Mathematical Structures in Computer Science (2023), **33**, pp. 33–45
doi:10.1017/S0960129523000105

PAPER

Local Yoneda completions of quasi-metric spaces†

Jing Lu and Bin Zhao*

School of Mathematics and Statistics, Shaanxi Normal University, Xi'an 710119, P.R. China
*Corresponding author. Email: zhaobin@snnu.edu.cn

(Received 16 March 2022; revised 24 February 2023; accepted 27 February 2023; first published online 24 March 2023)

Abstract

In this paper, we study quasi-metric spaces using domain theory. Given a quasi-metric space (X, d), we use $(\mathbf{B}(X, d), \leq^{d^+})$ to denote the poset of formal balls of the associated quasi-metric space (X, d). We introduce the notion of local Yoneda-complete quasi-metric spaces in terms of domain-theoretic properties of $(\mathbf{B}(X, d), \leq^{d^+})$. The manner in which this definition is obtained is inspired by Romaguera–Valero theorem and Kostanek–Waszkiewicz theorem. Furthermore, we obtain characterizations of local Yoneda-complete quasi-metric spaces via local nets in quasi-metric spaces. More precisely, we prove that a quasi-metric space is local Yoneda-complete if and only if every local net has a d-limit. Finally, we prove that every quasi-metric space has a local Yoneda completion.

Keywords: Quasi-metric space; local Yoneda-complete quasi-metric space; g-topology

1. Introduction

A quasi-metric on a nonempty set X is a map $d : X \times X \longrightarrow [0, +\infty]$ satisfying: $d(x, x) = 0$; $d(x, z) \leq d(x, y) + d(y, z)$; and $d(x, y) = d(y, x) = 0$ implies $x = y$. The pair (X, d) is then called a quasi-metric space. One motivation of domain theory is to study the properties of spaces and provide them with suitable computational models (Abramsky and Jung 1994; Gierz et al. 2003). The space of formal balls $\mathbf{B}(X, d)$ of a quasi-metric space (X, d) is probably the single most important artifact (Edalat and Heckmann 1998; Weihrauch and Schreiber 1981). Formal balls were introduced by Weihrauch and Schreiber (Weihrauch and Schreiber 1981). Edalat and Heckmann proved that a metric space is complete if and only if its poset of formal balls is a domain, and then they showed why formal balls were so important in the metric case (Edalat and Heckmann 1998). It is natural to ask whether the links between domain theory and quasi-metric space theory can be established in the style of Edalat and Heckmann. There are five important parallel theories in the setting of quasi-metric spaces and the formal ball model:

(1) [Romaguera–Valero Theorem] A quasi-metric space (X, d) is Smyth-complete if and only if $(\mathbf{B}(X, d), \leq^{d^+})$ is a continuous dcpo and its way-below relation is the relation $\prec$, defined by $(x, r) \prec (y, s)$ if and only if $d(x, y) < r - s$ (Romaguera and Valero 2010).

(2) [Kostanek–Waszkiewicz Theorem] A quasi-metric space (X, d) is Yoneda-complete if and only if $(\mathbf{B}(X, d), \leq^{d^+})$ is a dcpo (Kostanek and Waszkiewicz 2011).

(3) A quasi-metric space (X, d) is continuous Yoneda-complete if and only if $(\mathbf{B}(X, d), \leq^{d^+})$ is a domain (Goubault-Larrecq and Ng 2017).

†This work is supported by the National Natural Science Foundation of China (Grant Nos. 12101383, 11871320, 11531009) and the Fundamental Research Funds for the Central Universities (Grant No. GK202103006).

(4) A quasi-metric space (X, d) is algebraic Yoneda-complete if and only if $(\mathbf{B}(X, d), \leq^{d^+})$ is a domain with basis $\{(x, r) \mid x \text{ is finite}\}$ (Ali-Akbari et al. 2009).

(5) A Yoneda-complete quasi-metric space (X, d) is quasi-continuous if and only if $(\mathbf{B}(X, d), \leq^{d^+})$ is a quasi-continuous domain with a fin basis of $\{\{(x, s) \mid x \in G\} \mid G \in \mathscr{P}_f(X), s \in [0, +\infty)\}$, where $\mathscr{P}_f(X)$ is the set of all nonempty finite subsets of X (Ng and Ho 2019).

It is obvious from the above statements that we research on quasi-metric spaces and their spaces of formal balls, mostly focused on Yoneda-complete spaces, but those should have a meaning even in noncomplete spaces. One question arises naturally: can we give a characterization of properties of a non-Yoneda-complete quasi-metric space in terms of some order-theoretic properties of its poset of formal balls. Our objective is to answer this question. In this paper, we propose the notion of local Yoneda-complete quasi-metric spaces using the domain-theoretic properties of the poset of formal balls and prove that a quasi-metric space is local Yoneda-complete if and only if every local net has a d-limit.

Quasi-metric space unifies the metric space and partially ordered set, and this structure has been extensively studied by several people (Bonsangue et al. 1998; Lawvere 1973). Similar to the classical completion of metric spaces and posets, Smyth completions and Yoneda completions of quasi-metric spaces are fundamental in the study of quasi-metric spaces (Bonsangue et al. 1998; Künzi and Schellekens 2002; Vickers 2005). Smyth provided that there exists an idempotent Smyth completion to every quasi-metric space (Smyth 1987). Künzi and Schellekens (2002) prove that every quasi-metric space has a Yoneda completion. However, the fact remains that this Yoneda completion is not idempotent in general. In Ng and Ho (2017), Ng and Ho showed that every quasi-metric space (X, d) has an idempotent Yoneda completion via dcpo completion of $\mathbf{B}(X, d)$. After seeing that every quasi-metric space has a Smyth completion and Yoneda completion, one may ask: whether every quasi-metric space has a local Yoneda completion. In this paper, we shall provide a positive answer, that is, we prove that every quasi-metric space has a local Yoneda completion.

2. Quasi-Metric Spaces

Let (X, d) be a quasi-metric space, $(x_i)_{i\in I,\sqsubseteq}$ be a net, and $x \in X$. If for all $y \in X$, it holds that $d(x, y) = \limsup d(x_i, y)$, then $x \in X$ is called a d-limit of $(x_i)_{i\in I,\sqsubseteq}$, denoted by $x = \lim_d x_i$. A net $(x_i, r_i)_{i\in I,\sqsubseteq}$ in $X \times [0, +\infty)$ is *Cauchy-weighted* in (X, d) if $\bigwedge_{i\in I} r_i = 0$, and for each $i, i' \in I$, whenever $i \sqsubseteq i'$, $d(x_i, x_{i'}) \leq r_i - r_{i'}$. For any Cauchy-weighted net $(x_i, r_i)_{i\in I,\sqsubseteq}$ in (X, d), $x \in X$ is a d-limit of the Cauchy net $(x_i)_{i\in I,\sqsubseteq}$ if and only if for any $y \in X$, it holds that $d(x, y) = \bigvee_{i\in I} (d(x_i, y) - r_i)$.

Definition 1 (Ng and Ho 2017)**.** Let (X, d) and (X', d') be two quasi-metric spaces, and $f : X \longrightarrow X'$ be a mapping. We say that f is

(1) *limit-continuous* if it preserves Cauchy nets and their existing d-limits;

(2) *Y-continuous* if it is limit-continuous and nonexpansive, that is, for any $x, y \in X$, $d'(f(x), f(y)) \leq d(x, y)$.

Definition 2 (Goubault-Larrecq 2013)**.** Let (X, d) be a quasi-metric space. Define $\mathbf{B}(X, d) = X \times [0, +\infty)$. We call the elements of $\mathbf{B}(X, d)$ *formal balls*. On $\mathbf{B}(X, d)$, we can define a quasi-metric d^+ as follows:

$$\forall\, (x, r), (y, s) \in \mathbf{B}(X, d),\ \ d^+((x, r), (y, s)) = \max\,(d(x, y) - r + s, 0),$$

and as a result the induced order $\leq^{d^+}$ defined by

$$\forall\, (x, r), (y, s) \in \mathbf{B}(X, d),\ \ (x, r) \leq^{d^+} (y, s) \Longleftrightarrow d(x, y) \leq r - s.$$

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

Unless otherwise stated, throughout the paper, whenever an order is mentioned in the context of $\mathbf{B}(X, d)$, it is to be interpreted with respect to the induced order $\leq^{d^+}$ on $\mathbf{B}(X, d)$.

A quasi-metric space (X, d) is *standard* if and only if, for every directed family of formal balls $(x_i, r_i)_{i\in I}$, for every $s \in [0, +\infty)$, $(x_i, r_i)_{i\in I}$ has a supremum in $\mathbf{B}(X, d)$ if and only if $(x_i, r_i + s)_{i\in I}$ has a supremum in $\mathbf{B}(X, d)$.

