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EXTREME POINTS IN SPACES BETWEEN DIRICHLET AND VANISHING MEAN OSCILLATION

K.J. WIRTHS AND J. XIAO

For $p \in (0,\infty)$ define $Q_{p,0}(\partial\Delta)$ as the space of all Lebesgue measurable complex-valued functions f on the unit circle $\partial\Delta$ for which $\int_{\partial\Delta} f(z)|dz|/(2\pi) = 0$ and

$$\frac{1}{(2\pi)^2}\int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^{2-p}}|dz|\,|dw| = o(|I|^p)$$

as the open subarc I of $\partial\Delta$ varies. Note that each $Q_{p,0}(\partial\Delta)$ lies between the Dirichlet space and Sarason's vanishing mean oscillation space. This paper determines the extreme points of the closed unit ball of $Q_{p,0}(\partial\Delta)$ equipped with an appropriate norm.

1. INTRODUCTION

Denote by Δ and $\partial\Delta$ the open unit disk and the unit circle in the finite complex plane $\mathbb{C}$, respectively. For $p \in (0,\infty)$, let $Q_p(\partial\Delta)$ (respectively $Q_{p,0}(\partial\Delta)$) be the class of all Lebesgue measurable functions $f : \partial\Delta \to \mathbb{C}$ for which $\int_{\partial\Delta} f(z)|dz|/(2\pi) = 0$ and

$$S_p(f,I) := \frac{1}{(2\pi)^2}\int_I\int_I \frac{|f(z)-f(w)|^2}{|z-w|^{2-p}}|dz||dw| = O\big(|I|^p\big) \quad \Big(\text{respectively} \quad o\big(|I|^p\big)\Big)$$

as $I \subseteq \partial\Delta$ varies. Here and throughout this paper, I means an open subarc of $\partial\Delta$ and $|I|$ stands for the normalised arclength of $I \subseteq \partial\Delta$, that is, $|I| = \int_I |dz|/(2\pi)$. It is clear that $Q_{p,0}(\partial\Delta) \subseteq Q_p(\partial\Delta)$. For convenience, equip $f \in Q_p(\partial\Delta)$ with the following norm

$$\|f\|_{Q_p(\partial\Delta)} := \sup_{I\subseteq\partial\Delta}\left[\frac{S_p(f,I)}{|I|^p}\right]^{1/2}.$$

So, $Q_p(\partial\Delta)$ is a Banach space and $Q_{p,0}(\partial\Delta)$ is its closed subspace.

In the case $p \in (0,1)$ both classes were introduced in [**3**] and [**6**] when Essén, Nicolau, and Xiao studied the boundary behaviour of the holomorphic Q_p-spaces (see

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[1]). More important is that the article [9] (see [4] for another proof) proved that $Q_p(\partial\Delta) \subseteq BMO(\partial\Delta)$ and equality occurs as $p > 1$. In Section 4 of our current paper, we shall demonstrate that $Q_{p,0}(\partial\Delta) \subseteq VMO(\partial\Delta)$ and equality happens again as $p > 1$. Here $BMO(\partial\Delta)$ (respectively $VMO(\partial\Delta)$) is John-Nirenberg's [5] (respectively Sarason's [8]) space of functions with bounded respectively vanishing mean oscillation on $\partial\Delta$. More precisely, for $q \geqslant 1$ and a Lebesgue measurable function $f : \partial\Delta \to \mathbb{C}$ we say $f \in BMO(\partial\Delta)$ provided $\int_{\partial\Delta} f(z)|dz|/(2\pi) = 0$ and $\|f\|_{q-MO} = \sup_{z\in\partial\Delta} f_q^{\#}(z) < \infty$, where

$$f_q^{\#}(z) = \sup_{z\in I}\left[\frac{1}{2\pi|I|}\int_I\left|f(z) - \frac{1}{2\pi|I|}\int_I f(w)|dw|\right|^q|dz|\right]^{1/q},$$

and here, the supremum is taken over all open subarcs $I \subseteq \partial\Delta$ such that $z \in I$. Moreover we call $f \in VMO(\partial\Delta)$ if $f \in BMO(\partial\Delta)$ and

$$\lim_{\delta\searrow 0}\sup_{|I|<\delta}\frac{1}{2\pi|I|}\int_I\left|f(z) - \frac{1}{2\pi|I|}\int_I f(w)|dw|\right|^q|dz| = 0,$$

where the supremum ranges through all open subarcs $I \subseteq \partial\Delta$ with $|I| < \delta$.

