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DECAY RATES FOR SOME QUASI-BIRTH-AND-DEATH PROCESSES WITH PHASE-DEPENDENT TRANSITION RATES

ALLAN J. MOTYER, *University of Melbourne*
Department of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia.
Email address: a.motyer@ms.unimelb.edu.au

PETER G. TAYLOR, *University of Melbourne*
Department of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia.
Email address: p.taylor@ms.unimelb.edu.au

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DECAY RATES FOR SOME QUASI-BIRTH-AND-DEATH PROCESSES WITH PHASE-DEPENDENT TRANSITION RATES

BY ALLAN J. MOTYER AND PETER G. TAYLOR

Abstract

Recently, there has been considerable interest in the calculation of decay rates for models that can be viewed as quasi-birth-and-death (QBD) processes with infinitely many phases. In this paper we make a contribution to this endeavour by considering some classes of models in which the transition function is not homogeneous in the phase direction. We characterize the range of decay rates that are compatible with the dynamics of the process away from the boundary. In many cases, these rates can be attained by changing the transition structure of the QBD process at level 0. Our approach, which relies on the use of orthogonal polynomials, is an extension of that in Motyer and Taylor (2006) for the case where the generator has homogeneous blocks.

Keywords: Quasi-birth-and-death process; countable phase; decay rate; stationary distribution

2010 Mathematics Subject Classification: Primary 60K25

1. Introduction

A quasi-birth-and-death (QBD) process is a two-dimensional continuous-time Markov chain for which the generator has a block tridiagonal structure. The first variable of the QBD process is called the *level*, the second variable the *phase*.

The properties of QBD processes with finitely many possible values of the phase variable have been studied extensively. A comprehensive discussion can be found in the monographs of Neuts [11] and Latouche and Ramaswami [7]. It is well known that when such a process is positive recurrent, it possesses a stationary distribution which decays geometrically as the level is increased. The decay parameter is given by the spectral radius of Neuts's $\boldsymbol{R}$-matrix, which is strictly less than 1.

The situation is more complicated for a QBD process with countably many possible values of the phase. The $\boldsymbol{R}$-matrix is infinite dimensional and its spectral properties are not obvious.

When the transition structure of such a QBD process is homogeneous in both the level and the phase, it can be regarded as a random walk in the positive orthant. In recent years, there has been considerable interest in studying the decay properties of such models. Sufficient conditions under which the stationary distribution for the level process of a QBD process with countably many phases has a geometric tail were obtained by Takahashi *et al.* [14] and Haque *et al.* [5]. Miyazawa [9] essentially provided a complete study of the exact decay behaviour of such models in the direction of the axes. Similar conditions were obtained for specific examples by Foley and McDonald [4], Miyazawa [8], and Adan *et al.* [1]. In [2], Borovkov and Mogul´skiĭ studied the decay behaviour in arbitrary directions of 'N-partially homogeneous

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Markov chains', which have a homogeneous transition structure for states that are sufficiently far from the boundary.

In previous work [10], the authors considered a specific class of QBD processes with countably many phases and a skip-free phase process. The tridiagonal blocks in the generator matrix themselves each had a tridiagonal structure, and transition intensities were phase independent for positive phase. It was shown that there exist simple conditions on z for there to exist a z^{-1}-invariant measure of Neuts's $\boldsymbol{R}$-matrix, which is positive and in ℓ^1. These conditions define the set of possible decay rates for the process: the actual decay rate depends on the transition structure at level 0. This work generalized the analysis of Kroese *et al.* [6] for the two-queue tandem Jackson network.

This paper makes a contribution to the study of the behaviour of QBD processes with countably many phases by extending the results in [10] to some further specific classes of processes with a skip-free phase process, but where the transition intensities are now phase dependent. As far as the authors know, this is the first time that such an analysis has been extended to the fully nonhomogeneous case.

The rest of this paper is organized as follows. In Section 2 we summarize some general results for QBD processes. In Section 3 we consider models which result from simple transformations of a model with phase-homogeneous transition rates. In Section 4 we consider two specific classes of models that cannot be written in this form; here transition intensities may vary linearly with phase. The arguments in these sections utilize results about classes of polynomials, specifically the Charlier and Hermite polynomials, which were not considered in [10].

2. Background

We consider a level-independent QBD process with countably many phases. This is a continuous-time Markov chain $(Y_t, J_t, t \geq 0)$ on the state space $\{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\}$. The random variable Y_t is called the *level* of the process at time t and the random variable J_t is called the *phase* of the process at time t. With a lexicographical ordering of the states, the generator $\boldsymbol{Q}$ has a block tridiagonal representation:

$$\boldsymbol{Q} = \begin{pmatrix} \tilde{\boldsymbol{Q}}_1 & \boldsymbol{Q}_0 & & & \\ \boldsymbol{Q}_2 & \boldsymbol{Q}_1 & \boldsymbol{Q}_0 & & \\ & \boldsymbol{Q}_2 & \boldsymbol{Q}_1 & \boldsymbol{Q}_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The matrices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, and $\tilde{\boldsymbol{Q}}_1$ are square and of infinite size.