Let (X, d) be a quasi-metric space. A directed family $(x_i, r_i)_{i\in I}$ in $\mathbf{B}(X, d)$ is *translational complete* if for any $a \in [-\bigwedge_{i\in I} r_i, +\infty)$, $\bigvee_{i\in I} (x_i, r_i + a)$ exists.

Remark 3. Let (X, d) be a quasi-metric space. If $(x_i, r_i)_{i\in I}$ is a translational complete directed family in $\mathbf{B}(X, d)$ and has supremum (x, r), then $(x_i, r_i + a)_{i\in I}$ is a translational complete directed family in $\mathbf{B}(X, d)$ and has supremum $(x, r + a)$ for any $a \in [-\bigwedge_{i\in I} r_i, +\infty)$.

Let (X, d) and (X', d') be two quasi-metric spaces. A map $g : \mathbf{B}(X, d) \longrightarrow \mathbf{B}(X', d')$ is *Y-Scott continuous* if it preserves the suprema of translational complete directed families.

Proposition 4 (Ng and Ho 2017)**.** *Let (X, d) and (X', d') be two quasi-metric spaces, and $f : X \longrightarrow X'$ be a mapping. Define a mapping $\mathbf{B}(f) : \mathbf{B}(X, d) \longrightarrow \mathbf{B}(X', d')$ as follows:*

$$\forall\, (x, r) \in \mathbf{B}(X, d),\ \mathbf{B}(f)(x, r) = (f(x), r).$$

Then f is Y-continuous if and only if $\mathbf{B}(f)$ is a Y-Scott continuous map.

Definition 5 (Ng and Ho 2017)**.** Let (X, d) be a quasi-metric space. A subset C of $\mathbf{B}(X, d)$ is called *g-closed* if it is

(1) downward closed with respect to $\leq^{d^+}$;

(2) closed under the existing suprema of translational complete directed families, that is, whenever $(x_i, r_i)_{i\in I} \subseteq C$ is a translational complete directed family that has supremum (x, r), then (x, r) belongs to C.

We denote the collection of all g-closed subsets of $\mathbf{B}(X, d)$ by $\Gamma_g(\mathbf{B}(X, d))$. Then $\Gamma_g(\mathbf{B}(X, d))$ is a co-topology on $\mathbf{B}(X, d)$, that is, the family

$$\{U \subseteq \mathbf{B}(X, d) \mid \text{the complement of } U \text{ is a } g\text{-closed subset of } \mathbf{B}(X, d)\}$$

is a topology, called the *g-topology* on $\mathbf{B}(X, d)$. For any subset E of $B(X, d)$, let $cl_g(E)$ denote the g-closure of E. Define the Hausdorf-f-Hoare quasi-metric $d_{\mathcal{H}}$ on $\Gamma_g(\mathbf{B}(X, d))$ by $d_{\mathcal{H}}(A, B) = \bigvee_{(a,m)\in A} \bigwedge_{(c,r)\in B} d^+((a, m), (c, r))$. Then $d_{\mathcal{H}}(A, B) = 0$ if and only if $A \subseteq B$.

Remark 6. Let (X, d) be a Yoneda-complete quasi-metric space. Then C is a g-closed set if and only if C is a Scott closed set.

Let $\alpha : \Gamma_g(\mathbf{B}(X, d)) \longrightarrow [0, +\infty]$, $C \longmapsto \bigwedge_{(x,r)\in C} r$. Call $\alpha(C)$ the aperture of C. Let $t \geq 0$ and $C + t = \{(c, t + s) \mid (c, s) \in C\}$. Then $C + t$ is a g-closed set. Let $r \leq \alpha(C)$ and $C - r = \{(c, s - r) \mid (c, s) \in C\}$. Then $C - r$ is a g-closed set.

Proposition 7 (Ng and Ho 2017)**.** *Let (X, d) be a quasi-metric space and $E \subseteq \mathbf{B}(X, d)$. Then $cl_g(E) = cl_g(E - \alpha(E)) + \alpha(E)$.*

Definition 8 (Mislove 1999)**.** A poset P is called a *local dcpo* if every directed subset of P with an upper bound has a least upper bound.

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

3. Local Yoneda-Complete Quasi-Metric Spaces

Definition 9. Let (X, d) be a quasi-metric space. Then (X, d) is called *local Yoneda-complete* if the following conditions are satisfied:

(1) whenever a directed family $(x_i, r_i)_{i\in I}$ in $\mathbf{B}(X, d)$ has supremum (x, r), then $r = \bigwedge_{i\in I} r_i$;

(2) $\mathbf{B}(X, d)$ is a local dcpo.

Remark 10. (1) Let (X, d) be a Yoneda-complete quasi-metric space. Then (X, d) is a local Yoneda-complete quasi-metric space. But the converse may not be true as shown by Examples 11 (1) and (2).

(2) The conditions (1) and (2) in Definition 9 are independent as shown by Examples 11 (2) and (3).

Example 11. (1) $([0, +\infty), d)$ is local Yoneda-complete, not Yoneda-complete, where $d: [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty]$ as follows:

$$d(x, y) = \begin{cases} 0, & x \le y, \\ x - y, & y < x. \end{cases}$$

In fact, the map $(x, r) \longmapsto (x-r, -r)$ defines an order isomorphism from $\mathbf{B}([0, +\infty), d)$ onto $C = \{(a, b) \in \mathbb{R} \times (-\infty, 0] \mid a \ge b\}$ ordered component-wise. Since C is a local dcpo, we have that $\mathbf{B}([0, +\infty), d)$ is a local dcpo. Let $(x_i, r_i)_{i\in I}$ be a directed family in $\mathbf{B}([0, +\infty), d)$ with supremum (x, r). Then $\bigvee^{C}_{i\in I} (x_i-r_i, -r_i) = (x - r, -r)$, and thus $\bigvee_{i\in I} (-r_i) = -r$. Hence, $r = \bigwedge_{i\in I} r_i$. So we conclude that $([0, +\infty), d)$ is local Yoneda-complete. Obviously, C is not a dcpo. Then $\mathbf{B}([0, +\infty), d)$ is not a dcpo, and thus $([0, +\infty), d)$ is not Yoneda-complete.

(2) Every poset $(X, \le)$ can be seen as a quasi-metric space by letting

$$d_{\le}(x, y) = \begin{cases} 0, & x \le y, \\ +\infty, & \text{otherwise.} \end{cases}$$

On formal balls $(x, r) \le^{d_{\le}^{+}} (y, s)$ if and only if $x \le y$ and $r \ge s$. So $(x, r) \longmapsto (x, -r)$ defines an order isomorphism from $\mathbf{B}(X, d_{\le})$ onto $X \times (-\infty, 0]$ ordered component-wise. Obviously, $(X, d_{\le})$ satisfies the condition (1) in Definition 9. This implies that $(X, d_{\le})$ is a local Yoneda-complete quasi-metric space if and only if X is a local dcpo. Since $\mathbb{N}$ is a local dcpo, not a dcpo, we have that $(\mathbb{N}, d_{\preceq})$ is local Yoneda-complete, not Yoneda-complete, where $\mathbb{N}$ denotes the set of natural numbers with the usual order $\preceq$. Moreover, let $X = \mathbb{N} \cup \{\omega_1, \omega_2\}$, and let the partial order $\le$ on X as follows:

- for $x \in \mathbb{N}$, $x \le \omega_1, \omega_2$;
- $\omega_1 \le \omega_1, \omega_2 \le \omega_2$;
- for $x, y \in \mathbb{N}$, $x \le y$ if and only if $x \preceq y$.

That is, we add ω_1, ω_2 above all elements in $\mathbb{N}$. Then X is not a local dcpo, and thus $(X, d_{\le})$ does not satisfy the condition (2) in Definition 9.

(3) Let $X = \{0\} \cup \{\frac{1}{2^m} \mid m \in \mathbb{N}_+\}$ and d on $X \times X$ which is defined as follows:

$$d(x, y) = \begin{cases} 0, & x = y, \\ +\infty, & x < y, \\ \sqrt{2}, & 0 = y < x, \\ x - y, & 0 < y < x, \end{cases}$$

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

where $\mathbb{N}_+$ denotes the set of all positive integers. Then (X, d) is a quasi-metric space. Let $(x, r), (y, s) \in \mathbf{B}(X, d)$. If $(x, r) \leq^{d^+} (y, s)$, then $d(x, y) \leq r - s$, and thus $x \geq y$. First, we shall prove that $\mathbf{B}(X, d)$ is a local dcpo. Let $(x_i, r_i)_{i\in I}$ be a directed family in $\mathbf{B}(X, d)$ with an upper bound (x, r). Then $d(x_i, x) \leq r_i - r$ for any $i \in I$. There are three cases:

Case 1. There exists $i_0 \in I$ such that $x_{i_0} = 0$. Let $J = \{i \in I \mid (x_{i_0}, r_{i_0}) \leq^{d^+} (x_i, r_i)\}$. Then $(x_j, r_j)_{j\in J}$ is a cofinal subset of $(x_i, r_i)_{i\in I}$. Obviously, $\bigvee_{j\in J} (x_j, r_j) = (0, \bigwedge_{i\in I} r_i)$. Then $\bigvee_{i\in I} (x_i, r_i) = (0, \bigwedge_{i\in I} r_i)$.