Motivated by Axler-Shields' work [2], this paper is devoted to an investigation of the extreme points of the closed ball of $Q_{p,0}(\partial\Delta)$ (as well as $Q_p(\partial\Delta)$), but also extends those corresponding results on $VMO(\partial\Delta)$ (as well as $BMO(\partial\Delta)$). The main results of this note are presented in Section 2. Of particular interest are some examples of the extreme/nonextreme points provided in Section 3. In the meantime, it is worth mentioning that our functions g_n constructed as the extreme points of the closed unit ball of $(Q_{p,0}(\partial\Delta), \|\cdot\|_{Q_p(\partial\Delta)})$ are still extreme points of the closed unit ball of $(VMO(\partial\Delta), \|\cdot\|_{2-MO})$. Besides this, the method of constructing some nonextreme points of the closed unit ball of $(Q_{p,0}(\partial\Delta), \|\cdot\|_{Q_p(\partial\Delta)})$ is valid for the space $(VMO(\partial\Delta), \|\cdot\|_{q-MO})$. In other words, there are some nonextreme points in the closed unit ball of $(VMO(\partial\Delta), \|\cdot\|_{q-MO})$.

2. Results

First of all, let us determine the extreme points of the closed unit ball of $Q_{p,0}(\partial\Delta)$. For $f \in Q_p(\partial\Delta)$, $p \in (0,\infty)$, define the function $E_p(f,\cdot)$ on $\partial\Delta$ by

$$E_p(f,z) := \sup_{z\in I}\left[\frac{S_p(f,I)}{|I|^p}\right]^{1/2},$$

where the supremum ranges over all open subarcs $I \subseteq \partial\Delta$ such that $z \in I$. It is easy to establish the formula:

$$\|f\|_{Q_p(\partial\Delta)} = \sup_{z\in\partial\Delta} E_p(f,z).$$

This is due to an obvious fact that $E_p(f,\cdot)$ is lower semi-continuous, that is, $\{z \in \partial\Delta : E_p(f,z) > t\}$ is an open set for every $t > 0$. Letting $C(\partial\Delta)$ be the class of all continuous functions $f : \partial\Delta \to \mathbb{C}$, we can get further information on $E_p(f,\cdot)$. More precisely,

Lemma 2.1. *Let $p \in (0,\infty)$ and let $f \in Q_{p,0}(\partial\Delta)$. Then $E_p(f,\cdot) \in C(\partial\Delta)$.*

Proof: It suffices to prove that $E_p(f,\cdot)$ is upper semi-continuous too, namely that $\{z \in \partial\Delta : E_p(f,z) < t\}$ is an open set for every $t > 0$. To do this, fix $t > 0$ and let $z \in \partial\Delta$ obey $E_p(f,z) < t$. If $E_p(f,z) \neq 0$, then by $f \in Q_{p,0}(\partial\Delta)$, there exists a $\delta > 0$ such that

$$\sup_{I\subseteq\partial\Delta;|I|<\delta} \frac{S_p(f,I)}{|I|^p} \leqslant \left[E_p(f,z)\right]^2.$$

Now let $J \subseteq \partial\Delta$ be an open subarc centred at z whose arclength $|J| = \varepsilon$ is very small compared to δ. And, for $w \in J$ suppose $I \subseteq \partial\Delta$ contains w. When $|I| < \delta$, one has $S_p(f,I)/|I|^p < t^2$, thanks to the previous estimation. In the case $|I| \geqslant \delta$ take $K = J \cup I$. Then K is an open subarc containing z and hence

$$\frac{S_p(f,I)}{|I|^p} \leqslant \left(\frac{|K|}{|I|}\right)^p \frac{S_p(f,K)}{|K|^p} \leqslant \left(\frac{|K|}{|I|}\right)^p \left[E_p(f,z)\right]^2.$$

Further, putting $E_p(f,z) = t - \tau$, $\tau \in (0,t)$, we get that $S_p(f,I)/|I|^p < t^2$ and thus $E_p(f,w) < t$ when $w \in J$ and $|J| = \varepsilon < \delta\Big[\big(t/(t-\tau)\big)^{2/p} - 1\Big]$. On the other hand, if $E_p(f,z) = 0$ then via the analysis on the open subarc K, we can select an open subarc J centred at z such that $E_p(f,w) = 0 < t$ as $w \in J$. This completes the proof.

Theorem 2.2. *Let $p \in (0,\infty)$ and $f \in Q_{p,0}(\partial\Delta)$. Then f is an extreme point of the closed unit ball of $Q_{p,0}(\partial\Delta)$ if and only if $E_p(f,\cdot)$ is identical with 1.*

