Canad. Math. Bull. Vol. **67** (4), 2024, pp. 1092–1106
http://dx.doi.org/10.4153/S0008439524000602

On the sign changes of Dirichlet coefficients of triple product L-functions

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Abstract. Let f and g be two distinct normalized primitive holomorphic cusp forms of even integral weight k_1 and k_2 for the full modular group $SL(2,\mathbb{Z})$, respectively. Suppose that $\lambda_{f\times f\times f}(n)$ and $\lambda_{g\times g\times g}(n)$ are the n-th Dirichlet coefficient of the triple product L-functions $L(s, f\times f\times f)$ and $L(s, g\times g\times g)$. In this paper, we consider the sign changes of the sequence $\{\lambda_{f\times f\times f}(n)\}_{n\geq 1}$ and $\{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\}_{n\geq 1}$ in short intervals and establish quantitative results for the number of sign changes for $n\leq x$, which improve the previous results.

1 Introduction

Let H_k be the set of normalized primitive holomorphic cusp forms of even integral weight k for the full modular group $SL(2,\mathbb{Z})$, which are eigenfunctions of all the Hecke operators T_n. Then $f(z)\in H_k$ has a Fourier expansion at the cusp infinity

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{k-1}{2}}e(nz)\quad(\Im z>0), \tag{1.1}$$

where we normalize $f(z)$ so that $\lambda_f(1)=1$. From the theory of Hecke operators, the Fourier coefficient $\lambda_f(n)$ is real and satisfies the multiplicative property

$$\lambda_f(m)\lambda_f(n)=\sum_{d|(m,n)}\lambda_f\left(\frac{mn}{d^2}\right), \tag{1.2}$$

where $m\geq 1$ and $n\geq 1$ are any integers. In 1974, Deligne [1] proved the Ramanujan-Petersson conjecture: for all integers $n\geq 1$,

$$|\lambda_f(n)|\leq d(n), \tag{1.3}$$

where $d(n)$ is the number of positive divisors of n.

The sign changes of Fourier coefficients attached to automorphic forms is an important problem and has been studied extensively by several scholars. In [12], Knopp, Kohnen and Pribitkin showed $\{\lambda_f(n)\}_{n\geq 1}$ has infinitely many sign changes. After that, Ram Murty[20] first considered the sign changes of the sequence of Fourier coefficients in short intervals. Later, Meher, Shankhadhar, and Viswanadham [19] established lower bounds for the number of sign changes of the sequence $\{\lambda_f(n^j)\}_{n\geq 1}$ with $j=2,3,4$.

Received by the editors August 8, 2024; revised August 28, 2024; accepted August 29, 2024.
AMS subject classification: **11F11**, 11F30, 11F66.
Keywords: sign change, Dirichlet coefficient, cusp form.

However, the analogous questions of simultaneous sign changes of two cusp forms have also been investigated by a number of mathematicians. Let f and g be two different cusp forms. In [13], Kumari and Ram Murty considered the simultaneous sign changes problem about $\{\lambda_f(n)\lambda_g(n)\}_{n\geq 1}$. Later, Gun, Kumar, and Paul [3] studied this problem about $\{\lambda_f(n)\lambda_g(n^2)\}_{n\geq 1}$. Recently, Lao and Luo [14] also considered more general cases and obtained better results about $\{\lambda_f(n^i)\lambda_g(n^j)\}_{n\geq 1}$ for $i \geq 1, j \geq 2$. Further, Hua [5, 6] investigated the analogous problem over a certain integral binary quadratic form.

The triple product L-function $L(s, f\times f\times f)$ satisfies analogous analytic properties such as those of the Hecke L-functions, and its coefficients also change signs. In this paper, we investigate the sign change about the sequence $\{\lambda_{f\times f\times f}(n)\}_{n\geq 1}$ and $\{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\}_{n\geq 1}$ in short intervals and prove the following theorems.

Theorem 1.1 *Let $f \in H_k$ and $\lambda_{f\times f\times f}(n)$ be the n-th normalized Dirichlet coefficient of the triple product L-function $L(s, f\times f\times f)$. Then for $j \geq 2$ and any δ with*

$$1 - \frac{315}{40\sqrt{30} + 8442} = 0.963\cdots < \delta < 1,$$

the sequence $\{\lambda_{f\times f\times f}(n)\}_{n\geq 1}$ has at least one sign change for $n \in (x, x + x^{\delta}]$ for sufficiently large x. Moreover, the number of sign changes of the above sequence for $n \leq x$ is $\gg x^{1-\delta}$.

Remark 1.2 By comparison, in Theorem 1.1, our results about the number of sign changes for $n \leq x$ improve the results of Hua [7, Theorem 1.1].

Theorem 1.3 *Let $f \in H_{k_1}$, $g \in H_{k_2}$ be two different forms. Also let $\lambda_{f\times f\times f}(n)$ and $\lambda_{g\times g\times g}(n)$ be the n-th normalized Dirichlet coefficient of the triple product L-function $L(s, f\times f\times f)$ and $L(s, g\times g\times g)$, respectively. Then for any δ with*

$$1 - \frac{882}{400\sqrt{21} + 1771497} = 0.99950\cdots < \delta < 1,$$

the sequence $\{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\}_{n\geq 1}$ has at least one sign change for $n \in (x, x + x^{\delta}]$ for sufficiently large x. Moreover, the number of sign changes of the above sequence for $n \leq x$ is $\gg x^{1-\delta}$.

In Section 2, we give some preliminary lemmas. In Section 3, we prove three propositions which play an important part in proving Theorem 1.1 and Theorem 1.3. In Section 4 and Section 5, we complete the proofs of Theorem 1.1 and Theorem 1.3, respectively. And, throughout the paper, we denote by ε a sufficiently small positive constant, whose value may not be necessarily the same in all occurrences.

