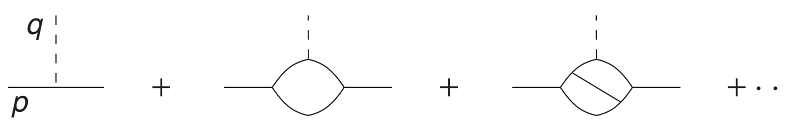
4

Electromagnetic interaction of hadrons

What do we mean when we talk about electromagnetic interactions of hadrons? The simplest of them is the interaction of hadrons with leptons. As you know there are particles – leptons – that do not participate in the strong interaction. They interact electromagnetically, and this interaction* can be described in the field theoretical (QFT) framework.

We suppose, and this is our main hypothesis, that a charged lepton interacts with hadrons only via a photon field. The second part of our hypothesis must contain something about the interaction of a photon with hadrons. Here we will say, once again within the QFT concept, that it is still a pointlike interaction (as in the usual QED) which has to be treated with account of all possible field-theoretical corrections:



(4.1)

Strong interaction being *strong*, the photon will always interact with a virtual particle in the intermediate state that depicts the internal structure of a dressed hadron.

Since QED amplitudes are small, we can restrict ourselves to the one-photon-exchange picture.

4.1 Electron–proton interaction

Consider the electron–proton scattering amplitude in the first order in $\alpha_{\rm em}$. Both particles have spin-$\frac{1}{2}$, and from general considerations we can

* as well as the weak one, with the advent of the Glashow–Weinberg–Salam theory. (ed.)

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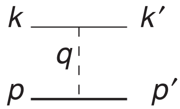
immediately write



$$= e_1 e_2 \big[\bar{u}(k')\gamma_\mu u(k)\big]\frac{1}{q^2}\big[\bar{U}(p')\Gamma_\mu U(p)\big]. \qquad (4.2)$$

What can we say about the vertex Γ_μ? It has got to be a vector since the photon has spin 1. We have at our disposal the proton momenta p_μ and p'_μ and the Dirac matrix γ_μ:

$$\Gamma_\mu = a(q^2)\gamma_\mu + b(q^2)\sigma_{\mu\nu}q^\nu \qquad \big(\sigma_{\mu\nu} \equiv \tfrac{1}{2}[\gamma_\mu, \gamma_\nu]\big). \qquad (4.3)$$

This is the most general structure that satisfies the on-mass-shell current conservation condition $q^\mu \cdot [\bar{U}(p')\Gamma_\mu U(p)] = 0$ (higher powers of γ-matrices reduce to (4.3) due to the Dirac equation $\bar{U}\hat{p'} = m\bar{U}$, $\hat{p}U = mU$).

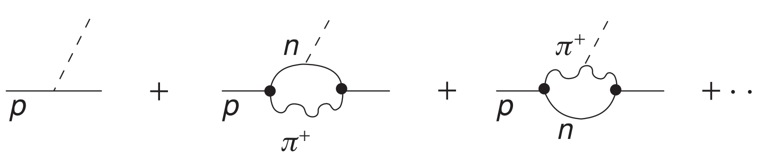
Already at this stage we have obtained a strong prediction that derives from the fact that the unknown functions a and b in (4.3) do not contain any s-dependence!

It is worthwhile to remark that in discussing electromagnetic interactions of hadrons we face a situation which is essentially different from that of the strong interaction. In pure hadron interactions we were basically dealing with real particles and on-mass-shell amplitudes. Now, by virtue of the *smallness* of the lepton–hadron interaction, we were able to select a single graph whose amplitude depends on the virtual momentum q^2. Formerly, hadron–hadron scattering amplitudes depended seriously on s and t. Now we have a serious dependence only on $t = q^2$ while the s-dependence turns out to be trivial as it is exclusively due to free particles' spinors.

This observation alone gives rise to the *Rosenblutte formula* checking which one directly verifies that e and p indeed interact only via the electromagnetic force. (The Rosenblutte formula is also being intensively checked experimentally in an attempt to find a possible difference between the leptons – an electron e and a muon μ.)

4.1.1 Electric charge

The photon–proton vertex will be determined by the ensemble of all the diagrams (4.1) which I would have to *calculate*, given a concrete QFT. For example, suppose there were only nucleons and pions. Then I would draw the bare interaction and corrections:



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The first thing to do is to calculate the *electric charge* – the value of the amplitude at $q \to 0$.

Bare proton. Suppose the proton possessed a bare electric charge, e_0. Then with account of higher-order radiative corrections it would get modified,

$$e_0 \implies e_0 \, Z_1^{-1} \, Z_2 \sqrt{Z_3} \equiv e_p,$$

where Z_1^{-1} is the vertex correction and the factor $Z_2 = (\sqrt{Z_2})^2$ comes from initial and final proton wave functions. After renormalization we would have the on-mass-shell condition

$$\Gamma_\mu(q = 0; \, p^2 = p'^2 = m_p^2) \; = \; e_p \cdot \gamma_\mu; \quad a(0) \equiv 1. \tag{4.4}$$

The Ward identity taught us that in the theory with a conserved current

$$Z_1 \; = \; Z_2, \quad \implies e_p = e_0 \cdot \sqrt{Z_3},$$

that is, charge renormalization is related to the photon only. This is exactly what we call *charge conservation*. Namely, if we plug in equal bare electric charges for the proton and the electron, then the renormalized physical charges will stay equal, irrespective of the nature of the charged particle and interactions it is subject to.

In fact we have only 'half a theory' of this important phenomenon, since (except for a few attempts) we have no pure theoretical reason for ascribing to electron and proton equal (and opposite) bare charges. This is just an experimental fact (and a very solid one in that).

Quarks. What if there is no bare proton at all? We can imagine the proton to be a bound state of some point-like constituents, quarks for example. Then we will have to work with *photon–quark* interaction amplitudes; the proton charge will be simply given by the sum of quark charges,

$$e_p = (e_1 + e_2 + e_3) + \cdots .$$

So essentially we would have an approach rather similar to the previous one but at the level of constituents. One way or another, we cannot move away from the field-theoretical concept of point-like interaction if we intend to keep things under control.