Case 2. $x_i \neq 0$ for any $i \in I$ and $\bigwedge_{i\in I} x_i = 0$. Since (x, r) is an upper bound of $(x_i, r_i)_{i\in I}$, we have that $x \leq x_i$ for any $i \in I$. Then $x = 0$, and thus $r + \sqrt{2} \leq r_i$ for any $i \in I$. Hence $(\bigwedge_{i\in I} r_i) - \sqrt{2} \geq 0$. Obviously, $(x_i, r_i) \leq^{d^+} (0, (\bigwedge_{i\in I} r_i) - \sqrt{2})$ for any $i \in I$. Let (y, s) be an upper bound of $(x_i, r_i)_{i\in I}$ in $\mathbf{B}(X, d)$. Then $y \leq x_i$ for any $i \in I$, and thus $y = 0$. This implies $\sqrt{2} + s \leq \bigwedge_{i\in I} r_i$, that is $d(0, 0) \leq (\bigwedge_{i\in I} r_i) - \sqrt{2} - s$, and hence $(0, (\bigwedge_{i\in I} r_i) - \sqrt{2}) \leq^{d^+} (y, s)$. Therefore, $\bigvee_{i\in I} (x_i, r_i) = (0, (\bigwedge_{i\in I} r_i) - \sqrt{2})$. In particular, $(\frac{1}{2^m}, \frac{1}{2^m} + \sqrt{2})_{m\in\mathbb{N}}$ is a directed family in $\mathbf{B}(X, d)$ and has an upper bound $(0, 0)$. So we conclude that $\bigvee_{m\in\mathbb{N}} (\frac{1}{2^m}, \frac{1}{2^m} + \sqrt{2}) = (0, 0)$.

Case 3. $x_i \neq 0$ for any $i \in I$ and $\bigwedge_{i\in I} x_i \neq 0$. Then there exists $m \in \mathbb{N}$ such that $\bigwedge_{i\in I} x_i = \frac{1}{2^m}$, and thus there exists $i_1 \in I$ such that $x_{i_1} = \frac{1}{2^m}$. For all i in I, there exists $j \in I$ such that (x_i, r_i), $(x_{i_1}, r_{i_1}) \leq^{d^+} (x_j, r_j)$, and thus $x_j = \frac{1}{2^m}$. So we conclude that $d(x_i, \frac{1}{2^m}) \leq r_i - r_j \leq r_i - \bigwedge_{i\in I} r_i$. This shows that $(\frac{1}{2^m}, \bigwedge_{i\in I} r_i)$ is an upper bound of $(x_i, r_i)_{i\in I}$. Let (y, s) be an upper bound of $(x_i, r_i)_{i\in I}$ in $\mathbf{B}(X, d)$. Then $d(x_i, y) \leq r_i - s$ for any $i \in I$. For all k in I, there exists $k_0 \in I$ such that $(x_k, r_k), (x_{i_1}, r_{i_1}) \leq^{d^+} (x_{k_0}, r_{k_0})$, and thus $x_{k_0} = \frac{1}{2^m}$ and $r_{k_0} \leq r_k$. So we conclude that $d(\frac{1}{2^m}, y) \leq r_{k_0} - s \leq r_k - s$. This implies $d(\frac{1}{2^m}, y) + s \leq \bigwedge_{i\in I} r_i$. Hence, $(\frac{1}{2^m}, \bigwedge_{i\in I} r_i) \leq^{d^+} (y, s)$. Therefore, $\bigvee_{i\in I} (x_i, r_i) = (\frac{1}{2^m}, \bigwedge_{i\in I} r_i)$.

Consequently, $\mathbf{B}(X, d)$ is a local dcpo. Yet since $\bigvee_{m\in\mathbb{N}} (\frac{1}{2^m}, \frac{1}{2^m} + \sqrt{2}) = (0, 0)$ as shown in Case 2 and $\bigwedge_{m\in\mathbb{N}} (\frac{1}{2^m} + \sqrt{2}) = \sqrt{2}$, we have that (X, d) does not satisfy the condition (1) in Definition 9.

Proposition 12. *A metric space (X, d) is local Yoneda-complete if and only if (X, d) is complete.*

Proof. Sufficiency. Let (X, d) be a complete metric space. Then (X, d) is a Yoneda-complete metric space, and thus (X, d) is local Yoneda-complete.

Necessity. Let (X, d) be a local Yoneda-complete metric space. We shall prove that $\mathbf{B}(X, d)$ is a dcpo. Let $(x_i, r_i)_{i\in I}$ be a directed family in $\mathbf{B}(X, d)$ and let $r_\infty = \bigwedge_{i\in I} r_i$. Then there exists $(y_n, s_n) \in (x_i, r_i)_{i\in I}$ such that $s_n < r_\infty + \frac{1}{n}$ for any $n \in \mathbb{N}_+$, where $\mathbb{N}_+$ denotes the set of all positive integers. Let $(x_1, r_1) = (y_1, s_1)$. For any $n > 1$, let (x_n, r_n) be an upper bound of (y_n, s_n) and (x_{n-1}, r_{n-1}). Then $(x_1, r_1) \leq^{d^+} (x_2, r_2) \leq^{d^+} \cdots \leq^{d^+} (x_n, r_n) \leq^{d^+} \cdots$, and thus $\cdots \leq r_n \leq r_{n-1} \leq \cdots \leq r_2 \leq r_1$. For a fixed $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}_+$ such that $r_n - r_m < \varepsilon$ for any $n_0 \leq n \leq m$. For all n in $\mathbb{N}_+$, there exists $l \in \mathbb{N}_+$ such that $n_0, n \leq l$, and by symmetry $d(x_l, x_{n_0}) \leq r_{n_0} - r_l < \varepsilon$, and thus $d(x_n, x_{n_0}) \leq d(x_n, x_l) + d(x_l, x_{n_0}) < r_n - r_l + \varepsilon \leq r_n + \varepsilon$. Hence $(x_n, r_n + \varepsilon) \leq^{d^+} (x_{n_0}, 0)$. So we conclude that $(x_{n_0}, 0)$ is an upper bound of the directed family $(x_n, r_n + \varepsilon)_{n\in\mathbb{N}_+}$. By hypothesis, we have that $\bigvee_{n\in\mathbb{N}_+} (x_n, r_n + \varepsilon)$ exists, denoted by (x, r). Then $r = \bigwedge_{n\in\mathbb{N}_+} (r_n + \varepsilon)$. Obviously, $(x, r - \varepsilon)$ is an upper bound of $(x_n, r_n)_{n\in\mathbb{N}_+}$. Let (y, s) be an upper bound of $(x_n, r_n)_{n\in\mathbb{N}_+}$. Then $(x_n, r_n + \varepsilon) \leq^{d^+} (y, s + \varepsilon)$ for any $n \in \mathbb{N}_+$, and thus $(x, r) \leq^{d^+} (y, s + \varepsilon)$. Hence $(x, r - \varepsilon) \leq^{d^+} (y, s)$, and so we conclude that $\bigvee_{n\in\mathbb{N}_+} (x_n, r_n) = (x, r - \varepsilon)$. Let $i \in I$. Then there exists $(z_n, t_n) \in (x_i, r_i)_{i\in I}$ such that $(x_n, r_n), (x_i, r_i) \leq^{d^+} (z_n, t_n)$ for any $n \in \mathbb{N}_+$. Hence

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

$$
\begin{aligned}
d(x_i, x) \le d(x_i, z_n) + d(z_n, x_n) + d(x_n, x) &\le r_i - t_n + r_n - t_n + r_n - (r - \varepsilon) \\
&= r_i - (r - \varepsilon) + 2(r_n - t_n) \\
&\le r_i - (r - \varepsilon) + 2(r_n - r_\infty) \\
&< r_i - (r - \varepsilon) + \tfrac{2}{n}.
\end{aligned}
$$

This implies $d(x_i, x) \le r_i - (r - \varepsilon)$. Then $(x_i, r_i) \le^{d^+} (x, r - \varepsilon)$, and thus $(x, r - \varepsilon)$ is an upper bound of $(x_i, r_i)_{i\in I}$. Suppose that (z, t) is an upper bound of $(x_i, r_i)_{i\in I}$. Then (z, t) is an upper bound of $(x_n, r_n)_{n\in\mathbb{N}_+}$, and thus $(x, r - \varepsilon) \le^{d^+} (z, t)$. Hence, $\bigvee_{i\in I} (x_i, r_i) = (x, r - \varepsilon)$. This shows that $\mathbf{B}(X, d)$ is a dcpo. Therefore, (X, d) is complete. □

Definition 13. Let (X, d) be a quasi-metric space, and $(x_i)_{i\in I,\sqsubseteq}$ be a net. If there exists $(r_i)_{i\in I,\sqsubseteq} \subseteq [0, +\infty)$ such that $(x_i, r_i)_{i\in I,\sqsubseteq}$ is a monotone net in $\mathbf{B}(X, d)$ and has an upper bound, then $(x_i)_{i\in I,\sqsubseteq}$ is called a *local net*.