Proof: Assume that f is an extreme point of the closed unit ball of $Q_{p,0}(\partial\Delta)$. Since $\|f\|_{Q_p(\partial\Delta)} \leqslant 1$, one has that $E_p(f,z) \leqslant 1$ for all $z \in \partial\Delta$. If $E_p(f,\cdot)$ is not identical with 1, then by Lemma 2.1, there is an open subarc $J \subseteq \partial\Delta$ such that $\sup_{z\in J} E_p(f,z) < 1$. Choose a function $g \in Q_{p,0}(\partial\Delta)$ such that $\|g\|_{Q_p(\partial\Delta)} \leqslant 1 - \sup_{z\in J} E_p(f,z)$, $g = 0$ outside J, and $g \not\equiv 0$. When $I \subseteq \partial\Delta$ is such that $I \cap J = \emptyset$, we obviously obtain

$$\frac{S_p(f+g,I)}{|I|^p} = \frac{S_p(f,I)}{|I|^p} \leqslant 1.$$

If $I \subseteq \partial\Delta$ ensures $I \cap J \neq \emptyset$, then

$$\left[\frac{S_p(f+g,I)}{|I|^p}\right]^{1/2} \leqslant \left[\frac{S_p(f,I)}{|I|^p}\right]^{1/2} + \left[\frac{S_p(g,I)}{|I|^p}\right]^{1/2} \leqslant \sup_{z\in J} E_p(f,z) + \|g\|_{Q_p(\partial\Delta)} \leqslant 1.$$

Thus $\|f+g\|_{Q_p(\partial\Delta)} \leqslant 1$. Similarly, $\|f-g\|_{Q_p(\partial\Delta)} \leqslant 1$. As $g \not\equiv 0$, those inequalities show that f is not an extreme point of the closed unit ball of $Q_{p,0}(\partial\Delta)$, contradicting the assumption.

Conversely, let $E_p(f,\cdot) \equiv 1$. If $z_0 \in \partial\Delta$, then there exists a sequence of open subarcs I_n containing z_0 such that $S_p(f,I_n)/|I_n|^p$ is convergent to 1. Without loss of generality, let $I_n := (a_n, b_n)$, intervals moving counterclockwise. Then, by passing to a subsequence, we may assume that $a_n \to a$ and $b_n \to b$ in the usual sense. After that, let $I_{z_0} \subseteq \partial\Delta$ be the open subarc determined by the interval (a,b), that is, $I_{z_0} := (a,b)$. In this sense, I_{z_0} is viewed as the limiting open subarc of I_n. Accordingly, $S_p(f,I_{z_0})/|I_{z_0}|^p = 1$. Although I_{z_0} does not necessarily contain z_0, it is easy to see that z_0 belongs to $\overline{I}_{z_0}$ — the closure of I_{z_0} — a closed subarc $[a,b]$ of $\partial\Delta$. Observe that since $f \in Q_{p,0}(\partial\Delta)$, I_n cannot get small and hence I_{z_0} is not empty.

Now suppose $g \in Q_{p,0}(\partial\Delta)$ and $\|f+g\|_{Q_p(\partial\Delta)} \leqslant 1$ and $\|f-g\|_{Q_p(\partial\Delta)} \leqslant 1$. In order to prove that f is an extreme point, we must show that $g \equiv 0$. Fix $z_0 \in \partial\Delta$ with the open subarc I_{z_0} constructed above. Thus,

$$\frac{f(z)-f(w)}{(|z-w|/|I_{z_0}|)^{(2-p)/2}}\Big|_{I_{z_0}\times I_{z_0}}$$

is an element of the unit sphere of $L^2\Big(I_{z_0}\times I_{z_0}, (2\pi|I_{z_0}|)^{-2}|dz||dw|\Big)$. From $\|f+g\|_{Q_p(\partial\Delta)} \leqslant 1$, we also know that

$$\frac{f(z)-f(w)+g(z)-g(w)}{(|z-w|/|I_{z_0}|)^{(2-p)/2}}\Big|_{I_{z_0}\times I_{z_0}}$$

is a member of the closed unit ball of $L^2\Big(I_{z_0}\times I_{z_0}, (2\pi|I_{z_0}|)^{-2}|dz||dw|\Big)$, and similarly when g is replaced by $-g$. Notice that

$$\frac{f(z)-f(w)}{(|z-w|/|I_{z_0}|)^{(2-p)/2}}\Big|_{I_{z_0}\times I_{z_0}} = \frac{[f(z)-f(w)]+[g(z)-g(w)]}{2(|z-w|/|I_{z_0}|)^{(2-p)/2}}\Big|_{I_{z_0}\times I_{z_0}} + \frac{[f(z)-f(w)]-[g(z)-g(w)]}{2(|z-w|/|I_{z_0}|)^{(2-p)/2}}\Big|_{I_{z_0}\times I_{z_0}}$$

and more importantly, that $L^2\Big(I_{z_0}\times I_{z_0}, (2\pi|I_{z_0}|)^{-2}|dz||dw|\Big)$ enjoys the property that every point of the unit sphere is an extreme point of the closed unit ball ([**7**, p. 84, problem 16]). So, the last equation implies that $g(z)-g(w)=0$ on $I_{z_0}\times I_{z_0}$, that is, g is a constant on I_{z_0}.