4.1.2 Magnetic moment

Let us consider a small momentum transfer and keep the first power of q in the expression for the vertex (4.3):

$$\Gamma_\mu \; \simeq \; e_p(\gamma_\mu \; + \; b(0)\sigma_{\mu\nu} \, q^\nu) \, . \tag{4.5}$$

Here we have extracted from the vertex the renormalized charge e_p and set $a(0) = 1$.

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What is the physical meaning of the linear term in (4.3)? The Dirac vertex γ_μ contains interaction with the charge as well as with the *magnetic moment*. Using the identity

$$\bar{u}(k')(k+k')_\mu u(k) = \bar{u}(k')\big[\,2m\gamma_\mu + \sigma_{\mu\nu}q^\nu\big]u(k),$$

we may rewrite the amplitude of electron scattering, $A^\mu\,\bar{u}(k')\,\gamma_\mu\,u(k)$, as

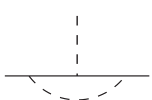
$$A^\mu\frac{(k+k')_\mu}{2m}\cdot\bar{u}(k')u(k)\;-\;A^\mu\cdot\bar{u}(k')\frac{\sigma_{\mu\nu}q^\nu}{2m}u(k).$$

With the electron at rest ($\mathbf{k}=0$), the first term does not contribute to scattering in a magnetic field: $A_0=0$, $\mathbf{A}\cdot\mathbf{q}=0\,(\mathbf{div}\,\mathbf{A}(x)=0)$. The second contribution survives and describes an interaction with the magnetic moment of a spin-$\frac{1}{2}$ charge.

The linear term from the proton (4.5) adds up with the Bohr *magneton* $e/2m$, resulting in the magnetic moment of the proton

$$\mu_p \;=\; -\frac{e_p}{2m_p}\big[\,1-2m_p b(0)\,\big]\,.$$

In QED we have seen that the magnetic moment of the electron acquires



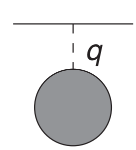
an 'anomalous' contribution $\alpha_{\rm em}/2\pi$ (Schwinger, 1962). This was a consequence of the internal structure of the electron that became apparent when radiative corrections had been taken into consideration. The magnetic moment of the proton also changes on account of the interaction, and not only electromagnetically, but of the strong one in the first place.

While the electromagnetic charge renormalizes in a universal way, independently of the proton's nature, the value of its magnetic moment depends crucially on the concrete properties of the strong interaction.

4.2 Form factors

The physical meaning of the functions $a(q^2)$ and $b(q^2)$ becomes apparent from the non-relativistic analogy.

Consider the scattering of a charge off an extended target, e.g. an atom. In quantum mechanics, the scattering amplitude is given by the Fourier integral of the Coulomb potential $V(r)$ which, in turn, one obtains by integrating the charge density $\rho(r')$ over the volume of the target:



$$f\;\propto\;\int d^3r\,V(r)\,\mathrm{e}^{-i\mathbf{q}\cdot\mathbf{r}},\qquad V(r)=\int d^3r'\,\frac{e^2}{|\mathbf{r}-\mathbf{r}'|}\rho(r').$$

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Combining the two expressions one arrives at

$$f \propto \frac{e^2}{|\mathbf{q}|^2} \cdot F(q^2), \qquad F(q^2) \equiv \int d^3r' \, \rho(r') \, e^{-i\mathbf{q}\cdot\mathbf{r}'}, \tag{4.6}$$

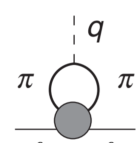
with F the *electric form factor* of the target atom. Analogously, our functions a and b are the proton form factors characterizing, correspondingly, the distribution of the *charge* and that of the electric *current* inside the proton.

In the case of a spinless object there is no preferred direction, and the magnetic moment is identically zero. Therefore, if we substitute a π-meson for a proton we will have only one (electric) form factor. Now the only vector at our disposal that satisfies the current conservation condition $q^\mu \Gamma_\mu = 0$ (for $p^2 = p'^2 = \mu^2$) is the sum of the pion momenta, $(p+p')_\mu$; hence, the photon–pion interaction vertex contains a single structure:

$$\Gamma^\pi_\mu \; = \; e \cdot a(q^2)(p+p')_\mu.$$

What else can be said about form factor(s)?

Let us look at the analytic properties of a form factor as a function of momentum transfer q^2. It is clear that the form factor is real for $q^2 < 0$ since no real process may occur in the intermediate (t-channel) state. At the same time, for positive virtuality above the two-pion threshold, $q^2 > 4\mu^2$, the form factor becomes complex-valued due to the $\gamma^* \to \pi^+\pi^-$ transition.



The relativistic theory allows us to link the scattering form factor to a completely different physical phenomenon, namely an *annihilation* of a pair of leptons into a pair of hadrons,

$$e^+e^- \;\to\; N\bar{N} \qquad \text{or} \qquad e^+e^- \;\to\; \pi^+\pi^-.$$

The very same function that describes an internal electromagnetic structure of a pion in the $e\pi$ scattering process in the region $q^2 < 0$, at $q^2 > (2\mu)^2$ determines the cross section of e^+e^- annihilation into two pions!

4.2.1 Analytic properties of pion form factor

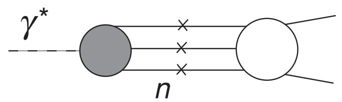
Knowing that the form factor is an analytic function of q^2, I can write

$$a(q^2) \; = \; 1 + \frac{q^2}{\pi} \int_{4\mu^2}^{\infty} \frac{dQ^2 \;\, \mathrm{Im}\, a(Q^2)}{Q^2(Q^2 - q^2)}. \tag{4.7}$$

I chose to write down the dispersion relation with *one subtraction* in order to exploit the knowledge of the normalization $a(0) = 1$ (which only helps the integral to converge faster).

As we know, the imaginary part is directly related to cross sections of real processes. The latter are subject to various restrictions which then must affect the form factor itself. Let us examine how serious these restrictions actually are by taking the pion form factor $a(q^2)$ as an example:

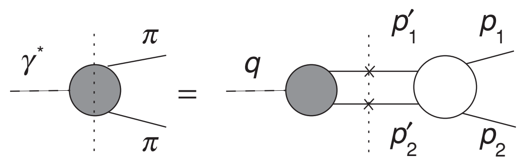
$$2\,\mathrm{Im}\,\Gamma_\mu = \sum_n \quad (4.8)$$



This equation tells us that the dispersion theory provides us with a system of linear equations for 'form factors' $\gamma^* \to (n * \pi)$, where the rôle of the kernel is played by the pure strong interaction amplitudes $(n * \pi) \to (m * \pi)$.