Theorem 14. *Let (X, d) be a quasi-metric space. Then every local net has a d-limit if and only if (X, d) is a local Yoneda-complete quasi-metric space.*

Proof. Necessity. Let (X, d) be a quasi-metric space where every local net has a d-limit. We start by proving part (2) of Definition 9. Let $(x_i, r_i)_{i\in I}$ be a directed family in $\mathbf{B}(X, d)$ with an upper bound. Define a preorder $\sqsubseteq$ on I as follows:

$$\forall\, i, j \in I,\ i \sqsubseteq j \Longleftrightarrow (x_i, r_i) \le^{d^+} (x_j, r_j).$$

Then $(x_i, r_i)_{i\in I,\sqsubseteq}$ is a monotone net in $\mathbf{B}(X, d)$ and has an upper bound, and thus $(x_i)_{i\in I,\sqsubseteq}$ is a local net. Hence, the net $(x_i)_{i\in I,\sqsubseteq}$ has a d-limit y. This implies $\bigwedge_{i\in I} \bigvee_{j\in I, i\sqsubseteq j} d(x_j, y) = 0$. Then for all $\varepsilon > 0$, there exists $i_0 \in I$ such that $\bigvee_{j\in I, i_0\sqsubseteq j} d(x_j, y) < \varepsilon$. Let $i \in I$. Then there exists $j \in I$ such that $i, i_0 \sqsubseteq j$, and thus

$$
\begin{aligned}
d(x_i, y) \le d(x_i, x_j) + d(x_j, y) &\le r_i - r_j + d(x_j, y) \\
&\le r_i - \textstyle\bigwedge_{i\in I} r_i + d(x_j, y) \\
&< r_i - \textstyle\bigwedge_{i\in I} r_i + \varepsilon.
\end{aligned}
$$

So we conclude that $d(x_i, y) \le r_i - \bigwedge_{i\in I} r_i$. Hence, $(y, \bigwedge_{i\in I} r_i)$ is an upper bound of $(x_i, r_i)_{i\in I}$. Let (z, s) be an upper bound of $(x_i, r_i)_{i\in I}$. Then $d(x_i, z) \le r_i - s$ for any $i \in I$, and thus $s \le \bigwedge_{i\in I} r_i$. Since $d(y, z) = \bigwedge_{i\in I} \bigvee_{j\in I, i\sqsubseteq j} d(x_j, z) \le \bigwedge_{i\in I} \bigvee_{j\in I, i\sqsubseteq j} (r_j - s) \le \bigwedge_{i\in I} (r_i - s) = (\bigwedge_{i\in I} r_i) - s$, we have that $(y, \bigwedge_{i\in I} r_i) \le^{d^+} (z, s)$. This shows that $\bigvee_{i\in I} (x_i, r_i) = (y, \bigwedge_{i\in I} r_i)$. Part (1) of Definition 9 is then a trivial consequence. Therefore, (X, d) is a local Yoneda-complete quasi-metric space.

Sufficiency. Let (X, d) be a local Yoneda-complete quasi-metric space, and let $(x_i)_{i\in I,\sqsubseteq}$ be a local net. Then, there exists $(r_i)_{i\in I,\sqsubseteq} \subseteq [0, +\infty)$ such that $(x_i, r_i)_{i\in I,\sqsubseteq}$ is a monotone net in $\mathbf{B}(X, d)$ and has an upper bound, and thus $(x_i, r_i)_{i\in I}$ is a directed family in $\mathbf{B}(X, d)$ and has an upper bound. Hence $\bigvee_{i\in I} (x_i, r_i)$ exists, denoted by (x, r). By hypothesis, we have that $r = \bigwedge_{i\in I} r_i$. Then $(x_i, r_i - \bigwedge_{i\in I} r_i)_{i\in I,\sqsubseteq}$ is a Cauchy-weighted net and $\bigvee_{i\in I} (x_i, r_i - \bigwedge_{i\in I} r_i) = (x, 0)$. Next, we shall prove that x is the d-limit of $(x_i)_{i\in I,\sqsubseteq}$. For any $z \in X$, since $d(x_i, z) \le d(x_i, x) + d(x, z) \le r_i - \bigwedge_{i\in I} r_i + d(x, z)$ for any $i \in I$, we have that $d(x_i, z) - r_i \le d(x, z) - \bigwedge_{i\in I} r_i$ for any $i \in I$. Obviously, $\bigvee_{i\in I} (d(x_i, z) - r_i)$ exists, denoted by s. Then $s \le d(x, z) - \bigwedge_{i\in I} r_i$. Suppose that $s < d(x, z) - \bigwedge_{i\in I} r_i$. Then $s < +\infty$. Since $d(x_i, z) - r_i \le s$ for any $i \in I$, we have that $d(x_i, z) \le r_i + s$ for any $i \in I$. Then $(x_i, r_i + s) \le^{d^+} (z, 0)$ for any $i \in I$, and thus $\bigvee_{i\in I} (x_i, r_i + s)$

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

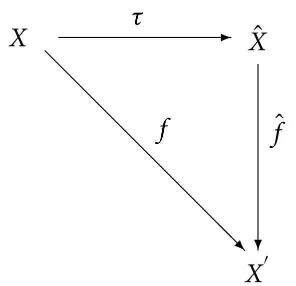
exists, denoted by $(y, \bigwedge_{i\in I} r_i + s)$. Hence $y = x$, that is $\bigvee_{i\in I}(x_i, r_i + s) = (x, \bigwedge_{i\in I} r_i + s)$. This implies $(x, \bigwedge_{i\in I} r_i + s) \leq^{d^+} (z, 0)$. Then $d(x, z) \leq \bigwedge_{i\in I} r_i + s$, which is a contradiction. So we conclude that $s = d(x, z) - \bigwedge_{i\in I} r_i$. Therefore, $d(x, z) = \bigvee_{i\in I}(d(x_i, z) - r_i) + \bigwedge_{i\in I} r_i = \bigvee_{i\in I}(d(x_i, z) - r_i + \bigwedge_{i\in I} r_i)$. This shows that x is the d-limit of $(x_i)_{i\in I,\sqsubseteq}$. □

Proposition 15. *Every local Yoneda-complete quasi-metric space is standard.*

Proof. Let (X, d) be a local Yoneda-complete quasi-metric space and let $(x_i, r_i)_{i\in I}$ be a directed family in $\mathbf{B}(X, d)$. By Lemma 7.4.25 of Goubault-Larrecq (2013) and Theorem 14, we have that (x, r) is the supremum of $(x_i, r_i)_{i\in I}$ if and only if x is the d-limit of $(x_i)_{i\in I,\sqsubseteq}$ and $r = \bigwedge_{i\in I} r_i$. For any $s \in [0, +\infty)$, then the existence of a supremum is equivalent for $(x_i, r_i)_{i\in I}$ and for $(x_i, r_i + s)_{i\in I}$, both being equivalent to the existence of a d-limit of the net $(x_i)_{i\in I,\sqsubseteq}$. Therefore, (X, d) is standard. □

4. The Local Yoneda Completions of Quasi-Metric Spaces

Definition 16. A *local Yoneda completion* of a quasi-metric space (X, d) is a local Yoneda-complete quasi-metric space $(\hat{X}, \hat{d})$, together with a Y-continuous map $\tau : X \longrightarrow \hat{X}$, such that for any local Yoneda-complete quasi-metric space (X', e) and Y-continuous map $f : X \longrightarrow X'$, there exists a unique Y-continuous map $\hat{f} : \hat{X} \longrightarrow X'$ such that $f = \hat{f} \circ \tau$, i.e., the following diagram commutes:



Definition 17. Let (X, d) be a quasi-metric space. A subset A of $\mathbf{B}(X, d)$ is called a *local g-set* if for any Y-continuous function $f : X \longrightarrow X'$ mapping into a local Yoneda complete quasi-metric space (X', d'), there exists a unique $(y_A, r_A) \in \mathbf{B}(X', d')$ such that $cl_g(\mathbf{B}(f)(A)) = \downarrow(y_A, r_A)$.