2 Preliminary and some lemmas

In this section, we will establish and recall some preliminary results for the proofs of Theorem 1.1 and Theorem 1.3. We first recall the definitions about some specific L-functions.

The Hecke L-function attached to $f \in H_k$ is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1}, \tag{2.1}$$

which converges absolutely for $\Re(s) > 1$. The local parameters $\alpha_f(p)$ and $\beta_f(p)$ satisfy

$$\alpha_f(p) + \beta_f(p) = \lambda_f(p) \quad \text{and} \quad |\alpha_f(p)| = |\beta_f(p)| = 1. \tag{2.2}$$

The j-th symmetric power L-function attached to $f \in H_k$ is defined as

$$L(s, \mathrm{sym}^j f) := \prod_p \prod_{m=0}^{j} \left(1 - \alpha_f(p)^{j-m} \beta_f(p)^m p^{-s}\right)^{-1}, \tag{2.3}$$

for $\Re(s) > 1$. Then, $L(s, \mathrm{sym}^j f)$ can be expressed as the Dirichlet series

$$L(s, \mathrm{sym}^j f) = \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s} = \prod_p \left(1 + \sum_{k \geq 1} \frac{\lambda_{\mathrm{sym}^j f}(p^k)}{p^{ks}}\right), \tag{2.4}$$

where $\lambda_{\mathrm{sym}^j f}(n)$ is a real multiplicative function, and

$$L(s, \mathrm{sym}^0 f) = \zeta(s), \qquad L(s, \mathrm{sym}^1 f) = L(s, f). \tag{2.5}$$

According to (2.1) and (2.3), we obtain

$$\lambda_{\mathrm{sym}^j f}(p) = \sum_{m=0}^{j} \alpha^{j-m}(p) \beta^m(p) = \lambda_f(p^j). \tag{2.6}$$

Remark 2.1 The result of Newton-Thorne [21] implies that $\mathrm{sym}^j f$ $(j \geq 1)$ is an automorphic cuspidal representation of $GL(j+1)$. This means that $L(s, \mathrm{sym}^j f)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $j+1$.

The Rankin–Selberg L-function associated with $\mathrm{sym}^i f$ and $\mathrm{sym}^j g$ is defined by

$$\begin{aligned} L(s, \mathrm{sym}^i f \times \mathrm{sym}^j g) &= \prod_p \prod_{u=0}^{i} \prod_{v=0}^{j} \left(1 - \frac{\alpha_f(p)^{i-u} \beta_f(p)^u \alpha_g(p)^{j-v} \beta_g(p)^v}{p^s}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)}{n^s}, \qquad \Re(s) > 1, \end{aligned} \tag{2.7}$$

where $\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(n)$ is a real multiplicative function, and

$$\lambda_{\mathrm{sym}^i f \times \mathrm{sym}^j g}(p) = \sum_{u=0}^{i} \sum_{v=0}^{j} \alpha_f(p)^{i-2u} \alpha_g(p)^{j-2v} = \lambda_{\mathrm{sym}^i f}(p) \lambda_{\mathrm{sym}^j g}(p). \tag{2.8}$$

In particular, we have

$$L(s, \mathrm{sym}^1 f \times \mathrm{sym}^1 g) = L(s, f \times g), \; L(s, \mathrm{sym}^2 \times \mathrm{sym}^1 g) = L(s, \mathrm{sym}^2 f \times g). \tag{2.9}$$

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Remark 2.2 Due to the works of Jacquet and Shalika [10] [11], Shahidi [26] [27], Rudnick and Sarnak [25], Lau and Wu [15] and Newton-Thorne [21], the Rankin-Selberg function $L\left(s, \mathrm{sym}^i f \times \mathrm{sym}^j g\right)$ ($f \in H_{k_1}, g \in H_{k_2}$ are two different forms) has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies a certain functional equation of Riemann zeta-type of degree $(i+1)(j+1)$.

The triple product L-function associated with f is defined by

$$L(s, f \times f \times f) = \prod_p \left(1 - \frac{\alpha_f(p)^3}{p^s}\right)^{-1} \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-3} \left(1 - \frac{\beta_f(p)^3}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-3}$$

$$:= \sum_{n=1}^{\infty} \frac{\lambda_{f\times f\times f}(n)}{n^s}, \qquad \Re(s) > 1, \tag{2.10}$$

where $\lambda_{f\times f\times f}$ is real and multiplicative.

Remark 2.3 Recalling that the triple product L-functions $L(s, f \times f \times f)$ are automorphic L-functions has been showed by Garrett [2], Piatetski-Shapiro and Rallis [24], etc. Furthermore, we learn that the L-function $L(f \times f \times f, s)$ has an analytic continuation as an entire function in the whole complex plane $\mathbb{C}$ and satisfies certain Riemann zeta-type functional equations of degree 8.

Thus for $i, j \geq 1$, $L(s, \mathrm{sym}^j f)$ and $L\left(s, \mathrm{sym}^i f \times \mathrm{sym}^j g\right)$ are also general L-functions in the sense of Perelli [23]. For general L-functions, we have the following result.

Lemma 2.4 *Suppose that $\mathfrak{L}(s)$ is a general L-function of degree m. Then, for any $\varepsilon > 0$, we have*

$$\mathfrak{L}(\sigma + it) \ll (1 + |t|)^{\max\{\frac{m(1-\sigma)}{2}, 0\} + \varepsilon}, \tag{2.11}$$

uniformly for $1/2 \leq \sigma \leq 1 + \varepsilon$ *and* $|t| \geq 1$. *And*

$$\int_1^T |\mathfrak{L}(\sigma + it)|^2 dt \ll T^{\max\{m(1-\sigma), 1\} + \varepsilon}, \tag{2.12}$$

uniformly for $\frac{1}{2} \leq \sigma \leq 2$ *and* $T \geq 1$.