We assume that the QBD process is irreducible, aperiodic, and positive recurrent, and denote the limiting probabilities by $\pi_{kj} := \lim_{t\to\infty} \mathrm{P}(Y_t = k, J_t = j)$. Let $\boldsymbol{\pi}_k = (\pi_{k0}, \pi_{k1}, \ldots)$ for $k = 0, 1, \ldots$ and $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots)$. Then, from [15],

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 \boldsymbol{R}^k, \qquad k \geq 0,$$

where the infinite-dimensional square matrix $\boldsymbol{R}$ is the minimal nonnegative solution to

$$\boldsymbol{Q}_0 + \boldsymbol{R}\boldsymbol{Q}_1 + \boldsymbol{R}^2\boldsymbol{Q}_2 = \boldsymbol{0}.$$

In this paper we investigate the problem of finding decay rates of the stationary distribution for some classes of QBD processes with countably many phases. As in [10], we derive conditions under which there exists an infinite-dimensional positive row vector $\boldsymbol{w} \in \ell^1$ and positive scalar $z < 1$ such that

$$\boldsymbol{w}\boldsymbol{R} = z\boldsymbol{w}. \tag{2.1}$$

That is, $\boldsymbol{w}$ is a z^{-1}-invariant measure of $\boldsymbol{R}$ [13, p. 205]. The significance of such a vector $\boldsymbol{w}$ and scalar z follows from Theorem 2.4 of [6], which states that if, additionally,

$$\boldsymbol{w}(\tilde{\boldsymbol{Q}}_1 + \boldsymbol{R}\boldsymbol{Q}_2) = \boldsymbol{0} \tag{2.2}$$

is satisfied then the QBD process is ergodic and has a stationary distribution given by

$$\boldsymbol{\pi}_k = K z^k \boldsymbol{w}, \qquad k \geq 0,$$

where K is a constant. That is, the stationary distribution has the level-phase independence property and decays at rate z. In general, we find that there is a range of values of $z \in (0, 1)$ for which there exists a positive $\boldsymbol{w} \in \ell^1$ satisfying (2.1). Since (2.2) may be satisfied by altering only $\tilde{\boldsymbol{Q}}_1$, Theorem 2.4 of [6] suggests that, by changing the transition structure of the QBD process at level 0, the stationary distribution may be forced to possess any decay rate from the range of z values. Furthermore, even when the process does not possess the level-phase independence property, there are many situations where the asymptotic decay rate is one of the values of z for which $\boldsymbol{R}$ has a z^{-1}-invariant measure; see, for example, [9].

In order to find a vector $\boldsymbol{w}$ and scalar z satisfying (2.1), we apply Theorem 5.4 of [12], which states that if $\boldsymbol{w} \in \ell^1$ and $z \in (0, 1)$ satisfy

$$\boldsymbol{w}(\boldsymbol{Q}_0 + z\boldsymbol{Q}_1 + z^2\boldsymbol{Q}_2) = \boldsymbol{0}, \tag{2.3}$$

with $\sum_{k=0}^{\infty} |w_k \boldsymbol{Q}_1(k, k)| < \infty$, then they also satisfy (2.1). This approach was used in [6] to study the two-queue tandem Jackson network and in [10] to study random walks in the quarter-plane. In both these cases, the matrices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, and $\boldsymbol{Q}_2$ are tridiagonal, allowing the use of orthogonal polynomials to determine when a solution of (2.3) is positive.

3. Simple transformations of the homogeneous case

Consider the class of QBD processes with countably many phases and blocks of the form

$$\boldsymbol{Q}_0 = \begin{pmatrix} \tilde{a}_1 & a_0 & & & \\ f(1)a_2 & f(1)a_1 & f(1)a_0 & & \\ & f(2)a_2 & f(2)a_1 & f(2)a_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.1a}$$

$$\boldsymbol{Q}_1 = \begin{pmatrix} \tilde{b}_1 & b_0 & & & \\ f(1)b_2 & f(1)b_1 & f(1)b_0 & & \\ & f(2)b_2 & f(2)b_1 & f(2)b_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.1b}$$

$$\text{and} \quad \boldsymbol{Q}_2 = \begin{pmatrix} \tilde{c}_1 & c_0 & & & \\ f(1)c_2 & f(1)c_1 & f(1)c_0 & & \\ & f(2)c_2 & f(2)3c_1 & f(2)c_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{3.1c}$$

where f is a positive function on $\mathbb{Z}_+ \setminus \{0\}$ such that $\xi \equiv \lim_{k\to\infty} f(k+1)/f(k)$ exists, $a_0, a_1, \tilde{a}_1, a_2, b_0, b_2, c_0, c_1, \tilde{c}_1, c_2 \geq 0$,

$$b_1 = -(a_0 + a_1 + a_2 + b_0 + b_2 + c_0 + c_1 + c_2)$$
$$\text{and} \quad \tilde{b}_1 = -(a_0 + \tilde{a}_1 + b_0 + c_0 + \tilde{c}_1).$$

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Let

$$\gamma_i(z) = a_i + b_i z + c_i z^2 \quad \text{for } i = 0, 1, 2, \tag{3.2}$$

$$\tilde{\gamma}_1(z) = \tilde{a}_1 + \tilde{b}_1 z + \tilde{c}_1 z^2, \tag{3.3}$$

$$\tau(z) = \gamma_1(z) + 2\sqrt{\gamma_0(z)\gamma_2(z)}, \tag{3.4}$$

$$\text{and} \quad \chi(z) = \tilde{\gamma}_1(z) + \frac{\gamma_0(z)\gamma_2(z)}{\tilde{\gamma}_1(z) - \gamma_1(z)}. \tag{3.5}$$

To avoid some trivial cases, we assume that both $\gamma_0(z) > 0$ and $\gamma_2(z) > 0$ for all $z > 0$.