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For each $z_0 \in \partial\Delta$ let J_{z_0} denote the largest open subarc covering I_{z_0} such that $g|J_{z_0}$ is constant. Obviously, as $z_0, w_0 \in \partial\Delta$, $J_{z_0} \cap J_{w_0} \neq \emptyset$ induces $J_{z_0} = J_{w_0}$, and so the collection $\{J_{z_0}\}$ can contain at most countably many different open subarcs, because $|\partial\Delta| = 1$. Relabel these disjoint open subarcs $\{K_n\}$. The condition $z_0 \in \overline{I}_{z_0}$ implies $\partial\Delta = \cup\overline{K}_n$ (where $\overline{K}_n$ stands for the closure of K_n). Thus $A := \partial\Delta \setminus \cup K_n$ consists of all endpoints of all the subarcs K_n. In particular, A is a closed, countable set. If $A = \emptyset$, then $\{K_n\}$ contains only one element, namely $\partial\Delta$, and thus g is a constant function on $\partial\Delta$. Note that $\int_{\partial\Delta} g(z)|dz|/(2\pi) = 0$. Accordingly, $g \equiv 0$.

It remains to consider the case: $A \neq \emptyset$. Using the Baire Category Theorem, we see that the non-empty, countable, closed set A must have an isolated point $c \in \partial\Delta$. Since $\partial\Delta = \cup\overline{K}_n$, we can conclude that there must be two disjoint open subarcs K_n and K_m such that c is an endpoint for them. Recall that $g|K_n$ and $g|K_m$ are constant. If the two constants do not coincide, then g has a jump discontinuity at c, which certainly contradicts the condition $g \in Q_{p,0}(\partial\Delta)$. If the two constants are the same, then g is constant on the open subarc $K_n \cup \{c\} \cup K_m$, which violates the maximality of the open subarcs $\{J_{z_0}\}_{z_0 \in \partial\Delta}$. Thus the case $A \neq \emptyset$ cannot occur, and therefore the proof of the theorem is finished. ∎

Next, we shall deal with the extreme points of the closed unit ball of $Q_p(\partial\Delta)$. This part may be considered as a consequence of the proof of the above theorem, although the forthcoming result looks quite different from that of the extreme points of the closed unit ball of $Q_{p,0}(\partial\Delta)$.

For $f \in Q_p(\partial\Delta)$ and $z \in \partial\Delta$, we shall use $E_p(f,z) = 1^+$ to denote that $E_p(f,z) = 1$ and there exists some open subarc I_z containing z that gives $S_p(f, I_z)/|I_z|^p = 1$. This means that the supremum defining $E_p(f,z)$ is attained.

Corollary 2.3. *Let $p \in (0,\infty)$ and let $f \in Q_p(\partial\Delta)$. If f is an extreme point of the closed unit ball of $Q_p(\partial\Delta)$ then there exist no two distinct points z_1 and z_2 in $\partial\Delta$ such that $E_p(f,z_1) < 1$ and $E_p(f,z_2) < 1$. Conversely, if $E_p(f,z) = 1^+$ for all $z \in \partial\Delta$ with one possible exception, then f is an extreme point of the closed unit ball of $Q_p(\partial\Delta)$.*

Proof: Let f be an extreme point of the closed unit ball of $Q_p(\partial\Delta)$. Without loss of generality, we may assume that $\|f\|_{Q_p(\partial\Delta)} = 1$ in that if $\|f\|_{Q_p(\partial\Delta)} < 1$ then we may select $g = (1-\varepsilon)f$ and $h = (1+\varepsilon)f$, where $0 < \varepsilon < \min\{1, \|f\|^{-1}_{Q_p(\partial\Delta)} - 1\}$, and hence it turns out from the equation $f = (g+h)/2$ that f cannot be an extreme point of the closed unit ball of $Q_p(\partial\Delta)$. In order to reach our goal, suppose otherwise that there are two distinct points $z_k \in \partial\Delta$, $k = 1,2$ such that $E_p(f,z_k) < 1$, $k = 1,2$; and use I_k, $k = 1,2$ to denote the two open subarcs of $\partial\Delta$ which have $\{z_1, z_2\}$ as endpoints. Define a function $g \in Q_p(\partial\Delta)$ by $g|_{I_1} = \varepsilon_1$, $g|_{I_2} = -\varepsilon_2$, where $\varepsilon_k > 0$,

$k = 1, 2$ are chosen so that $\int_{\partial\Delta} g(z)|dz| = 0$ and $\|g\|_{Q_p(\partial\Delta)} \leqslant 1 - \max_{k=1,2} E_p(f, z_k)$. Now, let I be an open subarc of $\partial\Delta$. If both z_1 and z_2 are not in I, then g is constant on I and thus $S_p(f+g, I)/|I|^p = S_p(f, I)/|I|^p \leqslant 1$. If one of z_k, say, z_1 lies in I, then

$$\left[\frac{S_p(f+g,I)}{|I|^p}\right]^{1/2} \leqslant \left[\frac{S_p(f,I)}{|I|^p}\right]^{1/2} + \left[\frac{S_p(g,I)}{|I|^p}\right]^{1/2} \leqslant E_p(f, z_1) + \|g\|_{Q_p(\partial\Delta)} \leqslant 1.$$

Accordingly, it follows that $\|f+g\|_{Q_p(\partial\Delta)} \leqslant 1$. Similarly, $\|f-g\|_{Q_p(\partial\Delta)} \leqslant 1$. Because f may be written as the sum of $(f+g)/2$ and $(f-g)/2$, the function f is not an extreme point of the closed unit ball of $Q_p(\partial\Delta)$, violating the given condition.