Proof See [23]. ∎

Lemma 2.5 *Let* $k = \frac{8}{63}\sqrt{15} = 0.4918\cdots$. *Then for any* $\varepsilon > 0$, *we have*

$$\zeta(\sigma + it) \ll t^{k(1-\sigma)^{3/2} + \varepsilon} \tag{2.13}$$

uniformly for $|t| \geq 1$ *and* $1/2 \leq \sigma \leq 1$.

Proof The bound is proved by Heath-Brown in [4, Theorem 5]. ∎

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Lemma 2.6 *For any $\varepsilon > 0$, we have*

$$L(\sigma + it, \mathrm{sym}^2 f) \ll (1 + |t|)^{\max\{\frac{6}{5}(1-\sigma),0\}+\varepsilon}, \tag{2.14}$$

uniformly for $\frac{1}{2} \le \sigma \le 2$ and $|t| \ge 1$.

Proof See [16, Corollary 1.2]. ∎

Suppose that π is a unitary cuspidal automorphic representation of $GL_r(\mathbb{A}_\mathbb{Q})$ and $L(s, \pi)$ is the automorphic L-function related to π. For $1/2 < \sigma < 1$, let $m(\sigma) \ge 2$ be the supremum of all numbers m such that

$$\int_1^T |L(s, \pi)|^m dt \ll T^{1+\varepsilon}. \tag{2.15}$$

Lemma 2.7 *Let $m(\sigma)$ be defined by (2.15). Then for each $1 - 1/r < \sigma < 1$ with $r \ge 4$, we have*

$$m(\sigma) \ge \frac{2}{r(1-\sigma)}. \tag{2.16}$$

Proof See [8, Theorem 1.1]. ∎

Newton and Thorne [21, 22] proved that $\mathrm{sym}^j f$ corresponds to a cuspidal automorphic representation of $GL_r(\mathbb{A}_\mathbb{Q})$ for all $j \ge 1$ with $f \in H_k$. As a result, we obtain the following lemma.

Lemma 2.8 *For $f \in H_k$ and any $\varepsilon > 0$, we have*

$$\int_1^T \left|L(23/25 + it, \mathrm{sym}^4 f)\right|^5 dt \ll T^{1+\varepsilon}, \tag{2.17}$$

uniformly for $T \ge 1$.

Proof According to Lemma 2.7, for $r = 5$, we take $\sigma = 23/25$. ∎

Lemma 2.9 *For $f \in H_k$, we have*

$$L(s, f \times f \times f) = L(s, f)^2 L(s, \mathrm{sym}^3 f). \tag{2.18}$$

Proof See [18, Lemma 2.1]. ∎

Lemma 2.10 *For $f \in H_k$ and $\Re(s) > 1$, let*

$$L(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{f\times f\times f}(n)}{n^s}. \tag{2.19}$$

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Then we have

$$L(s)=\zeta^5(s)L^9(s,\mathrm{sym}^2 f)L^5(s,\mathrm{sym}^4 f)L(s,\mathrm{sym}^6 f)U(s), \tag{2.20}$$

where the function $U(s)$ *is a Dirichlet series absolutely convergent in* $\Re(s)>1/2$ *and* $U(s)\neq 0$ *for* $\Re(s)=1$.

Proof See [17, Lemma 5]. ∎

Lemma 2.11 *Let* $f\in H_{k_1}$, $g\in H_{k_2}$ *be two different forms. For* $\Re(s)>1$, *let*

$$G(s)=\sum_{n=1}^{\infty}\frac{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)}{n^s}. \tag{2.21}$$

Then we have

(2.22)

$$G(s)=L(s,\mathrm{sym}^3 f\times\mathrm{sym}^3 g)L^2(s,f\times\mathrm{sym}^3 g)L^2(s,\mathrm{sym}^3 f\times g)L^4(s,f\times g)V(s),$$

where the function $V(s)$ *is a Dirichlet series absolutely convergent in* $\Re(s)>1/2$ *and* $V(s)\neq 0$ *for* $\Re(s)=1$.

Proof Noting that $\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)$ is multiplicative and satisfies the upper bound $O(n^{\varepsilon})$ due to (1.3), we obtain for $\Re(s)>1$,

$$G(s)=\prod_p\left(1+\frac{\lambda_{f\times f\times f}(p)\lambda_{g\times g\times g}(p)}{p^s}+\frac{\lambda_{f\times f\times f}(p^2)\lambda_{g\times g\times g}(p^2)}{p^{2s}}+\cdots\right). \tag{2.23}$$

From Lemma 2.9, we have

$$\lambda_{f\times f\times f}(p)=\lambda_{\mathrm{sym}^3 f}(p)+2\lambda_f(p). \tag{2.24}$$

Then,

$$\begin{aligned}\lambda_{f\times f\times f}(p)\lambda_{g\times g\times g}(p)&=\left(\lambda_{\mathrm{sym}^3 f}(p)+2\lambda_f(p)\right)\left(\lambda_{\mathrm{sym}^3 g}(p)+2\lambda_g(p)\right)\\&=\lambda_{\mathrm{sym}^3 f}(p)\lambda_{\mathrm{sym}^3 g}(p)+2\lambda_f(p)\lambda_{\mathrm{sym}^3 g}(p)+2\lambda_{\mathrm{sym}^3 f}(p)\lambda_g(p)+4\lambda_f(p)\lambda_g(p)\\&:=b(p).\end{aligned}$$