Theorem 3.1. *For the QBD process with the blocks in the generator $\boldsymbol{Q}$ given by (3.1), the system of equations (2.1) has positive solutions $\boldsymbol{w} \in \ell^1$ for $0 < z < 1$ if and only if z is such that*

(i) *if $(\tilde{\gamma}_1(z) - \gamma_1(z))^2 \le \gamma_0(z)\gamma_2(z)$ then $\tau(z) \le 0$, otherwise $\chi(z) \le 0$; and either*

(ii) *$\tau(z) \ge 0$ and $\gamma_0(z) < \kappa^2\gamma_2(z)$; or*

(iii) *$\tau(z) < 0$ and*

$$\kappa\gamma_2(z) > -\tilde{\gamma}_1(z) \quad \text{if } \chi(z) = 0, \qquad \varphi(z) > -\kappa\gamma_1(z) \quad \text{otherwise;}$$

where $\kappa = \min(\xi, 1)$, $\tau(z)$ and $\chi(z)$ are given by (3.4) and (3.5), respectively, $\gamma_0(z)$, $\gamma_1(z)$, $\gamma_2(z)$, and $\tilde{\gamma}_1(z)$ are given by (3.2) and (3.3), and

$$\varphi(z) = \kappa^2\gamma_2(z) + \min(\gamma_0(z), \kappa^2\gamma_2(z)).$$

These conditions are quadratic and quartic polynomial inequalities in z.

Proof of Theorem 3.1. Let $\boldsymbol{Q}(z) = \boldsymbol{Q}_0 + z\boldsymbol{Q}_1 + z^2\boldsymbol{Q}_2$, let $\boldsymbol{w} = \{w_k\}$ be the solution to

$$\boldsymbol{w}\boldsymbol{Q}(z) = \boldsymbol{0}, \tag{3.6}$$

and let $\boldsymbol{v} = \{v_k\}$ be the solution to

$$\boldsymbol{v}\hat{\boldsymbol{Q}}(z) = \boldsymbol{0},$$

where

$$\hat{\boldsymbol{Q}}(z) = \begin{pmatrix} \tilde{\gamma}_1(z) & \gamma_0(z) & & & \\ \gamma_2(z) & \gamma_1(z) & \gamma_0(z) & & \\ & \gamma_2(z) & \gamma_1(z) & \gamma_0(z) & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

as in Section 2 of [10]. Arbitrarily setting $w_0 = v_0$, it follows that $w_k = v_k/f(k)$ for $k = 0, 1, 2, \ldots$. Clearly, $\boldsymbol{w}$ is positive if and only if $\boldsymbol{v}$ is positive. Thus, we obtain condition (i), which is the condition in Theorem 3.1 of [10] that ensures that $\boldsymbol{v}$ is positive.

Since $q_k = -f(k)b_1$ for $k = 1, 2, \ldots$, the condition that $\sum_k |w_k| q_k < \infty$ is equivalent to requiring that $\boldsymbol{v} \in \ell^1$. In [10], it was shown that $\boldsymbol{v}$ takes one of three possible forms depending on the value of the discriminant

$$D(z) = \gamma_1^2(z) - 4\gamma_0(z)\gamma_2(z).$$

If $D(z) = 0$ then

$$v_k = u^k(1 + Ck) \quad \text{with} \quad u = \sqrt{\frac{\gamma_0(z)}{\gamma_2(z)}}$$

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and if $D(z) < 0$ then

$$v_k = (\cos(k\phi) + C\sin(k\phi))|u|^k \quad \text{with} \quad u = \sqrt{\frac{\gamma_0(z)}{\gamma_2(z)}},$$

where C_1, C_2, u_1, u_2, u, C, and ϕ are constants depending on the value of z. It follows that if $D(z) \le 0$ then $\boldsymbol{v} \in \ell^1$ if and only if $\gamma_0(z) < \gamma_2(z)$. Having observed that $w_k = v_k/f(k)$, it follows that if $D(z) \le 0$ then $\boldsymbol{w} \in \ell^1$ if and only if $\gamma_0(z) < \xi^2\gamma_2(z)$. Thus, both $\boldsymbol{v} \in \ell^1$ and $\boldsymbol{w} \in \ell^1$ if and only if $\gamma_0(z) < \kappa^2\gamma_2(z)$.

