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Model sets with Euclidean internal space

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Abstract. We give a characterization of inter-model sets with Euclidean internal space. This characterization is similar to previous results for general inter-model sets obtained independently by Baake, Lenz and Moody, and Aujogue. The new ingredients are two additional conditions. The first condition is on the rank of the abelian group generated by the set of internal differences. The second condition is on a flow on a torus defined via the address map introduced by Lagarias. This flow plays the role of the maximal equicontinuous factor in the previous characterizations.

Key words: Meyer sets, address map, dynamical systems, transverse groupoid, inter-model sets, Euclidean internal space
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1. *Introduction*

In the 1970s Meyer introduced some Delone sets in $\mathbb{R}^d$, now called Meyer sets, in connection with his work in harmonic analysis. He observed that each Meyer set can be embedded into another type of Delone set called a model set. This last collection is a subclass of Meyer sets defined by a simple geometric construction: they are the projection on the first coordinate of some part of a lattice in $\mathbb{R}^d \times H$ where H, the internal space, is a locally compact abelian group.

After the discovery of quasicrystals by Gratias *et al* [**DSC84**], model sets with Euclidean internal space were proposed as a geometric model for the atomic positions in a quasicrystal. Euclidean model sets and their associated dynamical systems played an important role in the mathematical diffraction theory of quasicrystals. Hof in [**Hof95**] proved that every repetitive regular inter-model set (see the definition in §2) has pure point diffraction, and then Schlottmann in [**Sch00**] generalized this result to repetitive regular inter-model sets with arbitrary locally compact abelian group as internal space.

Euclidean model sets are also important in the theory of Pisot substitution tilings. A central problem here has been to understand when the space generated by a Pisot substitution is topologically conjugate to the space generated by a Euclidean model set [**ABB+15**, **BK06**, **BST10**].

In [**Sch98**], Schlottmann gave a necessary and sufficient condition for a Delone set to be a general non-singular model set in terms of the recurrence structure of the Delone set, and he asked for a characterization of non-singular model sets with well-behaved internal space such as $\mathbb{R}^n$. We recall that every non-singular model set is a repetitive inter-model set (see Definition 2.4). A dynamical characterization of repetitive regular inter-model sets was given by Baake, Lenz and Moody in [**BLM07**], and then Aujogue [**Auj16a**] extended this characterization to arbitrary repetitive inter-model sets not necessarily regular. Both results apply to general repetitive inter-model sets but left open the question of characterizing repetitive inter-model sets with Euclidean internal space. In this paper, we answer this question by adding an algebraic and a dynamical property to the previous characterizations in [**Auj16a**, **BLM07**]. The first condition is given in terms of the rank of the abelian group generated by the set of differences of the Delone set, and the second condition is written in terms of a flow on a torus constructed from the address map introduced by Lagarias in [**Lag99**]. We call this flow the address system. We recall that every inter-model set is a Meyer set, and all the previous characterizations of inter-model sets are written in the form of what we need to add to a Meyer set in order to have an inter-model set. Our result states that all the information needed for being an inter-model set with Euclidean internal space is encoded in the rank of the group of differences and the dynamical relation between the dynamical system associated to the Meyer set and the address system.

In order to give a more detailed statement of our results we recall some definitions; see §2 for details.

A discrete subset Λ of $\mathbb{R}^d$ is a *Delone set* if it is uniformly discrete and relatively dense. It is *finitely generated* if the abelian group generated by $\Lambda - \Lambda$ is finitely generated, and it is *repetitive* if every pattern in Λ appears with bounded gaps. Given a Delone set Λ, its *hull* Ω_Λ is defined as the collection of all Delone sets whose local patterns agree with those of Λ up to translation. If Λ has *finite local complexity*, then the hull can be endowed with a topology which is metrizable and compact. The subset of the hull of all Delone sets containing 0 is called the *canonical transversal* of Ω_Λ and we denote it by Ξ_Λ. The group $\mathbb{R}^d$ acts on the hull continuously by translation, given a (topological) dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$. Some combinatorial properties of the Delone set translate into dynamical properties. For example, repetitivity of Λ is equivalent to minimality of $(\Omega_\Lambda, \mathbb{R}^d)$. It is well known in dynamical systems theory that there is a dynamical system with an equicontinuous action of $\mathbb{R}^d$ that is a factor (semi-conjugacy) of $(\Omega_\Lambda, \mathbb{R}^d)$ and it is maximal with to respect these properties. This dynamical system is unique up to topological conjugacy and we call it the *maximal equicontinuous factor* of $(\Omega_\Lambda, \mathbb{R}^d)$.

It is known that repetitivity implies finite local complexity (see, for instance, [**BG13**]) and that finite local complexity implies finitely generated (see [**Lag99**]). A Delone set Λ in $\mathbb{R}^d$ is a *Meyer* set if the set of differences $\Lambda - \Lambda$ is a Delone set.

Let Λ be a finitely generated Delone set in $\mathbb{R}^d$. The *rank* of Λ is the rank of the abelian group generated by Λ as a subset of $\mathbb{R}^d$. We denote this group by $\langle\Lambda\rangle$, and its rank by s. Let $\mathcal{B}$ be a basis of $\langle\Lambda\rangle$. Then the *address map* for Λ associated to $\mathcal{B}$ is the coordinate map with respect to the basis $\mathcal{B}$ from $\langle\Lambda\rangle$ to $\mathbb{Z}^s$.

Notice that since $\langle\Lambda\rangle$ is an abelian group and $\langle\Lambda-\Lambda\rangle\subseteq\langle\Lambda\rangle$, if $\langle\Lambda\rangle$ is finitely generated then $\langle\Lambda-\Lambda\rangle$ is finitely generated. On the other hand, for every x in Λ one has that $\langle\Lambda\rangle\subseteq\langle\{x\}\cup(\Lambda-\Lambda)\rangle$. Thus, if $\langle\Lambda-\Lambda\rangle$ is finitely generated then $\langle\Lambda\rangle$ is also finitely generated. Moreover, we get that $\operatorname{rank}\langle\Lambda-\Lambda\rangle\le\operatorname{rank}\langle\Lambda\rangle\le\operatorname{rank}\langle\Lambda-\Lambda\rangle+1$. Also observe that for every Λ_0 in Ξ_Λ we have $0\in\Lambda_0$ and thus

$$\langle\Lambda_0\rangle=\langle\Lambda_0-\Lambda_0\rangle. \tag{1.1}$$

In particular, if Λ is a repetitive Meyer set in $\mathbb{R}^d$ then all Delone sets Λ' in Ω_Λ have the same patterns and the set $\Lambda'-\Lambda'$ does not depend on Λ' and is the same for every Delone set in Ω_Λ.