On the other hand, if $w \in \partial\Delta$ is such that $E_p(f, z) = 1^+$ for all $z \in \partial\Delta \setminus \{w\}$, then to each $z \in \partial\Delta \setminus \{w\}$, there corresponds an open subarc I_z such that $z \in I_z$ and $S_p(f, I_z)/|I_z|^p = 1$. Now let $g \in Q_p(\partial\Delta)$ satisfy $\|f+g\|_{Q_p(\partial\Delta)} \leqslant 1$ and $\|f-g\|_{Q_p(\partial\Delta)} \leqslant 1$. To complete the proof, we must show $g \equiv 0$. Applying the same reasoning as in the argument for sufficiency of Theorem 2.2, we can prove that $g|I_z$ is a constant. Since g is locally constant on the connected set $\partial\Delta \setminus \{w\}$, $g \equiv 0$ as $\int_{\partial\Delta} g(z)|dz|/(2\pi) = 0$. ∎

3. Examples

In this section we present some examples of either extreme points or nonextreme points of the closed unit ball of $Q_{p,0}(\partial\Delta)$.

Example 3.1. (Extreme points.) Let $p \in (0, \infty)$, and for integers $n = \pm1, \pm2, \ldots$ let $f_n(z) = \lambda z^n$ where $z \in \partial\Delta$ and $|\lambda| \equiv 1$. Then $g_n = f_n/\|f_n\|_{Q_p(\partial\Delta)}$ are extreme points of the closed unit ball of $Q_{p,0}(\partial\Delta)$.

Proof: To make these examples more precise, by Theorem 2.2

$$E_p(g_n, z) = 1, \quad \forall z \in \partial\Delta.$$

In fact, some elementary calculations tell us that

$$S_p(f_n, I) = 2^{p+1} \int_0^{|I|} \big(|I| - t\big) \sin^{p-2}(\pi t) \sin^2(n\pi t)\, dt$$

and so that

$$E_p^2(f_n, z) = \sup_{|I| \in (0,1]} \frac{2^{p+1}}{|I|^p} \int_0^{|I|} \big(|I| - t\big) \sin^{p-2}(\pi t) \sin^2(n\pi t)\, dt.$$

Thus, $\|f_n\|_{Q_p(\partial\Delta)} = E_p(f_n, z)$ for each $z \in \partial\Delta$, and then $E_p(g_n, \cdot) \equiv 1$ follows. ∎

In the sequel, we point out that not all points on the closed unit ball of $Q_{p,0}(\partial\Delta)$ are extreme points. In fact, we consider a function first defined on $[0, 2\pi)$ with mean value zero and then extended periodically.

EXAMPLE 3.2. (Nonextreme points) Let $p \in (0,\infty)$, and for $\delta \in (0,1)$ let

$$f_\delta(e^{i\theta}) = \begin{cases} \dfrac{\theta}{\delta}, & \theta \in [0,\delta]; \\ 1, & \theta \in [\delta, 2\pi - \delta]; \\ \dfrac{2\pi - \theta}{\delta}, & \theta \in [2\pi - \delta, 2\pi). \end{cases}$$

Also put $g_\delta = f_\delta - 1 + \delta/(2\pi)$ on $[0,2\pi)$ and extend it 2π-periodically. Then there exists a $\delta > 0$ such that $g_\delta/\|g_\delta\|_{Q_p(\partial\Delta)}$ is a nonextreme point of the closed unit ball of $Q_{p,0}(\partial\Delta)$.

PROOF: A key observation is that g_δ is convergent to the zero-function as $\delta \to 0$. Because f_δ is a Lip1-function, $g_\delta/\|g_\delta\|_{Q_{p,0}(\partial\Delta)}$ is in $Q_{p,0}(\partial\Delta)$. However, we show that it is not an extreme point of the closed unit ball of $Q_{p,0}(\partial\Delta)$. By Theorem 2.2 we know this will be done if one can prove that $E_p(f_\delta,\cdot)$ is not a constant function for some δ. For this it suffices to verify $E_p(f_\delta,0) \neq E_p(f_\delta,\pi)$ for some δ. First, for any open subarc (or subinterval) $I = (a,b) \subseteq (0,\delta)$ we have

$$\frac{S_p(f_\delta,I)}{|I|^p} = \frac{2^p(2\pi)^{p-2}}{(b-a)^p\delta^2}\int_0^{b-a}\frac{(b-a-t)t^2}{\sin^{2-p}t/2}\,dt.$$