Consider for simplicity the region $4\mu^2 < q^2 < 16\mu^2$ where the two pion unitarity relation holds:

$$2\,\mathrm{Im}\,a(q^2)(p_1 - p_2)_\mu = \quad (4.9)$$



The kernel is the $\pi\pi \to \pi\pi$ scattering amplitude. Moreover, π is spinless ($s = 0$), while the total angular momentum of the $\pi\pi$ system, $\mathbf{J} = \boldsymbol{\ell} + \mathbf{s}$, must be equal the photon spin, $J = 1$. Hence, only one partial wave $\ell = 1$ of the $\pi\pi$ amplitude (P-wave) will contribute here.

Let us sketch how this selection occurs. The r.h.s. of (4.9) contains integration over the intermediate-state pion momentum:

$$\int \frac{d^4 p_1'}{(2\pi)^2}\, \delta_+(p_1'^2 - \mu^2)\delta_+(p_2'^2 - \mu^2)\,(p_1' - p_2')_\mu a(q^2) \cdot A^*(p_1', p_2'; p_1, p_2).$$

In the $\pi\pi$ centre-of-mass frame it reduces to the angular integral over the direction of the relative momentum $(p_1' - p_2')_\mu \Longrightarrow 2\mathbf{p}_c'$ which multiplies the scattering amplitude $A^*(\mathbf{p}_c, \mathbf{p}_c')$. Writing down the partial wave expansion of the latter,

$$A^*(\mathbf{p}_c, \mathbf{p}_c') = \sum_\ell (2\ell + 1) f_\ell^*(q^2) P_\ell(\cos\Theta_{\mathbf{p}_c\mathbf{p}_c'}),$$

we observe that the only term with $\ell = 1$ survives the integration:

$$(2\ell + 1) \int \frac{d\Omega'}{4\pi} 2\mathbf{p}_c' \cdot P_\ell(\cos\Theta_{\mathbf{p}_c\mathbf{p}_c'}) = 2\mathbf{p}_c \delta_{\ell,1}.$$

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This matches the structure of the l.h.s.,

$$(p_1 - p_2)_\mu \,\mathrm{Im}\, a(q^2) \quad \Longrightarrow \quad 2\mathbf{p}_c \,\mathrm{Im}\, a(q^2),$$

and the unitarity relation (4.9) takes the form

$$\mathrm{Im}\, a(q^2) \;=\; \tau a(q^2) f_1^*(q^2),$$

with τ the phase space volume (3.7b) of the two-pion state. Substituting the general solution of the two-particle unitarity condition (3.9)

$$f_\ell = \frac{1}{2i\tau}\left(\mathrm{e}^{2i\delta_\ell} - 1\right),$$

we have

$$\frac{a(q^2) - a^*(q^2)}{2i} = \tau \cdot \frac{-1}{2i\tau}\left(\mathrm{e}^{-2i\delta_1} - 1\right) a(q^2) \quad \Longrightarrow \quad \frac{a^*}{a} = \mathrm{e}^{-2i\delta_1}. \quad (4.10)$$

The unitarity condition simply tells us that the *phase* of the pion form factor equals that of the $\pi\pi$ scattering amplitude. The origin of the complexity of the form factor lies in *re-interaction* between pions in the final state.

Above the four-pion threshold the situations gets more complicated. Nevertheless, for the sake of simplicity, let us suppose that the relation (4.10) holds for all q^2 values. Then I would be able to calculate the form factor straight away! To this end consider the function $F = \ln a(q^2)$ and write the corresponding dispersion relation,

$$F(q^2) = \frac{1}{\pi}\int \frac{dQ^2}{Q^2 - q^2}\delta_1(Q^2) \;+\; \text{‘regular’}$$

with ‘regular’ marking a possible non-singular (analytic) piece. Then,

$$a(q^2) = P(q^2) \times \exp\left\{\frac{1}{\pi}\int \frac{dQ^2}{Q^2 - q^2}\delta_1(Q^2)\right\},$$

with P a polynomial in q^2. The latter can be replaced by a constant if I suppose a good behaviour at $q^2 \to \infty$. Then, making use of $a(0) = 1$, I would finally predict the form factor from the knowledge of the $\pi\pi$ scattering phase:

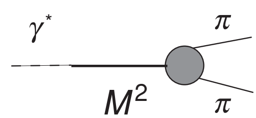
$$a(q^2) = \exp\left\{\frac{q^2}{\pi}\int_{4\mu^2}^{\infty} \frac{dQ^2}{Q^2(Q^2 - q^2)}\delta_1(Q^2)\right\}. \quad (4.11)$$

Unfortunately, literally this formula is incorrect since we have neglected many-particle channels which do essentially contribute at large Q^2. As a semi-quantitative estimate, however, (4.11) works reasonably well.

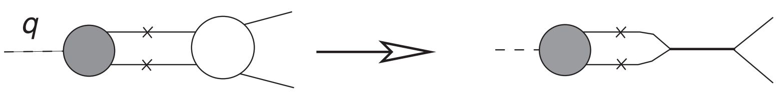
4.2.2 Pion radius and ρ meson

At large negative q^2 we expect hadron form factors to be falling fast since it is a truly point-like charge that can only give $a(q^2) = \mathcal{O}(1)$ in this limit. On the other hand, at large *positive* $q^2 \gg \mu^2$ it has to decrease too. This time, because a high-virtuality photon can produce many different multi-particle states, so that the probability of a given exclusive channel, $\gamma^* \to \pi\pi$, must fall. Therefore, $a(q^2)$ has to have somewhere a *maximum*.

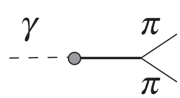
How could this be? An analytic function exhibiting a maximum makes us think of a nearby singularity. We saw that our form factor has the pion scattering phase as its source (in other words, strong interaction of pions). Suppose there is a *resonance* in the strong $\pi\pi$ interaction amplitude at some $q^2 = M^2$. Then this resonance will drive the behaviour of the form factor. Effectively, we will be looking for the process of the $\gamma^* \to \pi\pi$ transition via a resonance state.