Assume that Λ is a repetitive Meyer set in $\mathbb{R}^d$. Given a basis $\mathcal{B}$ of $\langle\Lambda-\Lambda\rangle$, let $\varphi:\langle\Lambda-\Lambda\rangle\to\mathbb{Z}^s$ be the coordinate map with respect to the basis $\mathcal{B}$. By (1.1), we have that for every Λ_0 in Ξ_Λ the address map of Λ_0 is equal to φ.

Lagarias proved in [**Lag99**] that if Λ is a Meyer set then there is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^s$ whose distance to the address map of Λ is uniformly bounded on the points of Λ. In fact, this property characterizes Meyer sets. Our first result gives the existence of one linear map that approximates the address map of all Delone sets in Ξ_Λ, and it also gives a linear flow on a torus that we use to characterize inter-model sets with Euclidean internal space.

Put $\|x\|_s$ for the Euclidean norm of x in $\mathbb{R}^s$.

PROPOSITION 1.1. (Address system) *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let s be the rank of $\langle\Lambda-\Lambda\rangle$. Let $\mathcal{B}$ be a basis of $\langle\Lambda-\Lambda\rangle$ and let $\varphi:\langle\Lambda-\Lambda\rangle\to\mathbb{Z}^s$ be the coordinate map with respect to the basis $\mathcal{B}$. There are an injective linear map $\ell:\mathbb{R}^d\to\mathbb{R}^s$ and a constant $C>0$ such that for every Λ_0 in Ξ_Λ and every $t\in\Lambda_0$ we have*

$$\|\varphi(t)-\ell(t)\|_s\le C.$$

Moreover, there is a linear flow $(\mathbb{T}^s,\mathbb{R}^d)$ defined by

$$(w,t)\in\mathbb{T}^s\times\mathbb{R}^d\longmapsto w+[\ell(t)]_{\mathbb{Z}^s},$$

and there is a homomorphism $\pi_{\mathrm{Ad}}:\Omega_\Lambda\to\mathbb{T}^s$ such that for every Λ' in Ω_Λ and every t in $\mathbb{R}^d$ we have $\pi_{\mathrm{Ad}}(\Lambda'-t)=\pi_{\mathrm{Ad}}(\Lambda')+[\ell(t)]_{\mathbb{Z}^s}$.

Notice that the dynamical system $(\mathbb{T}^s,\mathbb{R}^d)$ and the homomorphism π_{Ad} in Proposition 1.1 depend on the basis $\mathcal{B}$ chosen. However, if we change the basis, then the new system is topologically conjugate to the previous one. We call any of these dynamical systems an *address system* of Λ, and the map π_{Ad} an *address homomorphism* of Λ, which are well defined up to topological conjugacy. Observe that each coordinate of π_{Ad} in Proposition 1.1 gives a topological factor of $(\Omega_\Lambda,\mathbb{R}^d)$ onto the circle $\mathbb{T}$; however, $(\mathbb{T}^s,\mathbb{R}^d)$ is not necessarily a topological factor of $(\Omega_\Lambda,\mathbb{R}^d)$. The minimality of $(\Omega_\Lambda,\mathbb{R}^d)$

implies that $(\mathbb{T}^s, \mathbb{R}^d)$ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$ if and only it is minimal. Indeed, minimality of $(\Omega_\Lambda, \mathbb{R}^d)$ implies that $(\pi_{\mathrm{Ad}}(\Omega_\Lambda), \mathbb{R}^d)$ is minimal, and then $\pi_{\mathrm{Ad}}(\Omega_\Lambda) = \mathbb{T}^s$ if and only if $(\mathbb{T}^s, \mathbb{R}^d)$ is minimal. Finally, applying a well-known criterion for minimality of linear flows on the torus [**KH95**, Proposition 1.5.1], we have that if we denote by A the representative matrix of ℓ in the canonical basis and by A^T the transpose then we have that $(\mathbb{T}^s, \mathbb{R}^d)$ is minimal if and only if $\mathrm{Ker}(A^T) \cap \mathbb{Z}^s = \{0\}$, which gives a simple way to check minimality of the address system $(\mathbb{T}^s, \mathbb{R}^d)$.

The next theorem is the main result of the paper; it characterizes inter-model sets with Euclidean internal space.

THEOREM A. *A repetitive Meyer set Λ in $\mathbb{R}^d$ is an inter-model set with Euclidean internal space if and only if* $\mathrm{rank}(\langle \Lambda - \Lambda \rangle) > d$ *and there is an address system of Λ that is a topological factor of* $(\Omega_\Lambda, \mathbb{R}^d)$ *such that there is one point with a unique preimage under the factor map.*

By [**Pa76**, Proposition 1.1] (see also [**ABKL15**, Lemma 3.11]) and the previous theorem, an address system of a repetitive inter-model set Λ with Euclidean internal space is the maximal equicontinuous factor of $(\Omega_\Lambda, \mathbb{R}^d)$.

From Theorem A and [**BLM07**, Theorem 5] we obtain the following characterization for regular inter-model sets with Euclidean internal space. Observe that if an address system of Λ is minimal then it is also uniquely ergodic, since it is an equicontinuous system.

THEOREM B. *A repetitive Meyer set Λ in $\mathbb{R}^d$ is a regular inter-model set with Euclidean internal space if and only* $\mathrm{rank}(\langle \Lambda - \Lambda \rangle) > d$ *and there is an address system of Λ that is a topological factor of* $(\Omega_\Lambda, \mathbb{R}^d)$ *such that the set of points in the address system with a unique preimage under the factor map has full measure for the unique ergodic measure.*

For the proof of Theorem A, given a Meyer set, we construct a cut and project scheme (CPS) with a Euclidean internal space and a window, which we call the 'Lagarias CPS' and the 'minimal window', respectively. What we actually prove in Theorem A is that if Λ satisfies the necessary condition then it is an inter-model set generated by the Lagarias CPS and the minimal window. Using [**BLM07**, Theorem 5] again, we can give a more explicit version of Theorem B.