Thus

$$\sup_{I\subseteq(0,\delta)}\frac{S_p(f_\delta,I)}{|I|^p} \geqslant \frac{2^p(2\pi)^{p-2}}{\delta^{2+p}}\int_0^{\delta}\frac{(\delta-t)t^2}{\sin^{2-p}t/2}\,dt.$$

Furthermore, by Lemma 2.1 and the limit

$$\lim_{\delta\to0}\frac{1}{\delta^{p+2}}\int_0^{\delta}\frac{(\delta-t)t^2}{\sin^{2-p}t/2}\,dt = \frac{2^{2-p}}{(p+1)(p+2)},$$

we can find a $\delta_1 \in (0,1)$ such that for $\delta \in (0,\delta_1)$,

$$E_p(f_\delta,0) > 2^{-1}\Big[\frac{2^p\pi^{p-2}}{(p+1)(p+2)}\Big]^{1/2} := \frac{\mu_0}{2}.$$

Second, suppose $I \subseteq \partial\Delta$ is any open subarc containing π. If $|I| \leqslant (\pi-\delta)/(2\pi)$, then $S_p(f_\delta,I) = 0$. If $1 \geqslant |I| > (\pi-\delta)/(2\pi)$, then by the definition of f_δ,

$$S_p(f_\delta,I) \leqslant S_p(f_\delta,\partial\Delta) = \frac{1}{(2\pi)^2}\iint_\Omega\frac{|f_\delta(e^{i\phi}) - f_\delta(e^{i\psi})|^2}{|e^{i\phi}-e^{i\psi}|^{2-p}}\,d\phi d\psi,$$

where Ω is a domain defined by $\bigcup_{j=1}^{4} \Omega_j$:

$$\Omega_j = \begin{cases} \{(\phi,\psi) : 0 \leqslant \phi \leqslant \delta, 0 \leqslant \psi \leqslant 2\pi\}, & j=1; \\ \{(\phi,\psi) : 2\pi-\delta \leqslant \phi \leqslant 2\pi, 0 \leqslant \psi \leqslant 2\pi\}, & j=2; \\ \{(\phi,\psi) : 0 \leqslant \phi \leqslant 2\pi, 0 \leqslant \psi \leqslant \delta\} & j=3; \\ \{(\phi,\psi) : 0 \leqslant \phi \leqslant 2\pi, 2\pi-\delta \leqslant \psi \leqslant 2\pi\}, & j=4. \end{cases}$$

It is a completely elementary estimation to obtain a $\delta_2 \in (0,1)$ such that for $\delta \in (0,\delta_2)$,

$$\iint_{\Omega_j} \frac{\left|f_\delta(e^{i\phi}) - f_\delta(e^{i\psi})\right|^2}{|e^{i\phi}-e^{i\psi}|^{2-p}} d\phi d\psi \leqslant \left(\frac{\pi-\delta}{2\pi}\right)^p \left(\frac{2\pi\mu_0}{2}\right)^2, \quad j=1,2,3,4.$$

Hence

$$\frac{S_p(f_\delta,I)}{|I|^p} \leqslant \frac{1}{(2\pi)^2}\left(\frac{2\pi}{\pi-\delta}\right)^p \iint_\Omega \frac{\left|f_\delta(e^{i\phi}) - f_\delta(e^{i\psi})\right|^2}{|e^{i\phi}-e^{i\psi}|^{2-p}} d\phi d\psi \leqslant \left(\frac{\mu_0}{2}\right)^2.$$

Consequently, $E_p(f_\delta,\pi) \leqslant \mu_0/2$ whenever $\delta \in (0,\delta_2)$. Therefore there exists a $\delta_3 \in (0,\min\{\delta_1,\delta_2\})$ such that $E_p(f_{\delta_3},0) > \mu_0/2$ and $E_p(f_{\delta_3},\pi) \leqslant \mu_0/2$. This concludes the proof. ∎

Recall that the (boundary) Dirichlet space $D(\partial\Delta)$ consists of all Lebesgue measurable complex-valued functions f on $\partial\Delta$ for which $\int_{\partial\Delta} f(z)|dz|/(2\pi) = 0$ and

$$\|f\|_{D(\partial\Delta)} := \left[\frac{1}{(2\pi)^2}\int_{\partial\Delta}\int_{\partial\Delta} \frac{\left|f(z)-f(w)\right|^2}{|z-w|^2}|dz||dw|\right]^{1/2} < \infty.$$

It is clear that $D(\partial\Delta) \subseteq \bigcap_{p>0} Q_{p,0}(\partial\Delta)$, and that every point on the closed unit ball of $(D(\partial\Delta), \|\cdot\|_{D(\partial\Delta)})$ is an extreme point and vice versa. An explicit computation involving $E_p(f_\delta,\cdot)$ above reveals that for p small one has to choose δ small in order to get a nonextreme point of the closed unit ball of $(Q_{p,0}(\partial\Delta), \|\cdot\|_{Q_p(\partial\Delta)})$. This reflects the fact that $Q_{p,0}(\partial\Delta)$ approaches $D(\partial\Delta)$ as $p \searrow 0$ in some sense.