Substituting a resonance for $A_{\pi\pi}$ in the unitarity relation (4.9),



a rough guess for its contribution to the form factor would be

$a(q^2) =$  $\propto \dfrac{1}{M^2 - q^2}.$ (4.12a)

Strictly speaking, to build a realistic model I would have to analyze vertices and take into consideration their q^2 dependence away from the pole position. However, it suffices to invoke, once again, the restriction $a(0) = 1$ in order to reasonably fix the numerator in (4.12a):

$$a(q^2) \simeq \frac{M^2}{M^2 - q^2}. \tag{4.12b}$$

At small positive q^2, the slope of the q^2 dependence will tell me the *characteristic mass* the annihilation process goes through,

$$a(q^2) = 1 + \frac{q^2}{M^2} + \cdots . \tag{4.12c}$$

On the other hand, the same expansion can be carried out in the physical region of the scattering channel, $q^2 < 0$. Recall the non-relativistic expression for the charge form factor. Expanding (4.6) in $Q^2 = -q^2 > 0$

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we get

$$a(Q^2) = \int d^3r\,\rho(r)\,\mathrm{e}^{-i\mathbf{Q}\cdot\mathbf{r}} \simeq a(0) - \int d^3r \frac{(\mathbf{Q}\cdot\mathbf{r})^2}{2}\rho(r)$$
$$= 1 - \frac{Q^2}{2\cdot 3}\langle r^2\rangle\,. \tag{4.13a}$$

Here we have introduced the average squared radius of the distribution of the charge inside the pion,

$$\langle r^2\rangle \equiv \int d^3r\,\rho(r)\cdot r^2, \tag{4.13b}$$

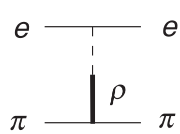
and used the spherical symmetry: $\langle r_z^2\rangle = \frac{1}{3}\langle r^2\rangle$.

Comparing the two expressions,

$$a(q^2) = 1 + \frac{q^2}{M^2} + \cdots \quad \Longleftrightarrow \quad a(q^2) = 1 + \frac{q^2\langle r^2\rangle}{6} + \cdots\,, \tag{4.14}$$

we conclude that the charge radius is directly related to the mass of the resonance in the annihilation channel.

In reality the P-wave pion–pion scattering is indeed dominated by the resonance – a vector meson ρ with a mass $m_\rho \simeq 750$ MeV. Firstly, this tells us that the annihilation process $e^+e^- \to \pi^+\pi^-$ should show a prominent peak at $q^2 \simeq m_\rho^2$ (and it does) and, secondly, that the position of this peak determines the electromagnetic radius of the pion:



In the case of nucleons the situation is somewhat more complicated. First of all, the mass of the $N\bar{N}$ state ($q^2 \sim 4\,\mathrm{GeV}^2$) is very large compared to the position of the $\pi\pi$ threshold ($q^2 \sim 0.1\,\mathrm{GeV}^2$). This pushes the physical region of the $e^+e^- \to N\bar{N}$ process far away from the scattering channel and from the first singularity. Besides, unlike pions which 'resonated' in only one meson ρ, a nucleon is also linked to another vector meson ω (with isospin 0). In reality the nucleon form factor behaves rather like $[M^2/(M^2 - q^2)]^2$, which may result from some destructive interference between various meson exchanges.

4.3 Isotopic structure of electromagnetic interaction

It is obvious that the electromagnetic interaction does not respect isotopic invariance:

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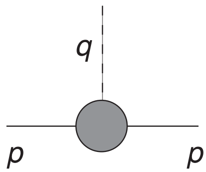
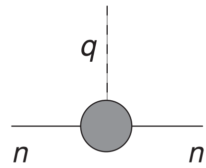


$a_p(0) = 1$



$a_n(0) = 0.$

Let us generalize the Dirac spinor U describing the proton wave function in (4.2), to represent the isotopic doublet of nucleons,

$$U = \begin{pmatrix} p \\ n \end{pmatrix}. \tag{4.15}$$

Then, with the help of Pauli matrices τ_i $(i = 1, 2, 3)$ and of the unit matrix in the 2×2 isotopic space, we can represent the electromagnetic vertex simultaneously for the proton and the neutron as

$$\Gamma_\mu \;=\; \bar{U}\left(\Gamma_\mu^{(0)} \cdot \mathbf{I} + \Gamma_\mu^{(1)} \cdot \boldsymbol{\tau}_3\right) U. \tag{4.16}$$

(We could not use $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ here since they would transfer a proton into a neutron, $p + \gamma \to n$, which we rather would not do.) Projecting onto the proton and neutron states, we get

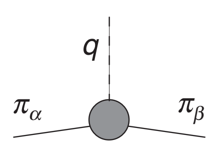
$$\Gamma_\mu^{(0)} + \Gamma_\mu^{(1)} = |p\rangle\Gamma_\mu\langle p|\,, \quad \Gamma_\mu^{(0)} - \Gamma_\mu^{(1)} = |n\rangle\Gamma_\mu\langle n|\,.$$

In particular, at $q^2 = 0$ this will give us

$$\Gamma_\mu^{(0)}(0) - \Gamma_\mu^{(1)}(0) \;=\; 0\,.$$

This isotopic beautification does not seem to bring us much profit. Still, from the point of view of the crossing channel, $\Gamma^{(0)}$ and $\Gamma^{(1)}$ may happen to acquire more fundamental meaning than the proton and neutron vertices themselves.

Look at the case of pions. Since pion π_α is a triplet in the $\mathbf{T}$ space $(\alpha, \beta = 1, 2, 3)$, we have now *three* independent diagonal matrices that can be used to construct the $\pi\pi\gamma$ interaction vertex:



$$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}^\dagger \left[\Gamma_\mu^{(0)} \cdot \mathbf{I} + \Gamma_\mu^{(1)} \cdot \mathbf{T}_3 \;+\; \Gamma_\mu^{(2)} \cdot \mathbf{T}_3^2\right] \begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}. \tag{4.17}$$

The term proportional to $\Gamma_\mu^{(k)}$ corresponds to isospin $\mathbf{T} = k$. For a $\pi\pi$ system the possibilities are: a scalar $(\mathbf{T} = 0)$, a vector $(\mathbf{T} = 1)$ and a tensor $(\mathbf{T} = 2)$ in the isotopic space.