THEOREM C. *A repetitive Meyer set Λ in $\mathbb{R}^d$ is a regular inter-model set with Euclidean internal space if and only* $\mathrm{rank}(\langle \Lambda - \Lambda \rangle) > d$*, there is an address system of Λ that is a topological factor of* $(\Omega_\Lambda, \mathbb{R}^d)$ *such that there is one point with a unique preimage under the factor map and the boundary of minimal window of Λ has measure zero.*

In order to put our results in context we mention an application to the theory of unimodular Pisot irreducible substitution tilings. For this purpose, given a tiling $\mathcal{T}$ of $\mathbb{R}$ by intervals, we identify $\mathcal{T}$ with the Delone set $\Lambda(\mathcal{T})$ in $\mathbb{R}$ obtained from the extreme points of the tiles in $\mathcal{T}$. It is well known that the hull of $\mathcal{T}$ with the action by translation of $\mathbb{R}$ is topologically conjugate to the hull of $\Lambda(\mathcal{T})$ with the action by translation of $\mathbb{R}$. When $\mathcal{T}$ is a periodic point of a unimodular Pisot irreducible substitution with the length of the tiles given by the coordinates of the eigenvector of the largest eigenvalue, one has that

$\mathcal{T}(\Lambda)$ is a Meyer set [**BK06**]. Using that the substitution is Pisot and irreducible, it is not difficult to prove that the length of the tiles forms a basis for the group of differences generated by $\Lambda(\mathcal{T})$ and that the address system associated to this basis is a factor of $(\Omega_{\Lambda(\mathcal{T})}, \mathbb{R})$. We call this address system the *canonical address system* of $\Lambda(\mathcal{T})$. Moreover, from [**BK06**, Theorem 1] it is not difficult to deduce that the canonical address system is the maximal equicontinuous factor of $(\Omega_{\Lambda(\mathcal{T})}, \mathbb{R})$. One has that the Lagarias CPS (see §4.2.1) constructed from $\Lambda(\mathcal{T})$ is exactly the geometric construction that gives rise to the Rauzy fractal modulo a linear change of coordinates, and the Rauzy fractal corresponds to the minimal window in the Lagarias CPS. Since it is known that the Rauzy fractal has zero measure boundary (see, for instance, [**BST10**]) we get that the minimal window has zero measure boundary. Then, using Theorems A and C, one can give another proof of the following known characterization of pure point unimodular Pisot irreducible substitution tilings as regular model sets with Euclidean internal space, and by the condition that the canonical address system has a point with unique preimage.

THEOREM 1.1. [**BK06**, Theorem 7.3, Corollary 9.4, and Remark 18.6] *Let $\Omega_{\mathcal{T}}$ be the hull of a unimodular Pisot irreducible substitution tiling $\mathcal{T}$ in $\mathbb{R}$. The following assertions are equivalent.*

(i) *$\Omega_{\mathcal{T}}$ has pure point dynamical spectrum.*
(ii) *$\Omega_{\Lambda(\mathcal{T})}$ is the hull of a regular model set with Euclidean internal space.*
(iii) *There is a point in the canonical address system of $\Lambda(\mathcal{T})$ with a unique preimage under the factor map.*

To prove Theorem 1.1, observe that (i) implies (iii). By Theorem C, we have that (iii) and the fact that the minimal window has zero measure boundary implies (ii). Finally, it is well known that the hull of a regular model set has pure point dynamical spectrum [**Hof95**].

There are constructions of Euclidean CPSs for Pisot type substitution in higher-dimensional Euclidean spaces; see, for instance, [**LAN18**]. It is a current subject of research to study the relation of those constructions and the Euclidean CPS proposed in this paper.

Finally, we remark that from Proposition 1.1, for every repetitive Meyer set Λ in $\mathbb{R}^d$, the dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$ has d continuous linearly independent eigenvalues in $\mathbb{R}^d$. This was first proved by Kellendonk and Sadun in [**KS14**] using pattern equivariant cohomological methods. Our proof relies on dynamical methods. Both proofs are basically the same and involve proving that some cocycles are coboundaries. But in our case we use a groupoid version of the classical Gottschalk–Hedlund theorem in dynamical systems, and in [**KS14**] the authors prove by hand that the cocycles that they define are coboundaries.

1.1. *Strategy of the proof of Theorem A.* The first step is to define an address system. This is done in Proposition 1.1 where we show that an address map on the canonical transversal minus the linear approximation defines a continuous cocycle on the transverse groupoid. This cocycle is bounded. Then we apply a groupoid version of the Gottschalk–Hedlund theorem [**Ren12**] to prove that the cocycle is a coboundary. Using this coboundary, we define the homomorphism. For the necessary condition we show that the maximal

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equicontinuous factor of the hull of an inter-model set with Euclidean internal space is topologically conjugate to the address system. To show that the condition is also sufficient is much more complicated. The first step is to prove that for a repetitive Meyer set the construction of an inter-model set given by Lagarias in [**Lag99**] gives a CPS that we call the Lagarias CPS†; see the definition in §2.4. Then we show that the maximal equicontinuous factor of the hull associated to the Lagarias CPS (for any window) is topologically conjugate to the address system. Then we use [**Auj16b**, Proposition 3.3] to prove that the closure of the projection in the internal space of the lifting of the Meyer set to the product space is a window for the Lagarias CPS. Finally, elaborating on the ideas in [**Auj16a**, Theorem 6.1], we show that if there is a point with a unique preimage under the maximal equicontinuous factor map of the hull of the Meyer set then the Meyer set is an inter-model set.

1.2. *Organization.* In §2 we give some definitions and results about the theory of aperiodic order related to Delone sets and the dynamical systems associated to the Delone sets. In §3 we prove Proposition 1.1. In §4.1 we prove the necessary condition of Theorem A. In §4.2.1, we describe the Lagarias CPS. The proof of the sufficient condition in Theorem A is in §4.2.2 and uses a result that we prove later in §5, the main technical lemma.