(2.25)

Define

(2.26)

$$G_1(s)=L(s,\mathrm{sym}^3 f\times\mathrm{sym}^3 g)L^2(s,f\times\mathrm{sym}^3 g)L^2(s,\mathrm{sym}^3 f\times g)L^4(s,f\times g).$$

Then it can be written as

$$G_1(s)=\prod_p\left(1+\sum_{k\geq 1}\frac{b(p^k)}{p^{ks}}\right). \tag{2.27}$$

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As a result,

$$G(s) = G_1(s) \times \prod_p \left(1 + \frac{\lambda_{f\times f\times f}(p^2)\lambda_{g\times g\times g}(p^2) - b(p^2)}{p^{2s}} + \cdots\right)$$
$$:= L(s, \mathrm{sym}^3 f \times \mathrm{sym}^3 g) L^2(s, f \times \mathrm{sym}^3 g) L^2(s, \mathrm{sym}^3 f \times g) L^4(s, f \times g) V(s), \tag{2.28}$$

where $V(s)$ converges absolutely and uniformly in the half-plane $\Re(s) > 1/2$. ■

Lemma 2.12 *Let $f \in H_{k_1}$, $g \in H_{k_2}$ be two different forms. For $\Re(s) > 1$, let*

$$H(s) = \sum_{n=1}^{\infty} \frac{\lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n)}{n^s}. \tag{2.29}$$

Then we have

$$H(s) = H_1(s) W(s), \tag{2.30}$$

where

$$\begin{aligned} H_1(s) =& \zeta^{25}(s) L^{45}(s, \mathrm{sym}^2 f) L^{45}(s, \mathrm{sym}^2 g) L^{25}(s, \mathrm{sym}^4 f) L^{25}(s, \mathrm{sym}^4 g) L^5(s, \mathrm{sym}^6 f) \\ & L^5(s, \mathrm{sym}^6 g) L(s, \mathrm{sym}^6 f \times \mathrm{sym}^6 g) L^5(s, \mathrm{sym}^6 f \times \mathrm{sym}^4 g) L^9(s, \mathrm{sym}^6 f \times \mathrm{sym}^2 g) \\ & L^5(s, \mathrm{sym}^4 f \times \mathrm{sym}^6 g) L^{25}(s, \mathrm{sym}^4 f \times \mathrm{sym}^4 g) L^{45}(s, \mathrm{sym}^4 f \times \mathrm{sym}^2 g) \\ & L^9(s, \mathrm{sym}^2 f \times \mathrm{sym}^6 g) L^{45}(s, \mathrm{sym}^2 f \times \mathrm{sym}^4 g) L^{81}(s, \mathrm{sym}^2 f \times \mathrm{sym}^2 g), \end{aligned}$$

and the function $W(s)$ is a Dirichlet series absolutely convergent in $\Re(s) > 1/2$ and $W(s) \neq 0$ for $\Re(s) = 1$.

Proof From (2.6) and Lemma 2.9, we have

$$\lambda_{f\times f\times f}(p) = \lambda_{\mathrm{sym}^3 f}(p) + 2\lambda_f(p) \quad \text{and} \quad \lambda_{\mathrm{sym}^j f}(p) = \lambda_f(p^j). \tag{2.31}$$

According to (1.2), we have

$$\begin{aligned} \lambda^2_{f\times f\times f}(p) &= \left(\lambda_f(p^3) + 2\lambda_f(p)\right)^2 = \lambda_f^2(p^3) + 4\lambda_f^2(p) + 4\lambda_f(p^3)\lambda_f(p) \\ &= \lambda_f(p^6) + 5\lambda_f(p^4) + 9\lambda_f(p^2) + 5 = \lambda_{\mathrm{sym}^6 f}(p) + 5\lambda_{\mathrm{sym}^4 f}(p) + 9\lambda_{\mathrm{sym}^2 f}(p) + 5. \end{aligned}$$

From (2.8),

$$\begin{aligned} &\lambda^2_{f\times f\times f}(p)\lambda^2_{g\times g\times g}(p) \\ &= \lambda_{\mathrm{sym}^6 f\times \mathrm{sym}^6 g}(p) + 5\lambda_{\mathrm{sym}^6 f\times \mathrm{sym}^4 g}(p) + 9\lambda_{\mathrm{sym}^6 f\times \mathrm{sym}^2 g}(p) + 5\lambda_{\mathrm{sym}^6 f}(p) \\ &+ 5\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^6 g}(p) + 25\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^4 g}(p) + 45\lambda_{\mathrm{sym}^4 f\times \mathrm{sym}^2 g}(p) + 25\lambda_{\mathrm{sym}^4 f}(p) \\ &+ 9\lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^6 g}(p) + 45\lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^4 g}(p) + 81\lambda_{\mathrm{sym}^2 f\times \mathrm{sym}^2 g}(p) + 45\lambda_{\mathrm{sym}^2 f}(p) \\ &+ 5\lambda_{\mathrm{sym}^6 g}(p) + 25\lambda_{\mathrm{sym}^4 g}(p) + 45\lambda_{\mathrm{sym}^2 g}(p) + 25. \end{aligned}$$

Now the lemma follows by standard argument like Lemma 2.11, so we omit it here. ■

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Lemma 2.13 *For $f \in H_k$ and any $\varepsilon > 0$, we have*

$$\sum_{n\le x} \lambda_{f\times f\times f}(n) \ll x^{7/10+\varepsilon}. \tag{2.32}$$

Proof See [18, Theorem 1.1]. ∎

3 The main proposition

In this section, we shall establish asymptotic formula of the sum

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n), \quad \sum_{n\le x} \lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n) \quad \text{and} \quad \sum_{n\le x} \lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n), \tag{3.1}$$

respectively (see Propositions 3.1, 3.2, 3.3). These asymptotic formulas are the key to prove Theorem 1.1 and Theorem 1.3.