It becomes clear that electromagnetic form factors of a particle belonging to some huge isotopic multiplet *could belong* to high-rank tensor structures in the $\mathbf{T}$ space. In reality, however, photons seem to couple

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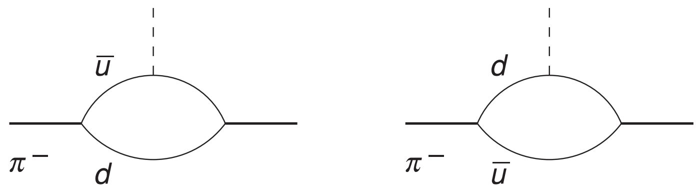
only with the $\mathbf{T} = 0$ and $\mathbf{T} = 1$ channels (though the accuracy of this experimental finding is presently not very high). In particular, in (4.17) for pions $\Gamma_\mu^{(2)} \simeq 0$.

How might one understand this phenomenon? Suppose that all hadrons were built of *isotopic doublets*, $\mathbf{T} = \frac{1}{2}$, like quarks. Then it is *quarks* which participate in the electromagnetic interaction,



and we have only an iso-scalar and an iso-vector in the photon channel:

$$\Gamma_{\text{em}} \sim \mathbf{I} + \boldsymbol{\tau}_3. \tag{4.18}$$

It is important to bear in mind that, since strong interactions respect the isospin symmetry, whichever diagrams I include to account for interactions between quarks, coupling of the photon to π-meson will retain the scalar + vector isotopic structure of (4.18) (in the first order in α_{em}).

This is an example of how studying electromagnetic interactions may produce highly non-trivial hints about the nature of hadrons.

4.4 Deep inelastic scattering

At large virtual-photon momentum transfer, $-q^2 \gg 1\,\text{GeV}^2$, electromagnetic proton form factors decrease fast, as a large inverse power of q^2.

This may look surprising at the first sight. Indeed, from the point of view of the dispersion relation,

$$F(q^2) = \frac{1}{\pi} \int \frac{dQ^2\ \text{Im}\, F(Q^2)}{Q^2 - q^2},$$

in order to ensure a fast falloff of F at large negative q^2, the imaginary part must oscillate, and in a very specific way.

4.4.1 Parton concept

Let us look at the problem from a classical perspective instead. It is easy to make $F(q^2)$ fast falling if we take the charge density $\rho(r)$ in the NQM formula (4.6) to be a smooth non-singular function.

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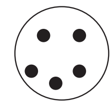
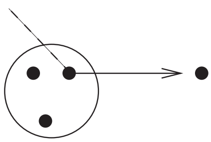
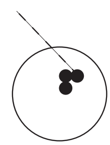


There exists another way to explain the smallness of elastic scattering. Imagine the proton consisting of some number of *weakly bound* point-like charges.



Then, physically, the scattering of an electron off such a composite object will *always* be inelastic: the photon will interact with one of the charges and kick it out, 'destroying' the target proton. This picture is similar to what happens to an atom: when an electron is kicked off from a core shell, leaving a vacancy, the excited atom 'decays' by emitting photons, Auger electrons, ...

The elastic proton scattering will only be possible in the configurations when all the quarks happen to be very close to each other, at small distances $(\Delta r)^2 \sim 1/|q^2| \ll \langle r^2 \rangle$. In this picture, the elastic channel suppression is due to the smallness of the probability of such small-distance configurations (equivalent to weak binding).



How to determine which answer is closer to reality? Since the q^2 behaviour of the elastic form factor does not help to discriminate the two pictures, let us look at more complex – and more interesting – *inelastic* processes. In the first picture the total inelastic cross section will be as small as the elastic one, since in the scenario of a smooth charge density ρ, a small-wavelength photon would simply find no-one to interact with, elastically or otherwise. In the second scenario, an *inelastic* cross section is not small at all. On the contrary, it is determined by the probability for the photon to interact with one of the internal point-like constituents of the proton.

4.4.2 DIS cross section

We will study the process called *deep inelastic lepton–proton scattering* (DIS); the word 'deep' stresses the fact that the a highly virtual photon with $Q^2 = -q^2 \gg \langle r^2 \rangle^{-1}$ penetrates *deep* into the proton's interior.

In non-relativistic quantum mechanics, electron scattering off an atom having N electrons is given by the transition matrix element

$$\rho^{0,n}_{\mathbf{k},\mathbf{k}'} \propto \int d^3\mathbf{r}\, e^{i\mathbf{q}\cdot\mathbf{r}} \int \prod d^3\mathbf{r}_i\, \psi^*_n(\mathbf{r}_i)\, \sum_{k=1}^{N} \frac{e^2}{|\mathbf{r}-\mathbf{r}_k|}\, \psi_0(\mathbf{r}_i)$$

$$\sim \frac{e^2}{\mathbf{q}^2} \int \prod d^3\mathbf{r}_i\, \psi^*_n(\mathbf{r}_i) \sum_{k=1}^{N} e^{i\mathbf{q}\cdot\mathbf{r}_k} \psi_0(\mathbf{r}_i).$$

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Here $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final electron momenta, and ψ_0 and ψ_n mark the wave functions of the initial ground state and of the excited final state of the atom, respectively.