2. *Preliminaries*

Let $\mathbb{R}^d$ be the Euclidean d-space endowed with its Euclidean norm that we denote by $\|\cdot\|_d$.

2.1. *Delone sets.* A subset Λ of $\mathbb{R}^d$ is called a Delone set if it is *uniformly discrete*, meaning that there is $r > 0$ such that every closed ball of radius r intersects Λ in at most one point; and *relatively dense*, which means that there is $R > 0$ such that every closed ball of radius R intersects Λ in at least one point.

Let Λ be a Delone set in $\mathbb{R}^d$. For every $t \in \mathbb{R}^d$, we denote by $\Lambda - t$ the Delone set $\{x - t \mid x \in \Lambda\}$.

For every $\rho > 0$ and every t in $\mathbb{R}^d$ denote by $B(t, \rho)$ the *open ball* in $\mathbb{R}^d$ of radius ρ and center t. A ρ*-patch* of Λ centered at $t \in \mathbb{R}^d$ is the set $\Lambda \cap \overline{B(t, \rho)}$. We consider two notions of long-range order for Delone sets: the first states that a Delone set Λ has *finite local complexity* if for every $\rho > 0$ it has a finite number of ρ-patches up to translation; and the second says that Λ is *repetitive* if for each $\rho > 0$ there is a number $M > 0$ such that each closed ball of radius M contains the center of a translated copy of every possible ρ-patch of Λ. Observe that every repetitive Delone set has finite local complexity; see [**BG13**, Proposition 5.6].

2.2. *Meyer sets and address map.* Let Λ be a Delone set in $\mathbb{R}^d$. We say that Λ is a *Meyer set* if there is a finite set F in $\mathbb{R}^d$ such that

$$\Lambda - \Lambda \subseteq \Lambda + F.$$

In [**Mey72**], Meyer proved that every model set is a Meyer set. The following characterization of Meyer set is used in the proofs of Proposition 1.1 and the main theorem.

† In the terminology of Lagarias, what we prove is that repetitivity implies that the CPS is irreducible; see §4.2.1.

THEOREM 2.1. [**Lag99**, Theorem 3.1] *Let Λ be a finitely generated Delone set in $\mathbb{R}^d$ with rank s. Then Λ is a Meyer set if and only if every address map*

$$\varphi : \langle\Lambda\rangle \to \mathbb{Z}^s,$$

is almost linear, that is, there are a unique linear map $\ell : \mathbb{R}^d \to \mathbb{R}^s$ and a constant $C > 0$ such that for every x in Λ we have

$$\|\varphi(x) - \ell(x)\|_s \leq C. \tag{2.1}$$

Remark 2.1. In the proof of [**Lag99**, Theorem 3.1] it was proved that ℓ is some kind of 'ideal address map' in the sense that if $\{v_1, \dots, v_s\}$ is the basis of $\langle\Lambda\rangle$ that we used to define the address map of Λ then for every t in $\mathbb{R}^d$ we have

$$\sum_{i=1}^{s} \ell_i(t) v_i = t. \tag{2.2}$$

2.3. *Dynamical systems and transverse groupoid.* Let $\Lambda \subseteq \mathbb{R}^d$ be a Delone set with finite local complexity. The *hull* of Λ is the collection of all Delone sets in $\mathbb{R}^d$ whose ρ-patches, for every $\rho > 0$, are also ρ-patches of Λ up to translation. We denote this set by Ω_Λ. There is a natural metrizable topology on Ω_Λ. Roughly speaking, two Delone sets are close in this topology if they agree on a large ball around the origin up to a small translation. In particular, for every Λ' in Ω_Λ a basis of open neighborhoods for Λ' is given by the following sets. First, for every $R > 0$ put

$$T(\Lambda', R) := \{\widetilde{\Lambda} \in \Omega_\Lambda \mid \widetilde{\Lambda} \cap \overline{B(0, R)} = \Lambda' \cap \overline{B(0, R)}\},$$

and for every $0 < \varepsilon < R/2$ we define the open neighborhood $N(\Lambda', \varepsilon, R)$ of Λ' by

$$N(\Lambda', \varepsilon, R) := \{\Lambda'' \in \Omega_\Lambda \mid \text{there exists } \widetilde{\Lambda} \in T(\Lambda', R),$$
$$\text{there exists } t \in B(0, \varepsilon), \Lambda'' = \widetilde{\Lambda} - t\};$$

for more details see, for example, [**FHK02**, **KL13**, **LM06**, **Sch00**]. If Λ has finite local complexity then its hull Ω_Λ is compact. Observe that the action by translation of $\mathbb{R}^d$ on Ω_Λ is continuous. Thus, we obtain a topological dynamical system denote by $(\Omega_\Lambda, \mathbb{R}^d)$. The *orbit* of x in Ω_Λ is the set $\{x - t \mid t \in \mathbb{R}^d\}$, and a subset A of Ω_Λ is called *invariant* if it is invariant by the action of $\mathbb{R}^d$. The dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$ is minimal if and only if the only closed invariant sets are the empty set and the whole space. It is well known that minimality is equivalent to the fact every point has a dense orbit, and in the context of Delone sets repetitivity is equivalent to minimality.

We recall that every topological dynamical system admits a *maximal equicontinuous factor*, that is, a topological factor with an equicontinuous action such that any other equicontinuous factor is a topological factor of it; see, for instance, [**BKS12**, **BK13**, **Kur03**]. For a topological dynamical system (X, G), where X is a compact metric space and G a locally compact abelian group, we denote by (X_{me}, G) its maximal equicontinuous factor. Given two minimal dynamical systems (X, G) and (Y, G), and a factor map $\pi : (X, G) \to (Y, G)$, we say that $\pi : (X, G) \to (Y, G)$ is an *almost automorphic extension*

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or that (X, G) is an *almost automorphic extension of* (Y, G) if there is a point in Y with a unique preimage under π.