3.1 Proof of Proposition 3.1

Proposition 3.1 *For $f \in H_k$ and any $\varepsilon > 0$, we have*

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n) = xP(\log x) + O_{f,\varepsilon}(x^{1-\frac{315}{40\sqrt{30}+8442}+\varepsilon}), \tag{3.2}$$

where $P(t)$ is a polynomial of degree 4.

Proof Recalling Lemma 2.10 and applying Perron's formula (see [9, Proposition 5.54]), we have

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} L(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.3}$$

where $b = 1 + \varepsilon$ and $3 \le T \le x$ is a parameter to be chosen later.

Then, we move the line of integration to the parallel segment with $\Re s = 23/25$. By Cauchy's residue theorem, we obtain

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n) = \operatorname*{Res}_{s=1}\left\{L(s)\frac{x^s}{s}\right\} + \frac{1}{2\pi i}\int_{\mathfrak{L}} L(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.4}$$

where $\mathfrak{L}$ is the contour joining $1+\varepsilon - iT$, $23/25 - iT$, $23/25 + iT$, $1+\varepsilon+iT$ with straight lines. The residue at $s = 1$ is equal to $xP(\log x)$, $P(t)$ is a polynomial of degree 4. We also have $U(s) \ll 1$ in that the absolutely convergence of $U(s)$ for $\Re(s) \ge 1/2 + \varepsilon$. Consequently, formula (3.4) can be written as

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n) = xP(\log x) + O\left(\mathcal{J}_1^h + \mathcal{J}_1^v + \frac{x^{1+\varepsilon}}{T}\right), \tag{3.5}$$

where

$$\mathcal{J}_1^h := \frac{1}{T}\int_{23/25}^{1+\varepsilon} |L(\sigma+iT)|x^\sigma d\sigma \ll \sup_{23/25\le\sigma\le 1+\varepsilon} x^\sigma T^{-1}|L(\sigma+iT)|, \tag{3.6}$$

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and

$$(3.7)\quad \mathcal{J}_1^v := x^{23/25} \int_1^T |L(23/25+it)| \frac{dt}{t} \ll x^{23/25+\varepsilon} \sup_{3 \le T_1 \le T} T_1^{-1} \int_{T_1}^{2T_1} |L(23/25+it)| dt.$$

By Lemma 2.5, $k = \frac{8}{63}\sqrt{15} = 0.4918\cdots$, for any $\varepsilon > 0$, we have

$$(3.8)\quad \zeta(23/25+it) \ll t^{\frac{\sqrt{2}k}{5} \times \frac{2}{25} + \varepsilon}.$$

Following from the well-known Phragmen-Lindelof principle, we obtain

$$(3.9)\quad \zeta(\sigma+it) \ll t^{\max\{\frac{\sqrt{2}k}{5}(1-\sigma), 0\}+\varepsilon},$$

uniformly for $23/25 \le \sigma \le 2$ and $|t| \ge 3$. According to Lemma 2.4, Lemma 2.6, and (3.9), we deduce that for $23/25 \le \sigma \le 1+\varepsilon$,

$$|L(\sigma+iT)| \ll T^{\{5\cdot\frac{\sqrt{2}k}{5}+9\cdot\frac{6}{5}+5\cdot\frac{5}{2}+\frac{7}{2}\}(1-\sigma)+\varepsilon} = T^{\{\sqrt{2}k+\frac{134}{5}\}(1-\sigma)+\varepsilon}.$$

Then it follows that

$$(3.10)\quad \mathcal{J}_1^h \ll T^{\sqrt{2}k+\frac{129}{5}+\varepsilon} \sup_{23/25 \le \sigma \le 1+\varepsilon} \left(\frac{x}{T^{\sqrt{2}k+\frac{134}{5}}} \right)^{\sigma} \ll x^{23/25+\varepsilon} T^{\frac{80\sqrt{30}+9009}{7875}} + \frac{x^{1+\varepsilon}}{T}.$$

For $\mathcal{J}_1^v$, we have

$$(3.11)\quad \mathcal{J}_1^v \ll x^{23/25+\varepsilon} \sup_{3 \le T_1 \le T} \frac{I_{1,1}(T_1)}{T_1} \int_{T_1}^{2T_1} \left| L(23/25+it, \mathrm{sym}^4 f) \right|^5 dt,$$

where

$$I_{1,1}(T_1) = \max_{T_1 \le t \le 2T_1} \zeta^5(23/25+it) L^9(23/25+it, \mathrm{sym}^2 f) L(23/25+it, \mathrm{sym}^6 f).$$

According to Lemma 2.4, Lemma 2.6, Lemma 2.8, and (3.8), we have

$$(3.12)\quad I_{1,1}(T_1) \ll T_1^{5\times\frac{8\sqrt{15}}{63}\times(\frac{2}{25})^{3/2}+9\times\frac{6}{5}\times\frac{2}{25}+\frac{7}{2}\times\frac{2}{25}+\varepsilon} = T_1^{\frac{80\sqrt{30}+9009}{7875}+\varepsilon},$$

and

$$(3.13)\quad \int_{T_1}^{2T_1} \left| L(23/25+it, \mathrm{sym}^4 f) \right|^5 dt \ll T_1^{1+\varepsilon}.$$

Consequently,

$$(3.14)\quad \mathcal{J}_1^v \ll x^{23/25+\varepsilon} T^{\frac{80\sqrt{30}+9009}{7875}}.$$