If $D(z) > 0$ then

$$v_k = C_1u_1^k + C_2u_2^k. \tag{3.7}$$

We first consider the case where either C_1 or C_2 is 0. This is the case if $\chi(z) = 0$. Only one of the u_1 and u_2 in (3.7) now has a nonzero coefficient, which takes the value $-\tilde{\gamma}_1(z)/\gamma_2(z)$. Thus, for both $\boldsymbol{v}$ and $\boldsymbol{w}$ to be in ℓ^1, we obtain the condition

$$\left|-\frac{\tilde{\gamma}_1(z)}{\gamma_2(z)}\right| < \kappa \quad \text{if } \chi(z) = 0.$$

If both C_1 and C_2 are nonzero, then $\boldsymbol{v} \in \ell^1$ if and only if u_1 and u_2 are in the interval $(-1, 1)$, and $\boldsymbol{w} \in \ell^1$ if and only if they are in $(-\xi, \xi)$. Thus, we require that u_1 and u_2 are in $(-\kappa, \kappa)$. The scalars u_1 and u_2 are roots of

$$g(u) = \gamma_2(z)u^2 + \gamma_1(z)u + \gamma_0(z).$$

Since $\gamma_2(z) > 0$ for $z > 0$, u_1 and u_2 are in $(-\kappa, \kappa)$ if and only if $f(\kappa) > 0$, $f(-\kappa) > 0$, $f'(\kappa) > 0$, and $f'(-\kappa) < 0$, giving the four inequalities

$$\kappa^2\gamma_2(z) + \kappa\gamma_1(z) + \gamma_0(z) > 0, \tag{3.8}$$

$$\kappa^2\gamma_2(z) - \kappa\gamma_1(z) + \gamma_0(z) > 0, \tag{3.9}$$

$$2\kappa^2\gamma_2(z) + \kappa\gamma_1(z) > 0, \tag{3.10}$$

$$2\kappa^2\gamma_2(z) - \kappa\gamma_1(z) > 0. \tag{3.11}$$

If $\kappa^2\gamma_2(z) \ge \gamma_0(z)$ then (3.10) follows from (3.8) and (3.11) follows from (3.9). Otherwise if $\kappa^2\gamma_2(z) < \gamma_0(z)$ then (3.8) follows from (3.10) and (3.9) follows from (3.11). Thus, these four inequalities can be written as

$$\kappa^2\gamma_2(z) + \min(\gamma_0(z), \kappa^2\gamma_2(z)) > \kappa|\gamma_1(z)|.$$

Finally, we note that $\gamma_1(z) < 0$ and $\tilde{\gamma}_1(z) < 0$, and that $D(z) > 0$ if $\tau(z) < 0$, $D(z) = 0$ if $\tau(z) = 0$, and $D(z) < 0$ if $\tau(z) > 0$.

4. Constant rates of phase increase, linearly increasing rates of phase decrease

Here we consider a class of QBD processes where the rate of phase decrease is linearly related to the phase variable. The other transition intensities are phase independent.

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Theorem 4.1. *For the QBD process with the blocks in the generator $\boldsymbol{Q}$ given by*

$$\boldsymbol{Q}_0 = \begin{pmatrix} a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ & 2a_2 & a_1 & a_0 & & \\ & & 3a_2 & a_1 & a_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\boldsymbol{Q}_1 = \begin{pmatrix} b_1^{(0)} & b_0 & & & & \\ b_2 & b_1^{(1)} & b_0 & & & \\ & 2b_2 & b_1^{(2)} & b_0 & & \\ & & 3b_2 & b_1^{(3)} & b_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\text{and} \quad \boldsymbol{Q}_2 = \begin{pmatrix} c_1 & c_0 & & & & \\ c_2 & c_1 & c_0 & & & \\ & 2c_2 & c_1 & c_0 & & \\ & & 3c_2 & c_1 & c_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\begin{aligned} b_1^{(i)} &= -(a_0 + a_1 + ia_2 + b_0 + ib_2 + c_0 + c_1 + ic_2) \\ &= b_1 - (i-1)(a_2 + b_2 + c_2) \\ &= b_1 - (i-1)\gamma_2(1) \quad \text{for } i = 0, 1, 2, \dots, \end{aligned}$$

the system of equations (2.1) has positive solutions $\boldsymbol{w} \in \ell^1$ for $0 < z < 1$ if and only if z is such that either

(i) $(z\gamma_2(1))^2 + z\gamma_2(1)\gamma_1(z) + \gamma_0(z)\gamma_2(z) = 0$; *or*

(ii) $(z\gamma_2(1))^2 + z\gamma_2(1)\gamma_1(z) + \gamma_0(z)\gamma_2(z) < 0$ *and* $0 < z < \min(a_2/c_2, 1)$.

Note that the phase process away from the level boundary, that is, the process with generator $\boldsymbol{Q}_0 + \boldsymbol{Q}_1 + \boldsymbol{Q}_2$, behaves as an M/M/$\infty$ queue. With z fixed such that $0 < |z| < 1$, the system of equations (3.6) reduces to

$$w_0(\gamma_1(z) + z\gamma_2(1)) + w_1\gamma_2(z) = 0, \tag{4.1}$$

$$w_n\gamma_0(z) + w_{n+1}(\gamma_1(z) - nz\gamma_2(1)) + (n+2)w_{n+2}\gamma_2(z) = 0, \qquad n \geq 0. \tag{4.2}$$

Consider the equations

$$\begin{aligned} P_0(x; z) &= 1, \\ \gamma_2(z)P_1(x; z) &= x - \gamma_1(z) - z\gamma_2(1), \\ (n+1)\gamma_2(z)P_{n+1}(x; z) &= (x - \gamma_1(z) + (n-1)z\gamma_2(1))P_n(x; z) - \gamma_0(z)P_{n-1}(x; z) \end{aligned}$$

for $n \geq 1$. For any given real and positive value of z, these equations define a sequence of orthogonal polynomials $P_n(x; z)$. Arbitrarily setting $w_0 = 1$, it is clear that $w_n = P_n(0; z)$.