The *transversal* of the hull is the closed subset

$$\Xi_\Lambda := \{x \in \Omega_\Lambda \mid 0 \in x\} \subseteq \Omega_\Lambda.$$

In general, the restriction of the action of $\mathbb{R}^d$ to Ξ_Λ is not defined. For this reason, to study the dynamical properties of the transversal we introduce the *transverse groupoid*,

$$\mathfrak{G}_\Lambda = \{(x, t) \in \Xi_\Lambda \times \mathbb{R}^d \mid x - t \in \Xi_\Lambda\} \subseteq \Xi_\Lambda \times \mathbb{R}^d.$$

This set, endowed with the induced topology from the product space $\Xi_\Lambda \times \mathbb{R}^d$, has the structure of a topological groupoid; see [**Ren80**] for the abstract definition of topological groupoids. Two elements (x, t) and (z, s) in $\mathfrak{G}_\Lambda$ are composable if and only if $x - t = z$, and the composition of (x, t) and (z, s) is defined by

$$(x, t) \cdot (z, s) = (x, t + s).$$

The inverse map $\cdot^{-1} : \mathfrak{G}_\Lambda \to \mathfrak{G}_\Lambda$ is defined by $(x, t)^{-1} = (x - t, -t)$ and the *domain* $d : \mathfrak{G}_\Lambda \to \Xi_\Lambda$ and *range* $r : \mathfrak{G}_\Lambda \to \Xi_\Lambda$ maps are defined by

$$d(x, t) = x \quad \text{and} \quad r(x, t) = x - t.$$

Notice that $d(\mathfrak{G}_\Lambda) = r(\mathfrak{G}_\Lambda) = \Xi_\Lambda$. In this context, the set Ξ_Λ is called the *unit space* of $\mathfrak{G}_\Lambda$.

We say that a subset E of the unit space is *invariant* by the groupoid $\mathfrak{G}$ if $E = r(d^{-1}(E))$. We recall the following definition from [**Ren80**].

Definition 2.2. A groupoid is *minimal* if the only open invariant subsets of its unit space are the empty set and the unit space itself.

The following result relates the minimality of $(\Omega_\Lambda, \mathbb{R}^d)$ to the minimality of the transverse groupoid.

PROPOSITION 2.3. *The topological groupoid $\mathfrak{G}_\Lambda$ is minimal if and only if the dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$ is minimal.*

Proof. First, observe that for every subset E of Ξ_Λ we have

$$r(d^{-1}(E)) = \{x - t \in \Xi_\Lambda \mid x \in E, t \in x\}. \tag{2.3}$$

Assume that the dynamical system $(\Omega_\Lambda, \mathbb{R}^d)$ is minimal. Suppose, by contradiction, that $E \subseteq \Xi_\Lambda$ is invariant by the groupoid $\mathfrak{G}_\Lambda$. Define

$$\widehat{E} := \{x - t \in \Omega_\Lambda \mid x \in E, t \in \mathbb{R}\}.$$

We have that $\widehat{E}$ is open in Ω_Λ and by (2.3) it is invariant for the $\mathbb{R}^d$-action on Ω_Λ. Then the complement of $\widehat{E}$ is an invariant non-empty closed set strictly contained in Ω_Λ, which contradicts the minimality of $(\Omega_\Lambda, \mathbb{R}^d)$.

Conversely, suppose that $(\Omega_\Lambda, \mathbb{R}^d)$ is not minimal. Let $C \subseteq \Omega_\Lambda$ be an invariant non-empty closed set strictly contained in Ω_Λ. Put $E = C^c \cap \Xi_\Lambda$. By (2.3), we have

$$E \subseteq r(d^{-1}(E)).$$

Since C is an invariant $\mathbb{R}^d$-action, we get $r(d^{-1}(E)) = E$. So, E is a non-empty open set strictly contained in Ξ_Λ invariant by the groupoid, and thus $\mathfrak{G}_\Lambda$ is not minimal. $\square$

2.4. *Cut and project scheme and inter-model sets.* A *cut and project scheme* over $\mathbb{R}^d$ is the data (H, L) of a locally compact σ-compact abelian group H, and a discrete set $L \subseteq \mathbb{R}^d \times H$ with compact quotient $(\mathbb{R}^d \times H)/L$ whose first coordinate projection on $\mathbb{R}^d$ is one-to-one and whose second coordinate projection on H is dense. A compact subset W of H that is the closure of its interior is called a *window* for the CPS. In the CPS the space $\mathbb{R}^d$ is called *the physical space*, the locally compact abelian group H is called *the internal space* and the set L *the lattice*. Following [**Auj16b**], a CPS can also be described as a triple (H, Γ, s_H) where H is a locally compact σ-compact abelian group, Γ a countable subgroup of $\mathbb{R}^d$ and $s_H : \Gamma \to H$ a group homomorphism with range $s_H(\Gamma)$ dense in H such that the graph

$$\mathcal{G}(s_H) := \{(\gamma, s_H(\gamma)) \in \mathbb{R}^d \times H \mid \gamma \in \Gamma\}$$

is a *lattice*, that is, a discrete and cocompact set. When H is a Euclidean space $\mathbb{R}^n$, for some positive integer n, we say that (H, Γ, s_H) is an *Euclidean CPS*.

Let (H, Γ, s_H) be a CPS with window W. For every w in H, the projection on $\mathbb{R}^d$ of the set $\mathcal{G}(s_H) \cap (\mathbb{R}^d \times (w + W))$ is called a *model set*. More generally, for every subset V of H and every w in H denote by $\curlywedge(w + V)$ the set

$$\curlywedge(w + V) := \{t \in \Gamma \mid s_H(t) \in w + V\}.$$

Definition 2.4. Let (H, Γ, s_H) be a CPS over $\mathbb{R}^d$ with window W. A Delone set $\Lambda \subseteq \mathbb{R}^d$ is called an *inter-model* set if there exist $t \in \mathbb{R}^d$ and $w \in H$ such that

$$\curlywedge(w + \mathrm{int}(W)) - t \subseteq \Lambda \subseteq \curlywedge(w + W) - t.$$

We say that an inter-model set Λ is *non-singular* or *generic* if there is (t, w) in $\mathbb{R}^d \times H$ such that

$$\curlywedge(w + \mathrm{int}(W)) - t = \Lambda = \curlywedge(w + W) - t.$$

Observe that this is equivalent to the fact that the boundary of $w + W$ does not intersect the projection of $\mathcal{G}(s_H)$ in H. Additionally, if the boundary of $w + W$ has zero Haar measure we say the inter-model set is *regular*.