Inserting (3.10) and (3.14) into (3.5), and taking $T = x^{\frac{315}{40\sqrt{30}+8442}}$, we obtain

$$(3.15)\quad \sum_{n \le x} \lambda_{f\times f\times f}^2(n) = xP(\log x) + O(x^{1-\frac{315}{40\sqrt{30}+8442}+\varepsilon}). \qquad \blacksquare$$

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3.2 Proof of Proposition 3.2

Proposition 3.2 *Let $f \in H_{k_1}$, $g \in H_{k_2}$ be two different forms. For any $\varepsilon > 0$, we have*

$$\sum_{n\leq x} \lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n) \ll x^{31/32+\varepsilon}. \tag{3.16}$$

Proof According to Lemma 2.11 and applying Perron's formula (see [9, Proposition 5.54]), we have

$$\sum_{n\leq x} \lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} G(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.17}$$

where $b = 1+\varepsilon$ and $3 \leq T \leq x$ is a parameter to be chosen later.

Then, we move the line of integration to the parallel segment with $\Re s = 1/2$. By Cauchy's residue theorem, we deduce that

$$\sum_{n\leq x} \lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n) = \frac{1}{2\pi i}\int_{\mathfrak{L}} G(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.18}$$

where $\mathfrak{L}$ is the contour joining $1+\varepsilon-iT$, $1/2-iT$, $1/2+iT$, $1+\varepsilon+iT$ with straight lines. $G(s)$ has no poles in the half-plane $\Re(s) > 1/2$ by using the analytic properties of Rankin-Selberg L-functions. We also have $V(s) \ll 1$ in that the absolutely convergence of $V(s)$ for $\Re(s) \geq 1/2+\varepsilon$. Consequently, formula (3.18) can be written as

$$\sum_{n\leq x} \lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n) = O\left(\mathfrak{I}_1^h + \mathfrak{I}_1^v + \frac{x^{1+\varepsilon}}{T}\right), \tag{3.19}$$

where

$$\mathfrak{I}_1^h := \frac{1}{T}\int_{1/2}^{1+\varepsilon} |G(\sigma+iT)|x^\sigma d\sigma \ll \sup_{1/2\leq\sigma\leq 1+\varepsilon} x^\sigma T^{-1}|G(\sigma+iT)|, \tag{3.20}$$

and

$$\mathfrak{I}_1^v := x^{1/2}\int_1^T |G(1/2+it)|\frac{dt}{t} \ll x^{1/2+\varepsilon}\sup_{3\leq T_1\leq T} T_1^{-1}\int_{T_1}^{2T_1} |G(1/2+it)|dt. \tag{3.21}$$

According to Lemma 2.4, we obtain for $1/2 \leq \sigma \leq 1+\varepsilon$

$$|G(\sigma+iT)| \ll T^{\frac{16+16+16+16}{2}(1-\sigma)+\varepsilon} = T^{32(1-\sigma)+\varepsilon}. \tag{3.22}$$

Therefore,

$$\mathfrak{I}_1^h \ll T^{31+\varepsilon}\sup_{1/2\leq\sigma\leq 1+\varepsilon}\left(\frac{x}{T^{32}}\right)^\sigma \ll x^{1/2+\varepsilon}T^{15} + \frac{x^{1+\varepsilon}}{T}. \tag{3.23}$$

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For $\mathcal{J}_1^{\nu}$, we get

$$\mathcal{J}_1^{\nu} \ll x^{1/2+\varepsilon} \sup_{1\le T_1\le T} \frac{I_{2,1}(T_1)}{T_1} \int_{T_1}^{2T_1} |L(1/2+it, f\times \mathrm{sym}^3 g)|^2 dt, \tag{3.24}$$

where

$$I_{2,1}(T_1) = \max_{\substack{T_1\le t\le 2T_1,\\ s_0=1/2+it}} L\left(s_0, \mathrm{sym}^3 f\times \mathrm{sym}^3 g\right) L^2\left(s_0, \mathrm{sym}^3 f\times g\right) L^4\left(s_0, f\times g\right). \tag{3.25}$$

By Lemma 2.4, we have

$$I_{2,1}(T_1) \ll T_1^{12+\varepsilon} \quad \text{and} \quad \int_{T_1}^{2T_1} \left|L\left(1/2+it, f\times \mathrm{sym}^3 g\right)\right|^2 dt \ll T_1^{4+\varepsilon}. \tag{3.26}$$

As a result,

$$\mathcal{J}_1^{\nu} \ll x^{1/2+\varepsilon} T^{15+\varepsilon}. \tag{3.27}$$

Inserting (3.23) and (3.27) into (3.19), and taking $T = x^{1/32}$, we obtain

$$\sum_{n\le x} \lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n) \ll x^{31/32+\varepsilon}. \tag{3.28}$$

■

3.3 Proof of Proposition 3.3

Proposition 3.3 *Let $f\in H_{k_1}$, $g\in H_{k_2}$ be two different forms. For any $\varepsilon>0$, we have*

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n) = xQ(\log x) + O\left(x^{1-\frac{882}{400\sqrt{21}+1771497}+\varepsilon}\right), \tag{3.29}$$

where $Q(t)$ is a polynomial of degree 24.