4. Appendix

The first result of this section is to illustrate the important basic relationship between $Q_{p,0}(\partial\Delta)$ and $VMO(\partial\Delta)$ mentioned in the introduction.

Proposition 4.1. *If $0 < p_1 < p_2 < \infty$ then $Q_{p_1,0}(\partial\Delta) \subseteq Q_{p_2,0}(\partial\Delta)$. In particular, if $p \in (1,\infty)$ then $Q_{p,0}(\partial\Delta) = VMO(\partial\Delta)$.*

Proof: It suffices to show that each $Q_{p,0}(\partial\Delta)$ coincides with $VMO(\partial\Delta)$ whenever $p > 1$. First, we verify $VMO(\partial\Delta) \subseteq Q_{p,0}(\partial\Delta)$. To do so, we observe the integrated Lip-character of $Q_{p,0}(\partial\triangle)$ (which may be worked out via the change of variables; see also [**4**, p. 579]): $f \in Q_{p,0}(\partial\triangle)$ if and only if $\lim_{\delta\to 0} F_p(f,\delta) = 0$, where for $\delta \in (0,1)$,

$$F_p(f,\delta) := \sup_{I\subseteq\partial\triangle, |I|<\delta} |I|^{-p} \int_0^{|I|} \sin^{p-2}\frac{\pi t}{2} \int_I \left| f\left(e^{i(s+t)}\right) - f\left(e^{is}\right)\right|^2 ds\, dt.$$

For convenience, we use $U \lesssim V$ to denote that there is a constant $c > 0$ such that $U \leqslant cV$. If $U \lesssim V$ and $V \lesssim U$ hold simultaneously then we say that $U \sim V$. In addition, we write rI $(r > 0)$ for the open arc with length $r|I|$ and the same centre as I, and f_J the average of f over $J \subseteq \partial\triangle$: $f_J = (2\pi|J|)^{-1}\int_J f\left(e^{it}\right) dt$. Now if f in $VMO(\partial\triangle)$ then for any small $\varepsilon > 0$ there is a $\delta \in (0, 1/3)$ such that as $|I| < \delta$,

$$\int_{3I} |f(e^{is}) - f_{3I}|^2\, ds < 2\pi\varepsilon|I|,$$

and hence

$$\int_0^{|I|} \sin^{p-2}\frac{\pi t}{2}\, dt \int_I \left|f\left(e^{is}\right) - f_{3I}\right|^2 ds \lesssim |I|^{p-1}\int_{3I}\left|f\left(e^{is}\right) - f_{3I}\right|^2 ds \lesssim \varepsilon|I|^p.$$

Consequently

$$\int_0^{|I|} \sin^{p-2}\frac{\pi t}{2}\int_I\left|f\left(e^{i(t+s)}\right) - f_{3I}\right|^2 ds\, dt \lesssim \varepsilon|I|^p.$$

So, $\lim_{\delta\to 0} F_p(f,\delta) = 0$, namely, $f \in Q_{p,0}(\partial\triangle)$.

Second, we show that $Q_{p,0}(\partial\triangle) \subseteq VMO(\partial\triangle)$. In case $p \in (1,2]$, the result follows immediately from the definition. It remains to consider the case $p > 2$. Let $f \in Q_{p,0}(\partial\triangle)$. Then for arbitrarily small $\varepsilon > 0$ there exists a $\delta \in (1, 1/2)$ such that $S_p(f,J) < \varepsilon|J|^p$ whenever $|J| < \delta$. Thus for $I \subseteq \partial\triangle$ with $|I| < \delta$, one has

$$\begin{aligned}
\int_I\int_I \left|f\left(e^{is}\right) - f\left(e^{it}\right)\right|^2 ds\, dt &\leqslant \sum_{k=1}^{\infty}\iint_{2^{-k}<\frac{|s-t|}{|I|}\leqslant 2^{1-k}} \frac{|e^{is}-e^{it}|^{2-p}\left|f\left(e^{is}\right) - f\left(e^{it}\right)\right|^2}{|e^{is}-e^{it}|^{2-p}}\, ds\, dt \\
&\lesssim \sum_{k=1}^{\infty}\left(\frac{|I|}{2^k}\right)^{2-p}\iint_{|s-t|/|I|\leqslant 2^{1-k}} \frac{\left|f\left(e^{is}\right) - f\left(e^{it}\right)\right|^2}{|e^{is}-e^{it}|^{2-p}}\, ds\, dt \\
&\lesssim \sum_{k=1}^{\infty}\left(\frac{|I|}{2^k}\right)^{2-p}\int_{2^{2-k}I}\int_{2^{2-k}I} \frac{\left|f\left(e^{is}\right) - f\left(e^{it}\right)\right|^2}{|e^{is}-e^{it}|^{2-p}}\, ds\, dt \\
&\lesssim \varepsilon|I|^2,
\end{aligned}$$

which implies that f in $VMO(\partial\triangle)$. Therefore, the proof is complete. □

The second conclusion of this section is an estimate of the distance from $f \in Q_p(\partial\triangle)$ to $Q_{p,0}(\partial\triangle)$.