Remark 2.5. Notice that by Baire's theorem, the fact that ∂W has empty interior and $s_H(\Gamma)$ is countable, the set

$$NS := H \setminus \bigcup_{\gamma^* \in s_H(\Gamma)} \gamma^* - \partial W$$

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is a dense G_δ-set in H. Moreover, for every w in H, the boundary of $w + W$ does not intersect the projection of $\mathcal{G}(s_H)$ in H if and only if $w \in NS$. In particular, for every (t, w) in $\mathbb{R}^d \times H$, the set $\curlywedge(w + W) - t$ is a non-singular inter-model set if and only if $w \in NS$.

We say that W' is *irredundant* if the equation $W' + w = W'$ holds only for $w = 0$ in H.

The following two results are well known in the theory of model sets and will be used in the proof of Theorem A.

PROPOSITION 2.6. *Let $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ be a CPS over $\mathbb{R}^d$ with window W. The class of generic model sets generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and window W gives a unique hull, denoted by Ω_{MS}, and the dynamical system $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ is minimal. Moreover, every element in Ω_{MS} is a repetitive inter-model generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and the window W. If the window W is irredundant then every repetitive inter-model set generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and the window W belongs to Ω_{MS}. In particular, for every repetitive inter-model set Λ generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and the irredundant window W we have that $\Omega_\Lambda = \Omega_{\mathrm{MS}}$.*

The first part of Proposition 2.6 follows from [**Rob07**, Proposition 5.18, Corollary 5.10] and [**LM06**, Proposition 4.4]. The part that assumes that the window is irredundant follows from [**Rob07**, Theorem 5.19] and the idea of the proof of [**LM06**, Proposition 4.6].

Let (H, Γ, s_H) be a CPS in $\mathbb{R}^d$ and consider the set $\mathbb{T}_{\mathcal{G}} := (\mathbb{R}^d \times H)/\mathcal{G}(s_H)$ with an action of $\mathbb{R}^d$ given by translation on the first coordinate. More precisely, for every $s \in \mathbb{R}^d$ and every $[(t, w)] \in \mathbb{T}_{\mathcal{G}}$ the action of s on $[(t, w)]$ is

$$[(t, w)] \cdot s := [(t, w)] + [(s, 0)].$$

THEOREM 2.2. *Let $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ be a CPS over $\mathbb{R}^d$, let W be an irrendundant window, and let Ω_{MS} be the hull of the repetitive inter-model sets generated by $(\mathfrak{H}, \mathfrak{L}, s_{\mathfrak{H}})$ and W. Then every point in Ω_{MS} is an inter-model set, and there exists a factor map $\pi : \Omega_{\mathrm{MS}} \to \mathbb{T}_{\mathcal{G}}$ such that for every Λ' in Ω_{MS} there is (t, w) in $\mathbb{R}^d \times H$ such that $\pi(\Lambda') = [(t, w)]$ if and only if*

$$\curlywedge(w + \mathrm{int}(W)) - t \subseteq \Lambda' \subseteq \curlywedge(w + W) - t. \tag{2.4}$$

Moreover, the map π is injective precisely on the subset of non-singular inter-model sets in Ω_{MS} and the dynamical system $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^d)$ is the maximal equicontinuous factor of $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$.

The proof of Theorem 2.2 is mainly in [**Sch00**]. The proof that $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^d)$ is the maximal equicontinuous factor of $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ follows from the fact that $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^d)$ is an equicontinuous factor and from the existence of points where π is injective; see, for instance, [**ABKL15**, Lemma 3.11].

2.5. *Torus parametrization.* Let X be a compact space and let $(X, \mathbb{R}^d)$ be a topological dynamical system. Consider a compact abelian group $\mathbb{K}$ with a minimal action of $\mathbb{R}^d$ coming from group multiplication via a group homomorphism from $\mathbb{R}^d$ into $\mathbb{K}$. A *torus parametrization* is a factor map $\pi : (X, \mathbb{R}^d) \to (\mathbb{K}, \mathbb{R}^d)$. A *section* of π is a map

$s : \mathbb{K} \to X$ such that $\pi \circ s$ is the identity on $\mathbb{K}$. A point $x \in X$ is called *singular* if the fiber $\pi^{-1}(\pi(x))$ contains more than one element. Otherwise, $x \in X$ is *called non-singular*. The set of non-singular points of X for π is denoted by $R_\pi(X)$. The following proposition was proved in [**BLM07**].

PROPOSITION 2.7. [**BLM07**, Proposition 3] *Let $\pi : X \to \mathbb{K}$ be a torus parametrization and let s be a section of π. Then s is continuous at all points of $\pi(R_\pi(X))$.*

3. *The address system*

In this section we prove Proposition 1.1. Given a repetitive Meyer set Λ in $\mathbb{R}^d$, we start by defining a continuous and bounded cocycle in the transverse groupoid of Λ (see Definition 3.1 below). We use a version of the Gottschalk–Hedlund theorem for groupoids to show that this cocycle is a coboundary. We use this cocycle and the map defining the coboundary to construct an equicontinuous dynamical system and homomorphism from $(\Omega_\Lambda, \mathbb{R}^d)$ into this equicontinuous system.