The $P_n(0; z)$ are positive for all n if and only if the zeros of all the $P_n(x; z)$ are less than 0. This enables us to study conditions for the positivity of $\boldsymbol{w}$ via the properties of the polynomials $P_n(0; z)$.

Lemma 4.1. *For $z > 0$, the sequence $\{P_n(x; z)\}$ satisfies the orthogonality relationship*

$$\int_{-\infty}^{b(z)} P_n(x; z) P_m(x; z)\, \mathrm{d}\phi^{(a)}(x) = \frac{1}{n!}\left(\frac{\gamma_0(z)}{\gamma_2(z)}\right)^n \delta_{n,m},$$

where $\delta_{n,m} = 1$ if $n = m$ and 0 otherwise, and $\phi^{(a)}$ is the step function whose jumps are

$$\mathrm{d}\phi^{(a)}(x) = \frac{\mathrm{e}^{-a} a^j}{j!}$$

when $x = b(z) - jz\gamma_2(1)$, $j \in \mathbb{Z}_+$, $a = \gamma_0(z)\gamma_2(z)/(z\gamma_2(1))^2$, and

$$b(z) = \gamma_1(z) + z\gamma_2(1) + \gamma_0(z)\frac{\gamma_2(z)}{z\gamma_2(1)}. \tag{4.3}$$

Proof. For fixed $z > 0$, let

$$C_n^{(a)}(x) = \left(-\frac{\gamma_2(z)}{z\gamma_2(1)}\right)^n n!\, P_n(b(z) - z\gamma_2(1)x; z). \tag{4.4}$$

It follows that

$$C_0^{(a)}(x) = 1 \quad \text{and} \quad C_{n+1}^{(a)}(x) = (x - n - a)C_n^{(a)}(x) - anC_{n-1}^{(a)}(x), \qquad n \geq 0.$$

The $C_n^{(a)}$s are *Charlier polynomials*, for which the orthogonalizing relationship is given (see [3, pp. 170–172]) by

$$\int_0^\infty C_m^{(a)}(x) C_n^{(a)}(x)\, \mathrm{d}\psi^{(a)}(x) = a^n n!\, \delta_{m,n}, \tag{4.5}$$

where $\psi^{(a)}$ is the step function whose jumps are

$$\mathrm{d}\psi^{(a)}(j) = \frac{\mathrm{e}^{-a} a^j}{j!} \quad \text{at } j = 0, 1, 2, \ldots.$$

Substituting (4.4) into (4.5) yields the result.

Lemma 4.2. *For each value of $z > 0$, $P_n(x; z)$ has n distinct real zeros $x_{n,1} < \cdots < x_{n,n}$ and these zeros interlace. That is, for all $n \geq 2$ and $i = 1, \ldots, n-1$,*

$$x_{n,i} < x_{n-1,i} < x_{n,i+1}.$$

Proof. The lemma follows from a well-known result for orthogonal polynomial sequences; see Theorem I.5.3 of [3].

The support of the measure $\phi^{(a)}$ and the limiting behaviour of the zeros of the $P_n(x; z)$ are related. Some useful results are stated in the following lemma.

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Lemma 4.3. *The sequences of smallest and largest zeros of the $P_n(x; z)$ possess the following properties:*

- $\{x_{n,1}\}_{n=1}^{\infty}$ *is an unbounded strictly decreasing sequence;*
- $\{x_{n,n}\}_{n=1}^{\infty}$ *is a strictly increasing sequence with limit $b(z)$.*

Proof. See Section II.4 of [3].

Lemma 4.4. *Let $z > 0$. Then $P_n(x; z)$ is positive for all n if and only if $x \geq b(z)$.*

Proof. The leading coefficient of $P_n(x; z)$ is positive for all n (since $\gamma_2(z) > 0$ for $z > 0$). This implies that $P_n(x; z)$ is positive for $x > x_{n,n}$. Since $x_{n,n}$ is strictly increasing, we know that $P_n(x; z)$ is positive for all n if $x \geq b(z)$. Conversely, $P_k(x; z)$ is negative for $x \in (x_{k-1,k}, x_{k,k})$ and so the interleaving property given in Lemma 4.2 implies that, for every $x < x_{n,n}$, $P_k(x; z)$ is less than 0 for at least one $k \in \{1, \dots, n\}$. Thus, if $x < b(z)$, $P_k(x; z)$ is less than 0 for at least one $k \in \mathbb{Z}_+$.

We are now in a position to say when the vector $\boldsymbol{w}$ which solves (4.1) and (4.2) is positive.

Lemma 4.5. *The vector $\boldsymbol{w}$ is positive if and only if $b(z) \leq 0$.*

Proof. This follows immediately from Lemma 4.4 and the fact that, for a given z, $w_n = P_n(0; z)$.