Proposition 18. Let (X, d) be a quasi-metric space, and A be a local g-set satisfying $\alpha(A) = 0$. If (Y, d') is a local Yoneda complete quasi-metric space and $f : X \longrightarrow Y$ is a Y-continuous function, then there exists a unique $(y_A, r_A) \in \mathbf{B}(Y, d')$ such that $cl_g(\mathbf{B}(f)(A)) = \downarrow(y_A, r_A)$ and $r_A = 0$.

Proof. By Definition 17, we only need to prove that $r_A = 0$. Let $(y, s) \in cl_g(\mathbf{B}(f)(A))$. Then $(y, s) \leq^{d^+} (y_A, r_A)$, and thus $r_A \leq s$. Hence $r_A \leq \alpha(cl_g(\mathbf{B}(f)(A))) \leq \alpha(A) = 0$, that is, $r_A = 0$. □

Let $\tilde{X} = \{A \subseteq \mathbf{B}(X, d) \mid A \text{ is a } g\text{-closed set satisfying } \alpha(A) = 0\}$, where the notion of g-closed sets is introduced in Definition 5. Define a mapping $\tilde{d} = d_{\mathcal{H}}|_{\tilde{X}\times\tilde{X}} : \tilde{X} \times \tilde{X} \longrightarrow [0, +\infty]$ as follows:

$$\forall (A, B) \in \tilde{X} \times \tilde{X},\ \tilde{d}(A, B) = \bigvee_{(a,m)\in A} \bigwedge_{(c,r)\in B} d^+((a, m), (c, r)).$$

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

Then $(\tilde{X},\tilde{d})$ is a Yoneda-complete quasi-metric space (see Ng and Ho 2017). We write

$$\hat{X}=\{A\subseteq \mathbf{B}(X,d)\mid A \text{ is a } g\text{-closed local } g\text{-set satisfying } \alpha(A)=0\}.$$

Then $\hat{X}\subseteq\tilde{X}$. Define the mapping $\hat{d}$ as the restriction of $\tilde{d}$ to $\hat{X}\times\hat{X}$. Then $(\hat{X},\hat{d})$ is a quasi-metric space.

Proposition 19. *Let (X,d) be a quasi-metric space, and $(A_{i_1},r_{i_1}),(A_{i_2},r_{i_2})\in\mathbf{B}(\tilde{X},\tilde{d})$. Then $(A_{i_1},r_{i_1})\leq^{\tilde{d}^+}(A_{i_2},r_{i_2})$ if and only if $A_{i_1}+r_{i_1}\subseteq A_{i_2}+r_{i_2}$.*

Proof. This immediately follows from Ng and Ho (2017, Lemma 3.14). □

Proposition 20. *Let (X,d) be a quasi-metric space, and $(A_i,r_i)_{i\in I}$ be a directed family in $\mathbf{B}(\tilde{X},\tilde{d})$. Then $\bigvee_{i\in I}^{\mathbf{B}(\tilde{X},\tilde{d})}(A_i,r_i)=(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)$.*

Proof. Obviously, $cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))$ is a g-closed set. Since

$$\begin{aligned}\alpha(cl_g(\textstyle\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)))&\leq\alpha(\textstyle\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))\\&=\textstyle\bigwedge_{i\in I}\alpha(A_i+r_i-\bigwedge_{i\in I}r_i)\\&=\textstyle\bigwedge_{i\in I}(r_i-\bigwedge_{i\in I}r_i)\\&=0,\end{aligned}$$

we have that $cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))\in\tilde{X}$. Since $(A_i,r_i)\leq^{\tilde{d}^+}(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)$ for any $i\in I$, we have that $(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)$ is an upper bound of $(A_i,r_i)_{i\in I}$ in $\mathbf{B}(\tilde{X},\tilde{d})$. Let (B,s) be an upper bound of $(A_i,r_i)_{i\in I}$ in $\mathbf{B}(\tilde{X},\tilde{d})$. Then $(A_i,r_i)\leq^{\tilde{d}^+}(B,s)$ for any $i\in I$, and thus $A_i+r_i\subseteq B+s$ for any $i\in I$. So we conclude that $\bigcup_{i\in I}(A_i+r_i)\subseteq B+s$, and hence $cl_g(\bigcup_{i\in I}(A_i+r_i))\subseteq B+s$. Since $\alpha(\bigcup_{i\in I}(A_i+r_i))=\bigwedge_{i\in I}r_i$, it follows from Proposition 7 that $cl_g(\bigcup_{i\in I}(A_i+r_i))=cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))+\bigwedge_{i\in I}r_i$. Therefore, $cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))+\bigwedge_{i\in I}r_i\subseteq B+s$, whence $(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)\leq^{\tilde{d}^+}(B,s)$ by Proposition 19. Therefore, $\bigvee_{i\in I}^{\mathbf{B}(\tilde{X},\tilde{d})}(A_i,r_i)=(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)$. □

Next, we show that the sub-poset $\mathbf{B}(\hat{X},\hat{d})$ of $\mathbf{B}(\tilde{X},\tilde{d})$ is closed under least upper bounds of bounded directed families.

Proposition 21. *Let (X,d) be a quasi-metric space, and $(A_i,r_i)_{i\in I}$ be a directed family in $\mathbf{B}(\hat{X},\hat{d})$ with an upper bound (A,r). Then $\bigvee_{i\in I}^{\mathbf{B}(\hat{X},\hat{d})}(A_i,r_i)=(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)$.*

Proof. Let $(A_i,r_i)_{i\in I}$ be a directed family in $B(\hat{X},\hat{d})$ with an upper bound (A,r). Then $(A_i,r_i)_{i\in I}$ is a directed family in $B(\tilde{X},\tilde{d})$. By Proposition 20, $\bigvee_{i\in I}^{B(\tilde{X},\tilde{d})}(A_i,r_i)=(cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i)),\bigwedge_{i\in I}r_i)$ holds. Next, we shall prove that $cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))\in\hat{X}$. It suffices to prove that $cl_g(\bigcup_{i\in I}(A_i+r_i-\bigwedge_{i\in I}r_i))$ is a local g-set. Let (Y,d') be a local Yoneda-complete

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

quasi-metric space and $f: X \longrightarrow Y$ be a Y-continuous function. Then there exists a unique $(y_{A_i}, r_{A_i}) \in B(Y, d^{'})$ such that $cl_g(\mathbf{B}(f)(A_i)) = \downarrow(y_{A_i}, r_{A_i})$ and $r_{A_i} = 0$ for any $i \in I$, and there exists a unique $(y_A, r_A) \in B(Y, d^{'})$ such that $cl_g(\mathbf{B}(f)(A)) = \downarrow(y_A, r_A)$ and $r_A = 0$. Let $i_1, i_2 \in I$. Then there exists $i_3 \in I$ such that $(A_{i_1}, r_{i_1}), (A_{i_2}, r_{i_2}) \leq^{\hat{d}^+} (A_{i_3}, r_{i_3})$, and thus $A_{i_1} + r_{i_1} \subseteq A_{i_3} + r_{i_3}$ and $A_{i_2} + r_{i_2} \subseteq A_{i_3} + r_{i_3}$. Hence $cl_g(\mathbf{B}(f)(A_{i_1} + r_{i_1})) \subseteq cl_g(\mathbf{B}(f)(A_{i_3} + r_{i_3}))$ and $cl_g(\mathbf{B}(f)(A_{i_2} + r_{i_2})) \subseteq cl_g(\mathbf{B}(f)(A_{i_3} + r_{i_3}))$. By Proposition 7, we have that $cl_g(\mathbf{B}(f)(A_i + r_i)) = cl_g(\mathbf{B}(f)(A_i)) + r_i$ for any $i \in I$. Then $cl_g(\mathbf{B}(f)(A_{i_1})) + r_{i_1} \subseteq cl_g(\mathbf{B}(f)(A_{i_3})) + r_{i_3}$ and $cl_g(\mathbf{B}(f)(A_{i_2})) + r_{i_2} \subseteq cl_g(\mathbf{B}(f)(A_{i_3})) + r_{i_3}$, and thus $\downarrow(y_{A_{i_1}}, 0) + r_{i_1} \subseteq \downarrow(y_{A_{i_3}}, 0) + r_{i_3}$ and $\downarrow(y_{A_{i_2}}, 0) + r_{i_2} \subseteq \downarrow(y_{A_{i_3}}, 0) + r_{i_3}$. Hence $(y_{A_{i_1}}, r_{i_1}), (y_{A_{i_2}}, r_{i_2}) \leq^{d^{'+}} (y_{A_{i_3}}, r_{i_3})$, and so we conclude that $(y_{A_i}, r_i)_{i \in I}$ is a directed family in $\mathbf{B}(Y, d^{'})$. Similarly, we can check that (y_A, r) is an upper bound of $(y_{A_i}, r_i)_{i \in I}$. Since $(Y, d^{'})$ is a local Yoneda-complete quasi-metric space, we have that $\bigvee_{i \in I} (y_{A_i}, r_i)$ exists. Then $(y_{A_i}, r_i - \bigwedge_{i \in I} r_i)_{i \in I}$ is a directed family in $B(Y, d^{'})$ and has an upper bound, hence there exists $b \in Y$ such that $\bigvee_{i \in I} (y_{A_i}, r_i - \bigwedge_{i \in I} r_i) = (b, 0)$. This implies $\mathbf{B}(f)(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)) \subseteq \downarrow(b, 0)$. Let B be a g-closed set satisfying $\mathbf{B}(f)(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)) \subseteq B$. Then $(y_{A_i}, r_i - \bigwedge_{i \in I} r_i) \in B$ for any $i \in I$. Obviously, $(y_{A_i}, r_i - \bigwedge_{i \in I} r_i)_{i \in I}$ is a translational complete directed set. Then $\bigvee_{i \in I} (y_{A_i}, r_i - \bigwedge_{i \in I} r_i) = (b, 0) \in B$, and thus $\downarrow(b, 0) \subseteq B$. Hence, $cl_g(\mathbf{B}(f)(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i))) = \downarrow(b, 0)$. This shows that $cl_g(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)) \in \hat{X}$. Therefore, $\bigvee_{i \in I}^{\mathbf{B}(\hat{X}, \hat{d})} (A_i, r_i) = (cl_g(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)), \bigwedge_{i \in I} r_i)$. □

