathcal{B}$ is similar to C_ψ acting on $\mathcal{B}(\Pi^+)$, where $\psi = \sigma\circ\zeta^{-1}\circ\varphi\circ\zeta\circ\sigma^{-1}$. Since ψ fixes ∞ as the unique fixed point and not an automorphism,

$$\psi(w) = w + w_0 \quad \text{for all } \operatorname{Im} w_0 > 0.$$

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Calculation shows that $w_0 = \psi(0) = 2i\varphi(0)/(\zeta - \varphi(0))$. Note that C_ψ is embedded in the holomorphic semigroup $\{C_{\psi_t} : t \in \Pi^+\}$, where $\psi_t = w + t|w_0|$. The inclusion $\sigma(C_\varphi) \subset \{e^{iw_0 t} : t \geq 0\} \cup \{0\}$ is exactly the same as the proof of Theorem 2.8. To prove the converse inclusion, for any point $e^{iw_0 t}, t \geq 0$,

$$C_\psi(e^{itw}) = e^{it(w+w_0)} = e^{itw_0} e^{itw}$$

and

$$\|e^{itw}\|_{\mathcal{B}(\Pi^+)} = \sup_{w\in\Pi^+} \operatorname{Im} w|ite^{itw}| = t \sup_{w\in\Pi^+} \operatorname{Im} w \cdot e^{-t \operatorname{Im} w} = \frac{1}{e} < \infty.$$

The proof is complete. □

At the end of this paper, for completeness, we would like to state the spectra of composition operators with linear fractional symbols acting on the Bloch space of the upper half-plane.

Corollary 3.5. *Let $\varphi : \Pi^+ \to \Pi^+$ be a linear fractional transformation. Then acting on the Bloch space $\mathcal{B}(\Pi^+)$, the spectrum of C_φ is:*

(i) $\overline{\{\varphi'(a)^n : n \geq 0\}}$ *if φ has an interior fixed point $a \in \Pi^+$ and an exterior fixed point;*
(ii) $\mathbb{T}$ *if φ is an automorphism without interior fixed point;*
(iii) $\overline{\mathbb{D}}$ *if φ has two distinct fixed points with only one on the boundary; and*
(iv) $\{e^{iw_0 t} : t \geq 0\} \cup \{0\}$, *where $w_0 = ((\varphi(i) - i)(b + i))/(b - \varphi(i))$, if φ is a nonautomorphism with the unique boundary fixed point b of 2-multiplicities.*

Remark. In the fourth case (iv) of Corollary 3.5, the value of w_0 is understood to be $\varphi(i) - i$ when the fixed point of φ is ∞.

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