We now consider whether $\boldsymbol{w} \in \ell^1$. There is an explicit formula for the Charlier polynomials:

$$C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k!\,(-a)^{n-k}.$$

By (4.4), we have, for $z > 0$,

$$\begin{aligned} w_n &= \frac{1}{n!}\left(\frac{-z\gamma_2(1)}{\gamma_2(z)}\right)^n C_n^{(a)}\left(\frac{b(z)}{z\gamma_2(1)}\right) \\ &= \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!}\left(\frac{\gamma_0(z)}{z\gamma_2(1)}\right)^{n-k}\left(\frac{1}{\gamma_2(z)}\right)^k \\ &\qquad \times (-b(z))(-b(z) + z\gamma_2(1)) \cdots (-b(z) + (k-1)z\gamma_2(1)). \end{aligned} \tag{4.6}$$

If the positivity condition $b(z) \leq 0$ is satisfied then all terms in the above summation are easily seen to be positive, which confirms the 'if' part of Lemma 4.5.

Lemma 4.6. *If $z > 0$ and $b(z) = 0$, then $\boldsymbol{w} \in \ell^1$.*

Proof. If $b(z) = 0$ then (4.6) reduces to

$$w_n = \frac{1}{n!}\left(\frac{\gamma_0(z)}{z\gamma_2(1)}\right)^n,$$

so clearly $\boldsymbol{w} \in \ell^1$.

Lemma 4.7. *If $z > 0$ and $b(z) < 0$, then $\boldsymbol{w} \in \ell^1$ if and only if $z < \min(a_2/c_2, 1)$ or $z > \max(a_2/c_2, 1)$.*

Proof. From (4.6) we have

$$\sum_{n=0}^{\infty}|w_n| = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(n-k)!\,k!}\left(\frac{\gamma_0(z)}{z\gamma_2(1)}\right)^{n-k}\left(\frac{1}{\gamma_2(z)}\right)^{k} \times (-b(z))(-b(z)+z\gamma_2(1))\cdots(-b(z)+(k-1)z\gamma_2(1))$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{n!\,k!}\left(\frac{\gamma_0(z)}{z\gamma_2(1)}\right)^{n}\left(\frac{1}{\gamma_2(z)}\right)^{k} \times (-b(z))(-b(z)+z\gamma_2(1))\cdots(-b(z)+(k-1)z\gamma_2(1))$$

$$= \exp\left(\frac{\gamma_0(z)}{z\gamma_2(1)}\right) \times \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\gamma_2(z)}\right)^{k}(-b(z))(-b(z)+z\gamma_2(1))\cdots(-b(z)+(k-1)z\gamma_2(1)).$$

Applying the ratio test for convergence gives the condition $\rho = z\gamma_2(1)/\gamma_2(z) < 1$. In the case where $\rho = 1$,

$$\sum_{n=0}^{\infty}|w_n| = \exp\left(\frac{\gamma_0(z)}{z\gamma_2(1)}\right)\sum_{k=0}^{\infty}(-1)^k\binom{b(z)/\gamma_2(z)}{k},$$

which does not converge (by the generalized version of the binomial theorem). Thus, for $z > 0$ and $b(z) < 0$, $\boldsymbol{w} \in \ell^1$ if and only if $z\gamma_2(1) < \gamma_2(z)$. This is satisfied for $0 < z < \min(a_2/c_2, 1)$ and $z > \max(a_2/c_2, 1)$.

The final condition of Theorem 5.4 of [12] for $\boldsymbol{w}$ to satisfy $\boldsymbol{wR} = z\boldsymbol{w}$ is $\sum_{k=0}^{\infty}|w_k|q_k < \infty$. Since $q_k = -\boldsymbol{Q}_1(k,k) = -b_1 + (k-1)\gamma_2(1)$, we require that both $\sum_{k=0}^{\infty}|w_k| < \infty$ and $\sum_{k=0}^{\infty}k|w_k| < \infty$.

Lemma 4.8. *If $z > 0$, $b(z) \le 0$, and $\boldsymbol{w} \in \ell^1$, then $\sum_{n=0}^{\infty}n|w_n| < \infty$.*

Proof. This is clear by inspection of the expressions obtained for w_n.

Proof of Theorem 4.1. This follows from the combination of the conditions of Lemmas 4.5–4.8 and Theorem 5.4 of [12], using the definition of $b(z)$ in (4.3).

Now we consider a variation on the class of QBD processes studied above. The intensity of transitions where the phase variable remains the same is now linearly related to the value of the phase variable.

Theorem 4.2. *For the QBD process with the blocks in the generator $\boldsymbol{Q}$ given by*

$$\boldsymbol{Q}_0 = \begin{pmatrix} a_1 & a_0 & & & \\ a_2 & 2a_1 & a_0 & & \\ & 2a_2 & 3a_1 & a_0 & \\ & & 3a_2 & 4a_1 & a_0 \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

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$$\boldsymbol{Q}_1 = \begin{pmatrix} b_1^{(0)} & b_0 & & & & \\ b_2 & b_1^{(1)} & b_0 & & & \\ & 2b_2 & b_1^{(2)} & b_0 & & \\ & & 3b_2 & b_1^{(3)} & b_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\text{and} \quad \boldsymbol{Q}_2 = \begin{pmatrix} c_1 & c_0 & & & & \\ c_2 & 2c_1 & c_0 & & & \\ & 2c_2 & 3c_1 & c_0 & & \\ & & 3c_2 & 4c_1 & c_0 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix},$$

where

$$\begin{aligned} b_1^{(i)} &= -(a_0 + b_0 + c_0 + (i+1)(a_1 + c_1) + i(a_2 + b_2 + c_2)) \\ &= (i+1)b_1 + (i-1)\gamma_0(1) - \gamma_1(1) \quad \text{for } i = 0, 1, 2, \ldots, \end{aligned}$$

the system of equations (2.1) has positive solutions $\boldsymbol{w} \in \ell^1$ *for* $0 < z < 1$ *if and only if* z *is such that*