Let $\mathscr{A} \subseteq \tilde{X}$. $\mathscr{A}$ satisfies **Condition** $(*)$ if, whenever for any directed family $(A_i, r_i)_{i \in I}$ in $\mathbf{B}(\mathscr{A}, d_{\mathscr{H}} |_{\mathscr{A} \times \mathscr{A}})$ with an upper bound (Z, t), then

$$\bigvee_{i \in I}^{\mathbf{B}(\tilde{X}, \tilde{d})} (A_i, r_i) \in \mathbf{B}(\mathscr{A}, d_{\mathscr{H}} |_{\mathscr{A} \times \mathscr{A}}).$$

Proposition 21 means that $\hat{X}$ satisfies **Condition** $(*)$. Let $\Psi(X) = \{\downarrow(x, 0) \mid x \in X\}$, and

$$cl_L(\Psi(X)) = \bigcap \{\mathscr{A} \subseteq \tilde{X} \mid \Psi(X) \subseteq \mathscr{A} \text{ and } \mathscr{A} \text{ satisfies } \textbf{Condition } (*)\}.$$

Define the mapping d_L as the restriction of $\tilde{d}$ to $cl_L(\Psi(X))$. Then $(cl_L(\Psi(X)), d_L)$ is a quasi-metric space.

Proposition 22. *Let (X, d) be a quasi-metric space. Then $(cl_L(\Psi(X)), d_L)$ is a local Yoneda-complete quasi-metric space.*

Proof. We start by proving part (2) of Definition 9. Let $(A_i, r_i)_{i \in I}$ be a directed family in $\mathbf{B}(cl_L(\Psi(X)), d_L)$ with an upper bound. If $\mathscr{A} \subseteq \tilde{X}$ with $\Psi(X) \subseteq \mathscr{A}$ satisfies **Condition** $(*)$, then $(A_i, r_i)_{i \in I}$ is a directed family in $\mathbf{B}(\mathscr{A}, d_{\mathscr{H}} |_{\mathscr{A} \times \mathscr{A}})$ and has an upper bound. So we conclude that $\bigvee_{i \in I}^{\mathbf{B}(\tilde{X}, \tilde{d})} (A_i, r_i) \in \mathbf{B}(\mathscr{A}, d_L)$. By Proposition 20, we have that $\bigvee_{i \in I}^{\mathbf{B}(\tilde{X}, \tilde{d})} (A_i, r_i) = (cl_g(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)), \bigwedge_{i \in I} r_i)$. Then $cl_g(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)) \in \mathscr{A}$, and thus $cl_g(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)) \in cl_L(\Psi(X))$. Hence $\bigvee_{i \in I}^{\mathbf{B}(cl_L(\Psi(X)), d_L)} (A_i, r_i) = (cl_g(\bigcup_{i \in I} (A_i + r_i - \bigwedge_{i \in I} r_i)), \bigwedge_{i \in I} r_i)$. Part (1) of Definition 9 is then a trivial consequence. Therefore, $(cl_L(\Psi(X)), d_L)$ is a local Yoneda-complete quasi-metric space. □

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

Proposition 23. *Let (X, d) be a quasi-metric space. Define a mapping $\zeta : X \longrightarrow cl_L(\Psi(X))$ as follows:*

$$\forall\, x \in X,\ \zeta(x) = \downarrow(x, 0).$$

Then ζ is a Y-continuous mapping.

Proof. Let $(x, r), (y, s) \in \mathbf{B}(X, d)$ satisfying $(x, r) \leq^{d^+} (y, s)$. Then $d(x, y) \leq r - s$. If $(a, m) \in \downarrow(x, 0)$, then $d^+((a, m), (x, 0)) = 0$, and thus

$$\bigwedge_{(b,n)\in\downarrow(y,0)} d^+((a, m), (b, n)) \leq d^+((a, m), (y, 0)) \leq d^+((x, 0), (y, 0)) \leq r - s.$$

Hence $\bigvee_{(a,m)\in\downarrow(x,0)} \bigwedge_{(b,n)\in\downarrow(y,0)} d^+((a, m), (b, n)) \leq r - s$. So we conclude that

$$d_L(\downarrow(x, 0), \downarrow(y, 0)) \leq r - s.$$

Therefore, $(\downarrow(x, 0), r) \leq^{d_L^+} (\downarrow(y, 0), s)$. This shows that $\mathbf{B}(\zeta)(x, r) \leq^{d_L^+} \mathbf{B}(\zeta)(y, s)$.

Let $(x_i, r_i)_{i\in I}$ be a translational complete directed family in $\mathbf{B}(X, d)$ with supremum (x, r). Then $r = \bigwedge_{i\in I} r_i$, and $(\downarrow(x_i, 0), r_i)_{i\in I}$ is a directed family in $\mathbf{B}(cl_L(\Psi(X)), d_L)$. Since $(x_i, r_i) \leq^{d^+} (x, r)$ for any $i \in I$, we have that $(\downarrow(x_i, 0), r_i) \leq^{d_L^+} (\downarrow(x, 0), r)$ for any $i \in I$. Let $(B, s) \in \mathbf{B}(cl_L(\Psi(X)), d_L)$ such that $(\downarrow(x_i, 0), r_i) \leq^{d_L^+} (B, s)$ for any $i \in I$. Then $s \leq \bigwedge_{i\in I} r_i$, and $(x_i, r_i) \in B + s$ for any $i \in I$. Since $B + s$ is a g-closed set, we have that $(x, r) \in B + s$. Then $(x, r - s) \in B$, and thus $\downarrow(x, r - s) \subseteq B$. Therefore, $(\downarrow(x, 0), r) \leq^{d_L^+} (B, s)$. So we conclude that $\bigvee^{\mathbf{B}(cl_L(\Psi(X)),d_L)}_{i\in I} (\downarrow(x_i, 0), r_i) = (\downarrow(x, 0), r)$. This shows that $\mathbf{B}(\zeta)$ is Y-Scott continuous, and hence ζ is a Y-continuous mapping. □

Proposition 24. *Let (X, d) be a quasi-metric space. Then $\hat{X} = cl_L(\Psi(X))$.*

Proof. By Proposition 21, we have that $\hat{X}$ satisfies **Condition** $(*)$. Obviously, $\Psi(X) \subseteq \hat{X}$. Then $cl_L(\Psi(X)) \subseteq \hat{X}$. Let $A \in \hat{X}$. By Proposition 22, we have that $(cl_L(\Psi(X)), d_L)$ is a local Yoneda-complete quasi-metric space. It follows from Proposition 23 that there exists $(M, s) \in \mathbf{B}(cl_L(\Psi(X)), d_L)$ such that $cl_g(\mathbf{B}(\zeta)(A)) = \downarrow(M, s)$. Then $s = 0$ and $A \subseteq M$. Let

$$\mathscr{B} = \{(Z, t) \in B(\tilde{X}, \tilde{d}) \mid d_{\mathscr{H}}(Z, A) \leq t\} \cap \mathbf{B}(cl_L(\Psi(X)), d_L).$$