We are interested in the cross section summed over all possible final states of the excited atom, $n \leq n_0$. If n_0 is large, the product of the final-state wave functions that enters the expression for the cross section can be simplified using the (*almost*) completeness relation:

$$\sum_{n=0}^{n_0} \big|\psi_n(\mathbf{r}')\big\rangle\langle\psi_n(\mathbf{r})\big| \;\simeq\; \sum_{n=0}^{\infty} \big|\psi_n(\mathbf{r}')\big\rangle\langle\psi_n(\mathbf{r})\big| \;=\; \delta(\mathbf{r}-\mathbf{r}'). \tag{4.19}$$

The cross section then reduces to

$$\frac{d\sigma}{d\mathbf{q}^2} \sim \frac{e^4}{\mathbf{q}^4} \int \prod d^3\mathbf{r}_i \, \psi_0^*(\mathbf{r}_i) \sum_{j,k=1}^{N} \mathrm{e}^{i\mathbf{q}\cdot(\mathbf{r}_j-\mathbf{r}_k)} \psi_0(\mathbf{r}_i). \tag{4.20}$$

When the photon wavelength is much smaller that the typical distance between the atomic electrons, $\mathbf{q}^2 \gg \langle \mathbf{r}_{jk}^2 \rangle^{-1}$, the *interference* terms with $j \neq k$ in (4.20) become negligible. We are left with the sum of N diagonal contributions:

$$\frac{d\sigma}{d\mathbf{q}^2} \sim \frac{e^4}{\mathbf{q}^4} \sum_{k=1}^{N} \psi_0^*(\mathbf{r}_i)\,\psi_0(\mathbf{r}_i) = \frac{e^4}{\mathbf{q}^4} \times N,$$

where we have used the normalization condition for the ground state wave function. Thus, the total inelastic eA cross section reduces to the sum of independent interactions of quasi-free individual electrons under two conditions, namely:

(1) the 'resolution' of the photon should be large enough to separate individual electrons inside the target atom, $\mathbf{q}^2 \gg R^{-2}$;

(2) the energy transferred to the atom should be sufficient to have a large enough number of excited states in order to employ the completeness relation (4.19).

Let us turn now to the relativistic theory and learn to write the corresponding cross section. The squared matrix element for the ep scattering process with the production of n particles is shown in Fig. 4.1. To obtain the cross section, we have to square the virtual photon–proton interaction amplitude A_μ bearing the photon index μ and convolute it with the corresponding electron scattering tensor $T_e^{\mu\nu}$. We also write down the phase space volume for $n+1$ particles (the scattered electron k' and n produced final state hadrons with momenta $\{p_i\}$), and include the square of

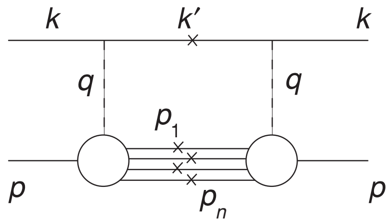


Fig. 4.1 Discontinuity of the forward photon–proton scattering amplitude.

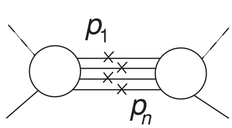
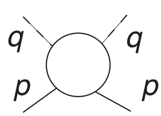
the photon propagator, $(1/q^2)^2$, and the flux factor $J_{ep} \simeq 4(pk)$:

$$d\sigma = \frac{e^2}{J_{ep}} \int \frac{d^4k'}{(2\pi)^3}\delta_+(k'^2 - m_e^2) \sum_n \prod_{i=1}^{n} \left[\int \frac{d^4p_i}{(2\pi)^3}\delta_+(p_i^2 - m_i^2) \right]$$
$$T_e^{\mu\nu} \frac{1}{q^4} A_\mu(p, q, \{p_i\}) A_\nu^*(p, q, \{p_i\}) \, (2\pi)^4 \delta\left(p + k - k' - \sum_{i=1}^{n} p_i \right). \tag{4.21}$$

If we *average* over initial electron polarizations, the electron tensor $T_e^{\mu\nu}$ is given by the expression

$$T_e^{\mu\nu} = \mathrm{Tr}\left[\frac{(\hat{k} + m_e)}{2} \cdot \gamma^\mu (\hat{k}' + m_e) \gamma^\nu \right] = 2(k^\mu k'^\nu + k'^\mu k^\nu) + g^{\mu\nu} q^2. \tag{4.22}$$

First comes an observation similar to the one we made when discussing the Chew–Low method (see (2.69)): we may replace the sum over produced hadrons by the imaginary part of the amplitude of the forward scattering:

$\sum_n$  $= 2\,\mathrm{Im}$  $\equiv e^2 W^{\mu\nu}(pq, q^2)$. (4.23)

The only peculiarity of this ‘optical theorem’ is that here one of the colliding objects is a virtual photon, so that the tensor $W_{\mu\nu}$ (bearing, once again, vector photon indices) depends on two variables rather than on the energy of the collision only, $W = W(pq, q^2)$. Due to conservation of the electromagnetic current, this tensor must be orthogonal to q^μ (and q^ν) and can be therefore represented as follows:

$$\frac{1}{2\pi} W_{\mu\nu} = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) W_1 + \frac{1}{M^2} \left(p_\mu - \frac{pq}{q^2} q_\mu \right) \left(p_\nu - \frac{pq}{q^2} q_\nu \right) W_2, \tag{4.24}$$

with W_1, W_2 known as the *structure functions*. (If the spin vector s of the initial particle is fixed, in $W_{\mu\nu}$ (as well as in $T_e^{\mu\nu}$) there appears an additional structure proportional to the anti-symmetric tensor $i\epsilon_{\mu\nu\alpha\beta} q^\alpha s^\beta$.)

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The differential cross section takes the form

$$d\sigma = \frac{\alpha^2}{4\pi}\frac{1}{q^4}T_e^{\mu\nu}W_{\mu\nu}\frac{d^3k'}{(pk)\,E'}. \tag{4.25}$$

Let us calculate the product of the tensors. The electron one, $T_e^{\mu\nu}$, also respects the current conservation; as becomes obvious from its expression equivalent to (4.22):

$$T_e^{\mu\nu} = (k+k')^\mu(k+k')^\nu - q^\mu q^\nu + g^{\mu\nu}q^2; \quad (k+k')\cdot q = k^2 - k'^2 = 0.$$

Therefore, we can drop the terms proportional to q_μ and/or q_ν from the hadron tensor when calculating the convolution:

$$(4.24) \implies -g_{\mu\nu}W_1 \;+\; \frac{p_\mu p_\nu}{M^2}W_2.$$

The convolution yields

$$\frac{1}{2\pi}T_e^{\mu\nu}W_{\mu\nu} = -4\Big[(kk') + q^2\Big]\,W_1 + \left[\frac{4(pk)(pk')}{M^2} + q^2\right]W_2. \tag{4.26a}$$

Using $q^2 = (k'-k)^2 = 2m_e^2 - 2(kk')$,

$$\frac{1}{2\pi}T_e^{\mu\nu}W_{\mu\nu} \simeq\; 4(kk')\,W_1 \;+\; 2\left[\frac{2(pk)(pk')}{M^2} - (kk')\right]W_2, \tag{4.26b}$$

where we have dropped the electron mass m_e as negligibly small.