(i) $\gamma_1(z) + z\gamma_0(1) < 0$; *and either*

(ii) $\gamma_1(z) - z(\gamma_0(1) + \gamma_1(1)) - \gamma_0(z)\gamma_2(z)/(\gamma_1(z) + z\gamma_0(1)) = 0$; *or*

(iii) $\gamma_1(z) - z(\gamma_0(1) + \gamma_1(1)) - \gamma_0(z)\gamma_2(z)/(\gamma_1(z) + z\gamma_0(1)) < 0$ *and* $z\gamma_0(1) + \gamma_1(z) + \gamma_2(z) > 0$.

Theorem 4.2 will be proved using a series of lemmas. For fixed z such that $0 < |z| < 1$, the system of equations (3.6) reduces to

$$w_0(\gamma_1(z) - z(\gamma_0(1) + \gamma_1(1))) + w_1\gamma_2(z) = 0,$$
$$w_{k-1}\gamma_0(z) + w_k((k+1)\gamma_1(z) + z((k-1)\gamma_0(1) - \gamma_1(1))) + (k+1)w_{k+1}\gamma_2(z) = 0$$

for $k \geq 1$. Again, we generalize these equations to

$$P_0(x; z) = 1, \tag{4.7}$$
$$\gamma_2(z)P_1(x; z) = x - \gamma_1(z) + z(\gamma_0(1) + \gamma_1(1)), \tag{4.8}$$
$$\begin{aligned} (n+1)\gamma_2(z)P_{n+1}(x; z) &= (x - (n+1)\gamma_1(z) - z((n-1)\gamma_0(1) - \gamma_1(1)))P_n(x; z) \\ &\quad - \gamma_0(z)P_{n-1}(x; z), \qquad n \geq 1. \end{aligned} \tag{4.9}$$

Lemma 4.9. *For* $z > 0$ *and* $\gamma_1(z) + z\gamma_0(1) \neq 0$, *the sequence* $\{P_n(x; z)\}$ *satisfies the orthogonality relationship*

$$\int_{\operatorname{supp}(\phi^{(a)}(x))} P_n(x; z)P_m(x; z)\,\mathrm{d}\phi^{(a)}(x) = \frac{1}{n!}\left(\frac{\gamma_0(z)}{\gamma_2(z)}\right)^n \delta_{n,m},$$

where $\phi^{(a)}$ *is the step function whose jumps are*

$$\mathrm{d}\phi^{(a)}(x) = \frac{\mathrm{e}^{-a} a^{(x-b(z))/(\gamma_1(z)+z\gamma_0(1))}}{((x - b(z))/(\gamma_1(z) + z\gamma_0(1)))!}$$

at $x = b(z) + k(\gamma_1(z) + z\gamma_0(1))$ *for* $k = 0, 1, 2, \ldots$, $a = \gamma_0(z)\gamma_2(z)/(\gamma_1(z) + z\gamma_0(1))^2$, *and*

$$b(z) = \gamma_1(z) - z(\gamma_0(1) + \gamma_1(1)) - \frac{\gamma_0(z)\gamma_2(z)}{\gamma_1(z) + z\gamma_0(1)}. \tag{4.10}$$

Proof. For fixed $z > 0$, let

$$C_n^{(a)}(x) = \left(\frac{\gamma_2(z)}{\gamma_1(z) + z\gamma_0(1)}\right)^n n!\, P_n((\gamma_1(z) + z\gamma_0(1))x + b(z); z). \tag{4.11}$$

We find that the $C_n^{(a)}(x)$ are Charlier polynomials. Substituting (4.11) into (4.5) yields the result.

Lemma 4.10. *The sequence of the largest zeros,* $\{x_{n,n}\}_{n=1}^{\infty}$, *of the* $P_n(x; z)$ *is a strictly increasing sequence. The sequence is unbounded if* $\gamma_1(z) + z\gamma_0(1) > 0$, *and has limit* $b(z)$ *if* $\gamma_1(z) + z\gamma_0(1) < 0$.

Proof. The limiting value of $x_{n,n}$ is given by the supremum of the support of the measure $\phi^{(a)}(x)$, if it exists. Then, by Lemma 4.9, when $\gamma_1(z) + z\gamma_0(1) > 0$, there is no limiting value of $x_{n,n}$, and when $\gamma_1(z) + z\gamma_0(1) < 0$, the limiting value is $b(z)$.

If $\gamma_1(z) + z\gamma_0(1) = 0$ then (4.7)–(4.9) become

$$\begin{aligned} P_0(x; z) &= 1, \\ \gamma_2(z)P_1(x; z) &= x - 2\gamma_1(z) + z\gamma_1(1), \\ (n+1)\gamma_2(z)P_{n+1}(x; z) &= (x - 2\gamma_1(z) + z\gamma_1(1))P_n(x; z) - \gamma_0(z)P_{n-1}(x; z), \qquad n \ge 1. \end{aligned}$$

Note that there is one value of $z \in (0, 1)$ for which this is the case.