Proof Recalling Lemma 2.12, we obtain

$$H_1(s) := \zeta^{25}(s)L^{45}(s,\mathrm{sym}^2 f)L^{45}(s,\mathrm{sym}^2 g)L^5(s,\mathrm{sym}^4 f\times \mathrm{sym}^6 g)H_2(s), \tag{3.30}$$

where

$$\begin{aligned} H_2(s) = {} & L^{25}(s,\mathrm{sym}^4 f)L^{25}(s,\mathrm{sym}^4 g)L^5(s,\mathrm{sym}^6 f)L^5(s,\mathrm{sym}^6 g)L(s,\mathrm{sym}^6 f\times \mathrm{sym}^6 g)\\ & L^5(s,\mathrm{sym}^6 f\times \mathrm{sym}^4 g)L^9(s,\mathrm{sym}^6 f\times \mathrm{sym}^2 g)L^{25}(s,\mathrm{sym}^4 f\times \mathrm{sym}^4 g)\\ & L^{45}(s,\mathrm{sym}^4 f\times \mathrm{sym}^2 g)L^9(s,\mathrm{sym}^2 f\times \mathrm{sym}^6 g)L^{45}(s,\mathrm{sym}^2 f\times \mathrm{sym}^4 g)\\ & L^{81}(s,\mathrm{sym}^2 f\times \mathrm{sym}^2 g) \end{aligned}$$

is a general L-function of degree 3626.

Applying Perron's formula (see [9, Proposition 5.54]), we have

$$\sum_{n\le x} \lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} H(s)\frac{x^s}{s} ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.31}$$

where $b = 1+\varepsilon$ and $3\le T\le x$ is a parameter to be chosen later.

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Then, we move the line of integration to the parallel segment with $\Re s = 34/35$. By Cauchy's residue theorem, we obtain

$$\sum_{n\le x}\lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n) = \operatorname*{Res}_{s=1}\left\{H(s)\frac{x^s}{s}\right\} + \frac{1}{2\pi i}\int_{\mathfrak{L}} H(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right), \tag{3.32}$$

where $\mathfrak{L}$ is the contour joining $1+\varepsilon-iT$, $34/35-iT$, $34/35+iT$, $1+\varepsilon+iT$ with straight lines. The residue at $s=1$ is equal to $xQ(\log x)$, $Q(t)$ is a polynomial of degree 24. We also have $W(s)\ll 1$ in that the absolutely convergence of $W(s)$ for $\Re(s)\ge 1/2+\varepsilon$. Consequently, formula (3.32) can be written as

$$\sum_{n\le x}\lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n) = xQ(\log x) + O\left(\mathcal{J}_3^h + \mathcal{J}_3^v + \frac{x^{1+\varepsilon}}{T}\right), \tag{3.33}$$

where

$$\mathcal{J}_3^h := \frac{1}{T}\int_{34/35}^{1+\varepsilon}|H(\sigma+iT)|x^\sigma d\sigma \ll \sup_{34/35\le\sigma\le 1+\varepsilon} x^\sigma T^{-1}|H(\sigma+iT)|, \tag{3.34}$$

and

(3.35)

$$\mathcal{J}_3^v := x^{34/35}\int_1^T |H(34/35+it)|\frac{dt}{t} \ll x^{34/35+\varepsilon}\sup_{3\le T_1\le T} T_1^{-1}\int_{T_1}^{2T_1}|H(34/35+it)|dt.$$

By Lemma 2.5, $k=\frac{8}{63}\sqrt{15}=0.4918\cdots$, for any $\varepsilon>0$, we have

$$\zeta(34/35+it)\ll t^{\frac{k}{\sqrt{35}}\times\frac{1}{35}+\varepsilon}. \tag{3.36}$$

Following from the well-known Phragmen-Lindelof principle, we obtain

$$\zeta(\sigma+it)\ll t^{\max\{\frac{k}{\sqrt{35}}(1-\sigma),0\}+\varepsilon}, \tag{3.37}$$

uniformly for $34/35\le\sigma\le 2$ and $|t|\ge 3$. According to Lemma 2.4, Lemma 2.6, and (3.37), we obtain for $34/35\le\sigma\le 1+\varepsilon$

$$|H(\sigma+iT)|\ll T^{\{25\cdot\frac{k}{\sqrt{35}}+45\cdot\frac{6}{5}+45\cdot\frac{6}{5}+5\cdot\frac{35}{2}+\frac{3626}{2}\}(1-\sigma)+\varepsilon} = T^{\{\frac{25k}{\sqrt{35}}+\frac{4017}{2}\}(1-\sigma)+\varepsilon}.$$

Therefore,

$$\begin{aligned}\mathcal{J}_3^h &\ll T^{\frac{25k}{\sqrt{35}}+\frac{4015}{2}+\varepsilon}\sup_{34/35\le\sigma\le 1+\varepsilon}\left(\frac{x}{T^{\frac{25k}{\sqrt{35}}+\frac{4017}{2}}}\right)^\sigma \\ &\ll x^{34/35+\varepsilon}T^{\{\frac{25k}{\sqrt{35}}+\frac{3947}{2}\}\cdot\frac{1}{35}} + \frac{x^{1+\varepsilon}}{T}.\end{aligned} \tag{3.38}$$

According to (3.30), we deduce that

$$\mathcal{J}_3^v \ll x^{34/35+\varepsilon}\sup_{3\le T_1\le T}\frac{I_{3,1}(T_1)}{T_1}\int_{T_1}^{2T_1}|L(34/35+it,\mathrm{sym}^4 f\times\mathrm{sym}^6 g)|^2 dt, \tag{3.39}$$

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where

$$I_{3,1}(T_1) = \max_{\substack{T_1 \le t \le 2T_1, \\ s_0 = 34/35 + iT_1}} \zeta^{25}(s_0) L^{45}(s_0, \mathrm{sym}^2 f) L^{45}(s_0, \mathrm{sym}^2 g) L^3(s_0, \mathrm{sym}^4 f \times \mathrm{sym}^6 g) H_2(s_0).$$