In the laboratory system where the target proton is at rest, $p = (M, \mathbf{0})$, we introduce the electron scattering angle Θ and approximate $(kk') \simeq EE'(1-\cos\Theta)$ in (4.26b) to get

$$\frac{1}{2\pi}T_e^{\mu\nu}W_{\mu\nu} \simeq 4EE'\left[2W_1\cdot\sin^2\frac{\Theta}{2} + W_2\cdot\cos^2\frac{\Theta}{2}\right]. \tag{4.27}$$

Substituting (4.27) into (4.25), the cross section becomes

$$\frac{d\sigma}{d\cos\Theta\,dE'} = \frac{4\pi\alpha^2E'^2}{q^4M}\left[W_2\cdot\cos^2\frac{\Theta}{2} + 2W_1\cdot\sin^2\frac{\Theta}{2}\right]. \tag{4.28}$$

By measuring the scattered electron momentum, one extracts the dependence of the structure functions $W_{1,2}(pq,q^2)$ by measuring the direction of the scattered electron momentum and the energy $\nu = E - E'$ transferred to the hadron system:

$$(pq) = M\nu, \quad q^2 \simeq -2EE'(1-\cos\Theta).$$

Let us make a comparison with the *elastic* proton scattering. Apart from the form factor in the photon–proton vertex,

$$\Gamma_\mu \sim e_p\,\gamma_\mu\cdot\Gamma_{\rm el}(q^2),$$

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the calculation of the hadron tensor becomes identical to the lepton one:

$$\begin{aligned}\frac{W^{\rm el}_{\mu\nu}}{2\pi} &= \left[2(p_\mu p'_\nu + p'_\mu p_\nu) + g_{\mu\nu}q^2\right]\Gamma^2_{\rm el}(q^2)\cdot\delta(2pq+q^2) \\ &= \left[(p+p')_\mu(p+p')_\nu - q_\mu q_\nu + g_{\mu\nu}q^2\right]\Gamma^2_{\rm el}(q^2)\,\delta\big(2pq+q^2\big).\end{aligned} \tag{4.29}$$

The final hadron state consists now of the recoiling proton only, and the delta-function puts it on the mass shell: $p'^2 - M^2 = (p+q)^2 - M^2 = 0$. It is easy to extract the functions W_i corresponding to elastic scattering. Observing that

$$q = p' - p \quad\Longrightarrow\quad p_\mu - \frac{pq}{q^2}q_\mu = \tfrac{1}{2}(p+p')_\mu;$$

by comparing (4.29) with the general decomposition (4.24) we derive

$$W_2^{\rm el} = 4M^2\,\Gamma^2_{\rm el}\cdot\delta\big(2pq+q^2\big), \quad W_1^{\rm el} = -\frac{q^2}{4M^2}W_2^{\rm el}. \tag{4.30}$$

Rewriting the phase space element in (4.28) terms of invariants,

$$\frac{E'^2}{M}\cdot d\cos\Theta\,dE' = \frac{E'^2}{M}\cdot\frac{dq^2}{2EE'}\,dq_0 = \frac{pk'}{pk}\cdot dq^2\frac{d(2pq)}{4M^2},$$

the differential cross section takes the form

$$\frac{d\sigma}{dq^2\,d(2pq)} = \frac{4\pi\alpha^2}{q^4}\cdot\frac{pk'}{pk}\cdot\left[\frac{W_2}{4M^2}\,\cos^2\frac{\Theta}{2} + \frac{W_1}{2M^2}\,\sin^2\frac{\Theta}{2}\right]. \tag{4.31}$$

When the energy of the incident electron is large, the scattering angle becomes very small, $\Theta^2 \simeq \left|q^2\right|M^2/(pk)^2 \propto |t|/s^2 \to 1$; in this limit

$$\frac{d\sigma^{\rm el}}{dq^2} \simeq \frac{4\pi\alpha^2}{q^4}\frac{W_2^{\rm el}}{4M^2}\,d(2pq) = \frac{4\pi\alpha^2}{q^4}\cdot\Gamma^2_{\rm el}(q^2). \tag{4.32}$$

Thus, $W_2^{\rm el}$ is nothing but the square of the elastic proton form factor.

An *inelasticity* of the interaction in a general case can be characterized by a dimensionless variable ω which measures the invariant mass of the final hadron system in units of the momentum transfer $\left|q^2\right|$:

$$W^2 = (p+q^2) - M^2 = -q^2(\omega - 1)\,, \quad \omega \equiv \frac{2(pq)}{-q^2} \geq 1\,.$$

In the elastic process, the invariant photon–proton energy is determined by the on-mass-shell condition $(p+q)^2 = 2pq + q^2 + M^2 = M^2$ corresponding *exactly* to $\omega = 1$. If we take ω not too close to unity, $\omega = \mathcal{O}(1)$,

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and keep it fixed while increasing $|q^2|$, the lepton–hadron interaction in this kinematics is called *deep inelastic scattering* (DIS). The inelasticity W^2 of such a process is proportional to the squared transferred momentum q^2 (the latter characterizing the 'hardness' of the process).

The inelastic cross section can be expressed then as

$$\frac{d\sigma}{dq^2\,d\omega} \simeq \frac{4\pi\alpha^2}{q^4} \cdot F_2(q^2, \omega), \tag{4.33}$$

where we have introduced the *scaling function* F_2 – an analogue of the squared form factor in (4.32):

$$F_2(q^2, \omega) = -\frac{q^2}{4M^2} W_2(pq, q^2); \quad F_2^{(\mathrm{el})}(q^2, \omega) = \Gamma_{\mathrm{el}}^2(q^2)\delta(\omega - 1).$$

The SLAC experiment has found that F_2 (and $F_1 = W_1$) becomes *independent* of q^2 starting from $|q^2| \sim 2-4\,\mathrm{GeV}^2$. This shows that the picture of a smooth charge distribution inside a proton cannot be correct, hence in such a case the 'inelastic form factor' would be falling with $|q^2|$ together with the elastic one.