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Proposition 4.2. *Let $p \in (0,\infty)$ and $f \in Q_p(\partial\Delta)$ with*

$$d\big(f, Q_{p,0}(\partial\Delta)\big) := \inf\big\{\|f-g\|_{Q_p(\partial\Delta)} : g \in Q_{p,0}(\partial\Delta)\big\}.$$

Then

$$d\big(f, Q_{p,0}(\partial\Delta)\big) \sim M_p(f) := \lim_{\delta\to 0}\sup_{I\subseteq\partial\Delta, |I|<\delta}\Big[\frac{S_p(f,I)}{|I|^p}\Big]^{1/2}.$$

Proof: Since $d\big(f, Q_{p,0}(\partial\Delta)\big) = 0$ whenever $f \in Q_{p,0}(\partial\Delta)$, it is easy to show that

$$M_p(f) \leqslant d\big(f, Q_{p,0}(\partial\Delta)\big), \quad f \in Q_p(\partial\Delta).$$

Regarding the reversed estimate, we define the function

$$f_r\big(e^{is}\big) = \frac{1}{2\pi}\int_{\partial\Delta} f(\eta)\frac{1-r^2}{|\eta - re^{is}|^2}|d\eta|$$

for $r \in (0,1)$ and $f \in Q_p(\partial\Delta)$. It is clear that $f_r \in Q_{p,0}(\partial\Delta)$ and

$$f(\zeta) - f_r(\zeta) = \frac{1}{2\pi}\int_{\partial\Delta}\Big[f(\zeta) - f\big(\zeta\overline{\lambda}\big)\Big]\frac{1-r^2}{|1-r\overline{\lambda}|^2}|d\lambda|, \quad \zeta = e^{is}, \quad \lambda = \zeta\overline{\eta}.$$

Setting $T_\lambda f(\zeta) = f\big(\zeta\overline{\lambda}\big)$ and using Minkowski's inequality, we see that for any small $\varepsilon > 0$,

$$\begin{aligned}
&\|f - f_r\|_{Q_p(\partial\Delta)} \\
&\qquad \lesssim \int_{\partial\Delta}\|f - T_\lambda f\|_{Q_p(\partial\Delta)}\frac{1-r^2}{|1-r\overline{\lambda}|^2}|d\lambda| \\
&\qquad \lesssim \int_{|\lambda|<\varepsilon}\|f - T_\lambda f\|_{Q_p(\partial\Delta)}\frac{1-r^2}{|1-r\overline{\lambda}|^2}|d\lambda| + \|f\|_{Q_p(\partial\Delta)}\int_{\varepsilon\leqslant|\lambda|\leqslant\pi}\frac{1-r^2}{|1-r\overline{\lambda}|^2}|d\lambda| \\
&\qquad := \mathrm{Term}_1 + \mathrm{Term}_2.
\end{aligned}$$

Suppose $\delta \in (0,1)$. By the Lebesgue Dominated Convergence Theorem we know that

$$\lim_{\lambda\to 0}\sup_{|I|\geqslant\delta}\frac{S_p(f - T_\lambda f, I)}{|I|^p} = 0.$$

Also the Triangle Inequality yields

$$\sup_{|I|<\delta}\frac{S_p(f - T_\lambda f, I)}{|I|^p} \lesssim \sup_{|I|<\delta}\frac{S_p(f,I)}{|I|^p}.$$

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Therefore, if $\varepsilon \to 0$ then

$$\text{Term}_1 \lesssim \sup_{|I|<\delta} \left[\frac{S_p(f,I)}{|I|^p} \right]^{1/2}.$$

And, if $r \to 1$ then $\text{Term}_2 \to 0$ and thus

$$d\big(f, Q_{p,0}(\partial\triangle)\big) \leqslant \lim_{r\to 1} \|f - f_r\|_{Q_p(\partial\triangle)} \lesssim M_p(f).$$

We are done. ∎

With the help of Proposition 4.1, we see that Proposition 4.2 extends Sarason's vanishing mean oscillation-version in [8]. Of course, Proposition 4.2 derives that $f \in Q_{p,0}(\partial\triangle)$ if and only if $d\big(f, Q_{p,0}(\partial\triangle)\big) = 0$.

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Institute of Analysis
TU-Braunschweig
D-38106 Braunschweig
Germany
e-mail: kjwirths@tu-bs.de

Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John's, NL A1C 5S7
Canada
e-mail: jxiao@math.mun.ca