Then $\mathscr{B}$ is a g-closed subset of $\mathbf{B}(cl_L(\Psi(X)), d_L)$. Since $\mathbf{B}(\zeta)(A) \subseteq \mathscr{B}$, we have that $cl_g(\mathbf{B}(\zeta)(A)) \subseteq \mathscr{B}$. Then $(M, 0) \in \mathscr{B}$, and thus $d_{\mathscr{H}}(M, A) = 0$. So we conclude that $M \subseteq A$. Therefore, $M = A \in cl_L(\Psi(X))$. This shows that $\hat{X} = cl_L(\Psi(X))$. □

Corollary 25. *Let (X, d) be a quasi-metric space. Then $(\hat{X}, \hat{d})$ is a local Yoneda-complete quasi-metric space.*

Proof. Immediately from Propositions 22 and 24. □

Proposition 26. *Let (X, d), (X', d') be two quasi-metric spaces, and $f : X \longrightarrow X'$ be a function. Then $\mathbf{B}(f)$ is a Y-Scott continuous map if and only if $\mathbf{B}(f)$ is continuous with respect to the g-topology.*

Proof. Only-if direction. Let C be a g-closed subset of $\mathbf{B}(X', d')$. Then $(\mathbf{B}(f))^{-1}(C)$ is downward closed. Let $(x_i, r_i)_{i\in I}$ be a translational complete directed family in $(\mathbf{B}(f))^{-1}(C)$ with supremum (x, r). Then $(f(x_i), r_i)_{i\in I} \subseteq C$. Since $\mathbf{B}(f)$ is a Y-Scott continuous map, we have that $(f(x_i), r_i)_{i\in I}$

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

is a translational complete directed family in $\mathbf{B}(X', d')$, and $\bigvee_{i\in I}(f(x_i), r_i) = \mathbf{B}(f)(x, r) = (f(x), r)$. Then $(f(x), r) \in C$, and thus $(x, r) \in (\mathbf{B}(f))^{-1}(C)$. Therefore, $(\mathbf{B}(f))^{-1}(C)$ is a g-closed set. This shows that $\mathbf{B}(f)$ is continuous with respect to the g-topology.

If direction. Let $(x_1, r_1), (x_2, r_2) \in \mathbf{B}(X, d)$ satisfying $(x_1, r_1) \leq^{d^+} (x_2, r_2)$. Clearly, $(f(x_2), r_2) \in \downarrow(f(x_2), r_2)$, and therefore, $(x_2, r_2) \in (\mathbf{B}(f))^{-1}(\downarrow(f(x_2), r_2))$. Since $\downarrow(f(x_2), r_2)$ is a g-closed set, so is $(\mathbf{B}(f))^{-1}(\downarrow(f(x_2), r_2))$. Thus, $(x_1, r_1) \in (\mathbf{B}(f))^{-1}(\downarrow(f(x_2), r_2))$, which implies $(f(x_1), r_1) \in \downarrow(f(x_2), r_2)$, whence $(f(x_1), r_1) \leq^{d'^+} (f(x_2), r_2)$.

Let $(x_i, r_i)_{i\in I}$ be a translational complete directed family in $\mathbf{B}(X, d)$ with supremum (x, r). Then $(f(x_i), r_i) \leq^{d'^+} (f(x), r)$ for any $i \in I$. Let (y, s) be an upper bound of $(f(x_i), r_i)_{i\in I}$. Then $(f(x_i), r_i) \in \downarrow(y, s)$ for any $i \in I$, and thus $(x_i, r_i) \in (\mathbf{B}(f))^{-1}(\downarrow(y, s))$ for any $i \in I$. Obviously, $\downarrow(y, s)$ is a g-closed set. Then $(\mathbf{B}(f))^{-1}(\downarrow(y, s))$ is a g-closed set, and thus $(x, r) \in (\mathbf{B}(f))^{-1}(\downarrow(y, s))$, that is, $(f(x), r) \leq^{d'^+} (y, s)$. Therefore, $\bigvee_{i\in I}(f(x_i), r_i) = (f(x), r)$. This implies that $(f(x_i), r_i)_{i\in I}$ is a translational complete directed family in $\mathbf{B}(X', d')$. So we conclude that $\mathbf{B}(f)$ is a Y-Scott continuous map. □

Theorem 27. *Let (X, d) be a quasi-metric space. Then $(\hat{X}, \hat{d})$ is the local Yoneda completion of (X, d).*

Proof. Let (X', d') be a local Yoneda-complete quasi-metric space, and $f : X \longrightarrow X'$ be a Y-continuous mapping. For all A in $\hat{X}$, there exists a unique $(y_A, r_A) \in \mathbf{B}(X', d')$ such that $cl_g(\mathbf{B}(f)(A)) = \downarrow(y_A, r_A)$ and $r_A = 0$. Define a mapping $f^* : \hat{X} \longrightarrow X'$ as follows:

$$\forall\, A \in \hat{X},\ f^*(A) = y_A.$$

Then f^* is well-defined.

Claim 1. $f^* \circ \zeta = f$.

Let $x \in X$. Then $f^*(\zeta(x)) = f^*(\downarrow(x, 0))$. Since

$$cl_g(\mathbf{B}(f)(\downarrow(x, 0))) = cl_g(\{(f(y), s) \mid (y, s) \leq^{d^+} (x, 0)\}) = \downarrow(f(x), 0),$$

we have that $f^*(\downarrow(x, 0)) = f(x)$. Then $f^* \circ \zeta = f$.

Claim 2. f^* is a Y-continuous mapping.

To prove this claim, we show that $\mathbf{B}(f^*)$ is Y-Scott continuous and then use Proposition 4. Let $(B_1, r), (B_2, s) \in \mathbf{B}(\hat{X}, \hat{d})$ satisfying $(B_1, r) \leq^{\hat{d}^+} (B_2, s)$. Then $B_1 + r \subseteq B_2 + s$, and thus $B_1 + r - s \subseteq B_2$. Therefore, $\mathbf{B}(f)(B_1 + r - s) \subseteq \mathbf{B}(f)(B_2)$, and so we conclude that $cl_g(\mathbf{B}(f)(B_1 + r - s)) \subseteq cl_g(\mathbf{B}(f)(B_2))$. Since $cl_g(\mathbf{B}(f)(B_1)) = \downarrow(y_{B_1}, 0)$, $cl_g(\mathbf{B}(f)(B_2)) = \downarrow(y_{B_2}, 0)$, and $cl_g(\mathbf{B}(f)(B_1 + r - s)) = cl_g(\mathbf{B}(f)(B_1)) + (r - s)$, we have that $(y_{B_1}, r - s) \leq^{d'^+} (y_{B_2}, 0)$. Then $d'(y_{B_1}, y_{B_2}) \leq r - s$. This shows that $\mathbf{B}(f^*)(B_1, r) \leq^{d'^+} \mathbf{B}(f^*)(B_2, s)$.

Let $(A_i, r_i)_{i\in I}$ be a translational complete directed family in $\mathbf{B}(\hat{X}, \hat{d})$ satisfying $\bigvee^{\mathbf{B}(\hat{X},\hat{d})}_{i\in I}(A_i, r_i)$ exists. By Proposition 21, we have that $\bigvee^{\mathbf{B}(\hat{X},\hat{d})}_{i\in I}(A_i, r_i) = (cl_g(\bigcup_{i\in I}(A_i + r_i - \bigwedge_{i\in I} r_i)), \bigwedge_{i\in I} r_i)$. Next, we shall prove that

$$\mathbf{B}(f^*)\Big(\bigvee^{\mathbf{B}(\hat{X},\hat{d})}_{i\in I}(A_i, r_i)\Big) = \bigvee^{\mathbf{B}(X',d')}_{i\in I}\mathbf{B}(f^*)(A_i, r_i),$$

that is, $(f^*(A), \bigwedge_{i\in I} r_i) = \bigvee_{i\in I}^{\mathbf{B}(X',d')} (f^*(A_i), r_i)$, where $A = cl_g(\bigcup_{i\in I}(A_i + r_i - \bigwedge_{i\in I} r_i))$. Since $(A_i, r_i) \leq^{\hat{d}^+} (A, \bigwedge_{i\in I} r_i)$ for any $i \in I$, we have that $(f^*(A_i), r_i) \leq^{d'^+} (f^*(A), \bigwedge_{i\in I} r_i)$ and $(f^*(A_i), r_i)_{i\in I}$ is a directed set. Since (X', d') is a local Yoneda-complete quasi-metric space, we have that $\bigvee_{i\in I}^{\mathbf{B}(X',d')} \mathbf{B}(f^*)(A_i, r_i)$ exists, denoted by $(y, \bigwedge_{i\in I} r_i)$. Thus $(y, \bigwedge_{i\in I} r_i) \leq^{d'^+} (f^*(A), \bigwedge_{i\in I} r_i)$, which implies $d'^+(y, f^*(A)) = 0$.