On the contrary, the observed *Bjorken scaling* regime $F(q^2, \omega) \simeq f(\omega)$ perfectly fits the second picture: (4.33) tells us that the cross section of an inelastic *ep* process equals that of the Rutherford elastic scattering off a point-like particle. It is a point-like charge inside the proton – a quasi-free 'parton' – that takes an impact.

The question arises, can we say anything about the parton spin? Let us ask ourselves, why did we get two structure functions in the first place, not ten?

The virtual photon linking the lepton with the hadron block has three polarizations $(e^\lambda q) = 0$, two orthogonal to the scattering plane $\{p, q\}$, and one lying in it:

$$\frac{g_{\mu\nu} - q_\mu q_\nu/q^2}{q^2} = \sum_{\lambda=1}^{3} e_\mu^\lambda(q) e_\nu^{\lambda *}(q); \quad (e^\lambda p) = 0,\ \lambda = 1, 2.$$

The photon does not change its polarization in the cause of scattering:

$$M^{\lambda\lambda'} = e_\mu^\lambda W^{\mu\nu} e_\nu^{\lambda' *} = M^\lambda \delta_{\lambda\lambda'}.$$

Invoking (4.24) we see that the scattering cross section $\sigma^\perp$ of a *transversal* photon ($\lambda = 1, 2$) is determined by the structure containing the $g_{\mu\nu}$ tensor that is, by the function W_1, while the *longitudinal* (in-plane) one, $\sigma^\parallel$ ($\lambda = 3$), – by a definite linear combination of W_1 and W_2.

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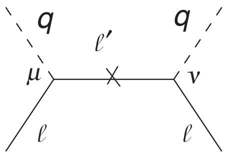

Imagine that some charged parton with momentum ℓ absorbs the virtual photon and scatters elastically. If the parton has spin *zero*, its electromagnetic vertex is proportional to the momentum, $\Gamma_\mu \propto (\ell+\ell')_\mu$, and we get

$$\frac{W_{\mu\nu}}{2\pi} \sim (\ell+\ell')_\mu \frac{\text{Im}}{\pi}\left[\frac{1}{m^2-\ell'^2-i\epsilon}\right](\ell+\ell')_\nu \sim \frac{4\ell_\mu\ell_\nu}{q^2}\delta\left(\frac{2\ell q}{q^2}+1\right).$$

In this case the longitudinal photon interacts with a normal cross section, while the transverse polarizations would be power suppressed (provided the parton inside the proton has a limited transverse momentum):

$$\frac{\sigma^{\|}}{\sigma_{\text{point}}} = \mathcal{O}(1)\,, \qquad \frac{\sigma^{\perp}}{\sigma_{\text{point}}} \sim \frac{\langle \boldsymbol{\ell}_\perp^2\rangle}{|q^2|} \ll 1.$$

On the other hand, for a spin-$\frac{1}{2}$ parton, the structure of $W_{\mu\nu}$ reproduces that of the electron tensor (4.22):

$$W_{\mu\nu} \propto \left(2\frac{\ell_\mu\ell'_\nu+\ell'_\mu\ell_\nu}{q^2}+g_{\mu\nu}\right)\delta\left(\frac{2\ell q}{q^2}+1\right).$$

Here, on the contrary, the *longitudinal* polarization gets suppressed,

$$\ell^\mu\cdot\left(\ell_\mu\ell'_\nu+\ell'_\mu\ell_\nu-g_{\mu\nu}(\ell\ell')\right) \propto \ell^2=m^2,$$

and we have a situation just *opposite* to the previous case

$$\frac{\sigma^{\perp}}{\sigma_{\text{point}}} = \mathcal{O}(1)\,, \qquad \frac{\sigma^{\|}}{\sigma_{\text{point}}} \sim \frac{\langle \boldsymbol{\ell}_\perp^2\rangle+m^2}{|q^2|} \ll 1\,.$$

The experiment shows

$$\sigma^{\perp} \gg \sigma^{\|}$$

$(\sigma^{\|}/\sigma^{\perp} \sim 1/5)$, hinting at spin-$\frac{1}{2}$ partons (quarks?).

The overall impression is that the picture with quarks in the rôle of partons stands up to scrutiny.

Two phenomena have to be understood:

(1) the fact that the inelastic cross section is not small, $F=\mathcal{O}(1)$; and

(2) the Bjorken scaling, $F(q^2,\omega)\simeq f(\omega)$.

If the Bjorken scaling phenomenon is verified by future more detailed and more accurate experiments† we will be facing a serious puzzle! Actually,

† To hope this would *not* is a sin, because it is *beautiful*, in the first place.

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the existence of this scaling challenges all we knew in 'the past'. While the observed scaling is apparently well explained by a naive parton model, it cannot hold from the field-theoretical point of view.

Indeed, imagine a proton consisting of 'points', subject to some QFT interaction. The Bjorken scaling emerges indeed in the toy model of scalar fields with the $\lambda\varphi^3$ interaction. Why? Because all the integrals in this theory converge in the ultraviolet momentum region, and at large $\left|q^2\right|$ all the corrections vanish leaving us with a point-like particle without form factor. Unfortunately, this is true not only for inelastic but for the elastic scattering as well.

In principle, we could get a falling elastic form factor back if we suppose that the fields φ may form bound states, in which case a falloff of $F_{\text{el}}^2(q^2)$ at large q^2 would be explained by the decay of a bound state into its point-like constituents. In spite of such an 'improvement', this model still does not suit us, for two reasons. Firstly, one cannot build real hadrons out of spinless particles, and, secondly, we would have $\sigma^{\parallel} \gg \sigma^{\perp}$, in contradiction with experiment.

As soon as we introduce *fermions*, the theory seizes to be superconvergent and becomes (at best) renormalizable. Ultraviolet logarithmic divergences appear that have to be renormalized, etc. But this means that interaction corrections are *never small* for any, whichever large, q^2.

In a logarithmic quantum field theory, inside a physical particle there are always virtual exchanges with arbitrarily large momenta, exceeding the DIS momentum transfer: $\left|q_{\text{virt}}^2\right| \gg \left|q^2\right|$. Under these circumstances, when probing a hadron with higher and higher 'resolution', we encounter more and more 'constituents'; hence, the exact Bjorken scaling regime simply *cannot hold*.