From Lemma 2.4, Lemma 2.6, and (3.36), we have

$$I_{3,1}(T_1) \ll T_1^{\{25\cdot\frac{k}{\sqrt{35}} + 45\cdot\frac{6}{5} + 45\cdot\frac{6}{5} + 3\cdot\frac{35}{2} + \frac{3626}{2}\}/35+\varepsilon} = T^{\{\frac{25k}{\sqrt{35}} + \frac{3947}{2}\}\cdot\frac{1}{35}+\varepsilon}, \tag{3.40}$$

and

$$\int_{T_1}^{2T_1} |L(34/35 + it, \mathrm{sym}^4 f \times \mathrm{sym}^6 g)|^2 dt \ll T_1^{1+\varepsilon}. \tag{3.41}$$

As a result,

$$\mathcal{J}_3^{\nu} \ll x^{34/35+\varepsilon} T^{\{\frac{25k}{\sqrt{35}} + \frac{3947}{2}\}\cdot\frac{1}{35}}. \tag{3.42}$$

Inserting (3.38) and (3.42) into (3.33), and taking $T = x^{\frac{882}{400\sqrt{21}+1771497}}$, we obtain

$$\sum_{n \le x} \lambda^2_{f\times f\times f}(n) \lambda^2_{g\times g\times g}(n) = xQ(\log x) + O\left(x^{1-\frac{882}{400\sqrt{21}+1771497}+\varepsilon}\right). \tag{3.43}$$

■

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by the argument of contradiction. Let

$$1 - \frac{315}{40\sqrt{30} + 8442} = 0.963\cdots < \delta < 1. \tag{4.1}$$

Suppose that the sequence $\{\lambda_{f\times f\times f}(n)\}_{n\ge 1}$ has the same sign in the interval$(x, x + x^{\delta}]$. Without loss of generality, let the sign is positive. Then by Lemma 2.13 and Deligne's bound (1.3), we obtain

$$\sum_{x \le n \le x + x^{\delta}} \lambda^2_{f\times f\times f}(n) \ll x^{\varepsilon} \sum_{x \le n \le x + x^{\delta}} \lambda_{f\times f\times f}(n) \ll x^{7/10+\varepsilon}. \tag{4.2}$$

According to Proposition 3.1, we deduce that

$$\begin{aligned} \sum_{x \le n \le x + x^{\delta}} \lambda^2_{f\times f\times f}(n) &= (x + x^{\delta}) P(\log(x + x^{\delta})) - xP(\log x) + O_{f,\varepsilon}(x^{1-\frac{315}{40\sqrt{30}+8442}+\varepsilon}) \\ &\ge (x + x^{\delta}) P(\log x) - xP(\log x) + O_{f,\varepsilon}(x^{1-\frac{315}{40\sqrt{30}+8442}+\varepsilon}) \\ &= x^{\delta} P(\log x) + O_{f,\varepsilon}(x^{1-\frac{315}{40\sqrt{30}+8442}+\varepsilon}) \gg x^{\delta}. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3), we get the contradiction. As a result, the sequence $\{\lambda_{f\times f\times f}(n)\}_{n\ge 1}$ has at least one sign change in the interval$(x, x + x^{\delta}]$ with $0.963\cdots < \delta < 1$. Therefore, the sequence $\{\lambda_{f\times f\times f}(n)\}_{n\ge 1}$ has at least $\gg x^{1-\delta}$ sign change in the interval$(x, x + x^{\delta}]$ for sufficiently large x.

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5 Proof of Theorem 1.2

In this section, we prove Theorem 1.3 by the argument of contradiction. Let

$$1-\frac{882}{400\sqrt{21}+1771497}=0.99950\cdots<\delta<1. \tag{5.1}$$

Suppose that the sequence $\{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\}_{n\geq 1}$ has the same sign in the interval$(x, x+x^{\delta}]$. Without loss of generality, let the sign is positive. Then by Proposition 3.2 and Deligne's bound (1.3), we obtain

$$\sum_{x\leq n\leq x+x^{\delta}}\lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n)\ll x^{\varepsilon}\sum_{x\leq n\leq x+x^{\delta}}\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\ll x^{31/32+\varepsilon}. \tag{5.2}$$

According to Proposition 3.3, we have

$$\begin{aligned}
&\sum_{x\leq n\leq x+x^{\delta}}\lambda^2_{f\times f\times f}(n)\lambda^2_{g\times g\times g}(n)\\
&=(x+x^{\delta})P(\log(x+x^{\delta}))-xP(\log x)+O_{f,\varepsilon}(x^{1-\frac{882}{400\sqrt{21}+1771497}+\varepsilon})\\
&\geq(x+x^{\delta})P(\log x)-xP(\log x)+O_{f,\varepsilon}(x^{1-\frac{882}{400\sqrt{21}+1771497}+\varepsilon})\\
&=x^{\delta}P(\log x)+O_{f,\varepsilon}\left(x^{1-\frac{882}{400\sqrt{21}+1771497}+\varepsilon}\right)\gg x^{\delta}.
\end{aligned} \tag{5.3}$$

From (5.2), (5.3), and $\frac{31}{32}=0.96\cdots$, we get the contradiction. As a result, the sequence $\{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\}_{n\geq 1}$ has at least one sign change in the interval $(x, x+x^{\delta}]$ with $0.99950\cdots<\delta<1$. Therefore, the sequence $\{\lambda_{f\times f\times f}(n)\lambda_{g\times g\times g}(n)\}_{n\geq 1}$ has at least $\gg x^{1-\delta}$ sign change in the interval$(x, x+x^{\delta}]$ for suffficiently large x.

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