Since $(f^*(A_i), r_i) \leq^{d'^+} (y, \bigwedge_{i\in I} r_i)$, we have that $(f^*(A_i), r_i - \bigwedge_{i\in I} r_i) \leq^{d'^+} (y, 0)$ for any $i \in I$. From this, we may conclude

$$\begin{aligned}
\mathbf{B}(f)(A_i + r_i - \textstyle\bigwedge_{i\in I} r_i) &\subseteq cl_g(\mathbf{B}(f)(A_i + r_i - \textstyle\bigwedge_{i\in I} r_i)) \\
&= cl_g(\mathbf{B}(f)(A_i)) + r_i - \textstyle\bigwedge_{i\in I} r_i \\
&= \downarrow(f^*(A_i), 0) + r_i - \textstyle\bigwedge_{i\in I} r_i \\
&= \downarrow(f^*(A_i), r_i - \textstyle\bigwedge_{i\in I} r_i) \\
&\subseteq \downarrow(y, 0)
\end{aligned}$$

for any $i \in I$. Since $\mathbf{B}(f)$ is a Y-Scott continuous mapping, it follows from Proposition 26 that $(\mathbf{B}(f))^{-1}(\downarrow(y, 0))$ is a g-closed set. Hence, $A = cl_g(\bigcup_{i\in I}(A_i + r_i - \bigwedge_{i\in I} r_i)) \subseteq (\mathbf{B}(f))^{-1}(\downarrow(y, 0))$, and thus $cl_g(\mathbf{B}(f)(A)) \subseteq \downarrow(y, 0)$. By the definition of f^*, this implies $\downarrow(f^*(A), 0) \subseteq \downarrow(y, 0)$, and therefore, $(f^*(A), 0) \leq^{d'^+} (y, 0)$, which implies $d'^+(f^*(A), y) = 0$. Together with $d'^+(y, f^*(A)) = 0$ from above, $f^*(A) = y$ follows. Therefore, $\bigvee_{i\in I}^{\mathbf{B}(X',d')} (f^*(A_i), r_i) = (f^*(A), \bigwedge_{i\in I} r_i)$. Hence f^* is a Y-continuous mapping.

Claim 3. f^* is a unique Y-continuous mapping such that $f^* \circ \zeta = f$.

Suppose that there exists a Y-continuous mapping $g : \hat{X} \longrightarrow X'$ such that $g \circ \zeta = f$. Let $\mathscr{C} = \{A \in \hat{X} \mid f^*(A) = g(A)\}$. Then $\mathscr{C} \subseteq \hat{X} \subseteq \tilde{X}$ and $\Psi(X) \subseteq \mathscr{C}$. Next, we shall prove that $\mathscr{C}$ satisfies the **Condition** $(*)$. Let $(A_i, r_i)_{i\in I}$ be a directed family in $\mathbf{B}(\mathscr{C}, d_{\mathscr{H}}|_{\mathscr{C}\times\mathscr{C}})$ with an upper bound (B, s). By Propositions 20 and 21, we have that $\bigvee_{i\in I}^{\mathbf{B}(\tilde{X},\tilde{d})} (A_i, r_i) = (A, \bigwedge_{i\in I} r_i)$ and $A \in \hat{X}$, where $A = cl_g(\bigcup_{i\in I}(A_i + r_i - \bigwedge_{i\in I} r_i))$. Obviously, $(A_i, r_i - \bigwedge_{i\in I} r_i)_{i\in I,\sqsubseteq}$ is a Cauchy weighted net in $\mathbf{B}(\hat{X}, \hat{d})$. Let $Z \in \hat{X}$. Then, $\hat{d}(A_i, Z) \leq \hat{d}(A_i, A) + \hat{d}(A, Z) \leq r_i - \bigwedge_{i\in I} r_i + \hat{d}(A, Z)$ for any $i \in I$, and thus $\hat{d}(A_i, Z) - r_i + \bigwedge_{i\in I} r_i \leq \hat{d}(A, Z)$ for any $i \in I$. Let s be an upper bound of $(\hat{d}(A_i, Z) - r_i + \bigwedge_{i\in I} r_i)_{i\in I}$. If $s = +\infty$, then $\hat{d}(A, Z) \leq s$. If $s < +\infty$, then $\hat{d}(A_i, Z) - r_i + \bigwedge_{i\in I} r_i \leq s$ for any $i \in I$, and thus $\hat{d}(A_i, Z) \leq s + r_i - \bigwedge_{i\in I} r_i$ for any $i \in I$. Hence, $(A_i, r_i + s) \leq^{\hat{d}^+} (Z, \bigwedge_{i\in I} r_i)$ for any $i \in I$, and so we conclude that $(A, \bigwedge_{i\in I} r_i + s) \leq^{\hat{d}^+} (Z, \bigwedge_{i\in I} r_i)$. This implies $\hat{d}(A, Z) \leq s$. Therefore, $\bigvee_{i\in I}(\hat{d}(A_i, Z) - r_i + \bigwedge_{i\in I} r_i) = \hat{d}(A, Z)$. By Lemma 2.6 (i) of Ng and Ho (2017), A is the d-limit of $(A_i)_{i\in I}$. Since f^* and g are Y-continuous, we have that $f^*(A) = f^*(\lim_d A_i) = \lim_d f^*(A_i) = \lim_d g(A_i) = g(\lim_d A_i) = g(A)$. So we conclude $A \in \mathscr{C}$. It follows from Proposition 24 that $\hat{X} = cl_L(\Psi(X)) = \mathscr{C}$, and therefore $f^* = g$.

Consequently, $(\hat{X}, \hat{d})$ is the local Yoneda completion of (X, d). □

Acknowledgments. The authors would like to thank the referee for the numerous and very helpful suggestions that have improved this paper substantially.

https://doi.org/10.1017/S0960129523000105 Published online by Cambridge University Press

References

Abramsky, S. and Jung, A. (1994). Domain theory. In: *Handbook of Logic in Computer Science*, Oxford, Clarendon Press.

Ali-Akbari, M., Honari, B., Pourmahdian, M. and Rezaii, M. M. (2009). The space of formal balls and models of quasi-metric spaces. *Mathematical Structures in Computer Science* **19** 337–355.

Bonsangue, M. M., Van Breugel, F. and Rutten, J. J. M. M. (1998). Generalized metric spaces: Completion, topology, and powerdomains via the Yoneda embedding. *Theoretical Computer Science* **193** 1–51.

Edalat, A. and Heckmann, R. (1998). A computational model for metric spaces. *Theoretical Computer Science* **193** 53–73.

Gierz, G. and Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M. and Scott, D. S. (2003). Continuous lattices and domains. In: *Encyclopedia of Mathematics and its Applications*, Cambridge, Cambridge University Press.

Goubault-Larrecq, J. (2013). *Non-Hausdorff Topology and Domain Theory*, New Mathematical Monographs, Cambridge, Cambridge University Press.

Goubault-Larrecq, J. and Ng, K. M. (2017). A few notes on formal balls. *Logical Methods in Computer Science* **13** (4) 1–34.

Kostanek, M. and Waszkiewicz, P. (2011). The formal ball model for $\mathcal{Q}$-categories. *Mathematical Structures in Computer Science* **21** (1) 41–64.

Künzi, H. and Schellekens, M. (2002). On the Yoneda completion of a quasi-metric space. *Theoretical Computer Science* **278** (1–2) 159–194.

Lawvere, F. W. (1973). Metric spaces, generalized logic, and closed categories. *Rendiconti del Seminario Matematico e Fisico di Milano* **43** 135–166.

Mislove, M. W. (1999). Local dcpos, local cpos and local completions. *Electronic Notes in Theoretical Computer Science* **20** 399–412.

Ng, K. M. and Ho, W. K. (2017). Yoneda completion via a DCPO completion of its poset of formal balls. *Electronic Notes in Theoretical Computer Science* **333** 103–121.

Ng, K. M. and Ho, W. K. (2019). Quasi-continuous Yoneda complete quasi-metric space. *Electronic Notes in Theoretical Computer Science* **345** 185–217.

Romaguera, S. and Valero, O. (2010). Domain theoretic characterisations of quasi-metric completeness in terms of formal balls. *Mathematical Structures in Computer Science* **20** 453–472.

Smyth, M. B. (1987). Quasi-uniformities: Reconciling domains with metric spaces. *Lecture Notes in Computer Science* **298** 236–253.

Vickers, S. (2005). Localic completion of generalized metric spaces I. *Theory and Applications of categories* **14** 328–356.

Weihrauch, K. and Schreiber, U. (1981). Embedding metric spaces into CPO's. *Theoretical Computer Science* **16** (1) 5–24.

Cite this article: Lu J and Zhao B (2023). Local Yoneda completions of quasi-metric spaces. *Mathematical Structures in Computer Science* **33**, 33–45. https://doi.org/10.1017/S0960129523000105