3.1. *Defining a cocycle on the groupoid.*

Let $\Lambda \subseteq \mathbb{R}^d$ be a repetitive Meyer set. Let $\mathcal{B} = \{v_1, \ldots, v_s\} \subseteq \mathbb{R}^d$ be a basis for $\langle \Lambda - \Lambda \rangle$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^d$ be the coordinate map with respect to the basis $\mathcal{B}$. Recall that by the repetitivity of Λ for every $x \in \Xi_\Lambda$ we have that $\langle x - x \rangle = \langle \Lambda - \Lambda \rangle$, and thus the address map of x associated to $\mathcal{B}$ is equal to φ. Note that for all t and t' in $\langle \Lambda - \Lambda \rangle$ we have

$$\varphi(t + t') = \varphi(t) + \varphi(t'). \tag{3.1}$$

From Theorem 2.1, for every $x \in \Xi_\Lambda$ there is a unique linear map $\ell_x : \mathbb{R}^d \to \mathbb{R}^s$ such that

$$\xi_x := \sup_{t \in x} \|\varphi(t) - \ell_x(t)\|_s < +\infty. \tag{3.2}$$

Definition 3.1. Let H be an abelian group. A *cocycle* on the topological groupoid $\mathfrak{G}_\Lambda$ with values in H is a map $c : \mathfrak{G}_\Lambda \to H$ such that for all composable pairs (x, t) and (z, s) in $\mathfrak{G}_\Lambda$ one has

$$c((x, t) \cdot (z, s)) = c((x, t)) + c((z, s)).$$

We define the maps $\Phi : \mathfrak{G}_\Lambda \to \mathbb{Z}^s$ and $L : \mathfrak{G}_\Lambda \to \mathbb{R}^s$ as follows: for every $(x, t) \in \mathfrak{G}_\Lambda$,

$$\Phi(x, t) := \varphi(t) \quad \text{and} \quad L(x, t) := \ell_x(t).$$

The aim of this subsection is to show that $L - \Phi$ defines a continuous cocycle on $\mathfrak{G}_\Lambda$. For this, we first prove that L does not depend on the first coordinate. The proof of the continuity is at the end of the subsection.

PROPOSITION 3.2. *There is a linear map $\ell : \mathbb{R}^d \to \mathbb{R}^s$ such that for all $(x, t) \in \mathfrak{G}_\Lambda$ we have $L(x, t) = \ell(t)$.*

The proof of this proposition is given at the end of this subsection after some lemmas.

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LEMMA 3.3. *Let Λ' be a relatively dense set in $\mathbb{R}^d$. The set $\{t/\|t\|_d \mid t \in \Lambda'\}$ is dense in the boundary of the Euclidean unitary ball centered on the origin. In particular, for all linear maps $T : \mathbb{R}^d \to \mathbb{R}^s$ we have that*

$$\|T\|_{\mathrm{op}} = \sup_{t \in x} \left\| T\left(\frac{t}{\|t\|_d}\right) \right\|_s,$$

where $\| \cdot \|_{\mathrm{op}}$ is the operator norm.

Proof. Put $D := \{t/\|t\|_d \mid t \in \Lambda'\}$. By contradiction, suppose the set D is not dense in the boundary of $B(0, 1)$. So there exists an open set in the relative topology which contains no elements of D. If we project this open set towards infinity, it generates a cone that contains Euclidean balls of size arbitrarily large and where there are no points of Λ'. This contradicts the fact that Λ' is relatively dense. □

LEMMA 3.4. *For all $(x, t) \in \mathfrak{G}_\Lambda$, we have $\ell_x = \ell_{x-t}$.*

Proof. Fix (x, t) in $\mathfrak{G}_\Lambda$. Let $u \in \mathbb{R}^d$ be such that $u \in x - t$. In particular, $t + u \in x$. By (3.2), we have

$$\|\varphi(u) - \ell_{x-t}(u)\|_s \leq \xi_{x-t} \quad \text{and} \quad \|\varphi(u) - \ell_x(t + u)\|_s \leq \xi_x.$$

Using these inequalities and (3.1), we get

$$\begin{aligned} \|\ell_x(t + u) - \ell_{x-t}(u)\|_s &\leq \|\varphi(t + u) - \ell_x(t + u)\|_s + \|\varphi(t + u) - \ell_{x-t}(u)\|_s \\ &\leq \xi_x + \|\varphi(t) + \varphi(u) - \ell_{x-t}(u)\|_s \\ &\leq \xi_x + \|\varphi(t)\|_s + \xi_{x-t}. \end{aligned}$$

Dividing both sides of this last inequality by $\|u\|_d$, we obtain

$$\left\| \ell_x\left(\frac{t}{\|u\|_d}\right) + \ell_x\left(\frac{u}{\|u\|_d}\right) - \ell_{x-t}\left(\frac{u}{\|u\|_d}\right) \right\|_s \leq \frac{\xi_x + \|\varphi(t)\|_s + \xi_{x-t}}{\|u\|_d}.$$

Taking the limit as $\|u\|_d \to +\infty$, we have

$$\lim_{\substack{\|u\|_d \to +\infty \\ u \in x-t}} \left\| (\ell_x - \ell_{x-t})\left(\frac{u}{\|u\|_d}\right) \right\|_s = 0.$$

This, together with Lemma 3.3, implies that $\|\ell_x - \ell_{x-t}\|_{\mathrm{op}} = 0$, and thus concludes the proof of the lemma. □

Proof of Proposition 3.2. Fix y in Ξ_Λ. We prove that for every x in Ξ_Λ we have $\ell_x = \ell_y$. By (3.1), (3.2) and Lemma 3.4, for t' in y we have

$$\begin{aligned} \xi_{y-t'} &= \sup_{t \in y-t'} \|\varphi(t) - \ell_{y-t'}(t)\|_s \\ &= \sup_{t \in y-t'} \|\varphi(t) - \ell_y(t)\|_s \\ &= \sup_{t+t' \in y} \|\varphi(t + t') - \varphi(t') - \ell_y(t)\|_s \end{aligned} \tag{3.3}$$

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$$\begin{aligned}&= \sup_{t+t'\in y} \|\varphi(t+t') - \varphi(t') - \ell_y(t+t'-t')\|_s \\ &= \sup_{t+t'\in y} \|\varphi(t+t') - \varphi(t') - \ell_y(t+t') + \ell_y(t')\|_s \\ &\leq 2\xi_y.\end{aligned}$$

Fix x in Ξ_Λ. By minimality, there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ such that $y - t_n$ converges to x in Ξ_Λ. Fix $t \in x$ and consider $\epsilon > 0$ such that $\|t\| \leq 1/\epsilon$. There is $N \in \mathbb{N}$ such that for all $n > N$ we have

$$(y - t_n) \cap \overline{B\left(0, \frac{1}{\epsilon}\right)} = x \cap \overline{B\left(0, \frac{1}{\epsilon}\right)}.$$