Lemma 4.11. *When* $z > 0$ *and* $\gamma_1(z) + z\gamma_0(1) = 0$, *the sequence* $\{P_n(x; z)\}$ *satisfies the orthogonality relationship*

$$\int_{-\infty}^{\infty} P_m(x)P_n(x)\exp\left(-\frac{(x - 2\gamma_1(z) + z\gamma_1(1))^2}{\gamma_0(z)\gamma_2(z)}\right)\mathrm{d}x = \frac{\sqrt{\pi}}{n!}\left(\frac{\gamma_0(z)}{\gamma_2(z)}\right)^n \delta_{m,n}.$$

Proof. For fixed $z > 0$, let

$$H_n(x) = \left(\frac{2\gamma_2(z)}{\gamma_0(z)}\right)^{n/2} n!\, P_n(x\sqrt{2\gamma_0(z)\gamma_2(z)} + 2\gamma_1(z) - z\gamma_1(1); z). \tag{4.12}$$

It follows that $H_n(x)$ is a Hermite polynomial, satisfying the recurrence relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \qquad n \ge 0,$$

with $H_0(x) = 1$, and the orthogonality relationship

$$\int_{-\infty}^{\infty} H_m(x)H_n(x)\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}2^n n!\,\delta_{m,n}. \tag{4.13}$$

Substitution of (4.12) into (4.13) yields the result.

We have the following as a consequence.

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Lemma 4.12. *For $\gamma_1(z) + z\gamma_0(1) = 0$, the sequence of the largest zeros, $\{x_{n,n}\}_{n=1}^{\infty}$, of the $P_n(x; z)$ is an unbounded strictly increasing sequence.*

Proof. This follows from the fact that the support of the orthogonalizing measure of the $P_n(x; z)$ is the real line.

Combining Lemmas 4.9–4.12 and following reasoning similar to that in Lemma 4.4 we get the following.

Lemma 4.13. *For $z > 0$, the vector $\boldsymbol{w}$ is positive if and only if $\gamma_1(z)+z\gamma_0(1) < 0$ and $b(z) \le 0$.*

We now consider whether $\boldsymbol{w} \in \ell^1$. For $z > 0$ and $\gamma_1(z) + z\gamma_0(1) \neq 0$, we have, by (4.11),

$$w_n = \frac{1}{n!}\left(\frac{\gamma_1(z) + z\gamma_0(1)}{\gamma_2(z)}\right)^n C_n^{(a)}\left(-\frac{b(z)}{\gamma_1(z) + z\gamma_0(1)}\right). \tag{4.14}$$

Lemma 4.14. *For $z > 0$, if $b(z) = 0$ and $\gamma_1(z) + z\gamma_0(1) \neq 0$, then $\boldsymbol{w} \in \ell^1$.*

Proof. From (4.14) we get

$$w_n = \frac{1}{n!}\left(\frac{-\gamma_0(z)}{\gamma_1(z) + z\gamma_0(1)}\right)^n,$$

so clearly $\boldsymbol{w} \in \ell^1$.

Lemma 4.15. *If $z > 0$, $b(z) < 0$, and $\gamma_1(z) + z\gamma_0(1) < 0$, then $\boldsymbol{w} \in \ell^1$ if and only if $z\gamma_0(1) + \gamma_1(z) + \gamma_2(z) > 0$.*

Proof. From (4.14) we get

$$\sum_{n=0}^{\infty}|w_n| = \exp\left(\frac{-\gamma_0(z)}{\gamma_1(z) + z\gamma_0(1)}\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{1}{\gamma_2(z)}\right)^k(-b(z))(-b(z) - \gamma_1(z) - z\gamma_0(1))\cdots$$
$$\times\,(-b(z) - (k-1)(\gamma_1(z) + z\gamma_0(1))).$$

Applying the ratio test for convergence gives the condition $\rho = (-\gamma_1(z)-z\gamma_0(1))/\gamma_2(z) < 1$. In the case where $\rho = 1$,

$$\sum_{n=0}^{\infty}|w_n| = \exp\left(\frac{-\gamma_0(z)}{\gamma_1(z) + z\gamma_0(1)}\right)\sum_{k=0}^{\infty}(-1)^k\binom{b(z)/\gamma_2(z)}{k},$$

which does not converge. Rearranging the expression for $\rho < 1$ gives the result.

Again, to satisfy the final condition of Theorem 5.4 of [12], we require that both

$$\sum_{k=0}^{\infty}|w_k| < \infty \quad \text{and} \quad \sum_{k=0}^{\infty}k|w_k| < \infty.$$

The following result is obtained by inspection of the expressions for w_k.

Lemma 4.16. *For $z > 0$, $b(z) \le 0$, and $\gamma_1(z)+z\gamma_0(1) < 0$, if $\boldsymbol{w} \in \ell^1$ then $\sum_{n=0}^{\infty} n|w_n| < \infty$.*

Finally, we are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. The proof follows from the combination of the conditions of Lemmas 4.13–4.16 and Theorem 5.4 of [12], using the definition of $b(z)$ in (4.10).

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ALLAN J. MOTYER, *University of Melbourne*

Department of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia.
Email address: a.motyer@ms.unimelb.edu.au

PETER G. TAYLOR, *University of Melbourne*

Department of Mathematics and Statistics, University of Melbourne, VIC 3010, Australia.
Email address: p.taylor@ms.unimelb.edu.au