In particular, for all $n > N$ we get $t \in y - t_n$. Then, using Lemma 3.4 and (3.3), for every t in x we have

$$\|\varphi(t) - \ell_y(t)\|_s = \|\varphi(t) - \ell_{y-t_n}(t)\|_s \leq 2\xi_y.$$

By uniqueness of the map ℓ_x, we conclude the proof of the proposition. □

LEMMA 3.5. *The map $L - \Phi$ is a continuous cocycle on $\mathfrak{G}_\Lambda$.*

Proof. By (3.1) and Proposition 3.2, we have that $L - \Phi$ is a cocycle. Now we prove the continuity of $L - \Phi$. Consider a sequence $\{(x_n, t_n)\}_{n\in\mathbb{N}}$ in $\mathfrak{G}_\Lambda$ that converges to (x, t) in $\mathfrak{G}_\Lambda$. By definition of convergence in the groupoid, we have that $\{x_n\}_{n\in\mathbb{N}} \subseteq \Xi_\Lambda$ converges to $x \in \Xi_\Lambda$, and $\{t_n\}_{n\in\mathbb{N}}$ converges to t in $\mathbb{R}^d$. Let ϵ be a positive real number less than the uniformly discrete radius of Λ such that $\|t\|_d < 1/2\epsilon$. There is a positive integer N such that for all $n \geq N$ we have

$$x_n \cap \overline{B\left(0, \frac{1}{\epsilon}\right)} = x \cap \overline{B\left(0, \frac{1}{\epsilon}\right)}, \quad \|t_n - t\|_d < \epsilon \quad \text{and} \quad \|t_n\|_d < \frac{1}{\epsilon}. \tag{3.4}$$

By definition of the groupoid $\mathfrak{G}_\Lambda$, for all n in $\mathbb{N}$ we have that $t_n \in x_n$, and also $t \in x$. By (3.4), for every $n \geq N$ we get $t_n = t$. Then, for every $n \geq N$, we have

$$L(t_n) = \ell(t) \quad \text{and} \quad \Phi(x_n, t_n) = \varphi(t_n) = \varphi(t) = \Phi(x, t),$$

which implies the continuity of $L - \Phi$. □

3.2. *Proof of Proposition 1.1.* We use the following version of the Gottschalk–Hedlund theorem, due to Jean Renault, to find continuous eigenvalues of $\mathfrak{G}_\Lambda$. This version is adapted to our context from [**Ren80**, Theorem 1.4.10] and appears in [**Ren12**].

THEOREM 3.1. *Let G be a minimal topological groupoid with compact unit space X. For a continuous cocycle $c : G \to \mathbb{R}^d$ the following properties are equivalent.*

(1) *There exists a continuous function $g : X \to \mathbb{R}^d$ such that*

$$c = g \circ r - g \circ d.$$

(2) *There exists $x \in X$ such that $c(d^{-1}(x))$ is relatively compact.*
(3) *$c(G)$ is relatively compact.*

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Proof of Proposition 1.1. Let $\Lambda \subseteq \mathbb{R}^d$ be a repetitive Meyer set. Let $\mathcal{B} = \{v_1, \ldots, v_s\} \subseteq \mathbb{R}^d$ be a basis for $\langle \Lambda - \Lambda \rangle$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^d$ be the coordinate map with respect to the basis $\mathcal{B}$. Let L and Φ be as in §3.1. We check that $\mathfrak{G}_\Lambda$ and the cocycle $L - \Phi : \mathfrak{G}_\Lambda \to \mathbb{R}^s$ verify the hypotheses of Theorem 3.1. By Proposition 2.3, the groupoid is minimal. By Lemma 3.5, the map $L - \Phi$ is a continuous cocycle. Let ℓ be the linear map given by Proposition 3.2. By (3.2), for every $x \in \Xi_\Lambda$ the set

$$(L - \Phi)(d^{-1}(x)) = \{\ell(t) - \varphi(t) \mid t \in x\}$$

is bounded. By Theorem 3.1, there is a continuous map $F : \Xi_\Lambda \to \mathbb{R}^s$ such that for every (x, t) in $\mathfrak{G}_\Lambda$ we have

$$\ell(t) - \varphi(t) = L(x, t) - \Phi(x, t) = F \circ r(x, t) - F \circ d(x, t) = F(x - t) - F(x). \tag{3.5}$$

Since F is continuous and the space Ξ_Λ is compact there is a constant $C > 0$ such that the inequality in the first part of Proposition 1.1 holds.

Now we check that ℓ is injective. By contradiction suppose that the kernel of ℓ has dimension greater than 1. Hence, there is an infinite subset of Λ such that the address map is bounded on this infinite set, which gives a contradiction.

Finally, we construct the address system. Denote by $\mathbb{T}^s$ the torus $\mathbb{R}^s/\mathbb{Z}^s$. Since ℓ is linear the following map defines an equicontinuous action of $\mathbb{R}^d$ on $\mathbb{T}^s$:

$$(w, t) \in \mathbb{T}^s \times \mathbb{R}^d \longmapsto w + [\ell(t)]_{\mathbb{Z}^s}.$$

Now we define $\pi_{\mathrm{Ad}} : \Omega_\Lambda \to \mathbb{T}^s$ as follows. For every $y \in \Omega_\Lambda$ there exist $x \in \Xi_\Lambda$ and $t \in \mathbb{R}^d$ such that $y = x - t$. Put

$$\pi_{\mathrm{Ad}}(y) := [F(x)]_{\mathbb{Z}^s} + [\ell(t)]_{\mathbb{Z}^s}.$$

We verify that π_{Ad} is well defined. Indeed, suppose that for $y \in \Omega_\Lambda$ there are $x_1, x_2 \in \Xi_\Lambda$ and $t_1, t_2 \in \mathbb{R}^d$ such that $y = x_1 - t_1 = x_2 - t_2$. Thus, $x_1 = x_2 - (t_2 - t_1)$, and by (3.5) we have that

$$F(x_1) = F(x_2) + \ell(t_2 - t_1) - \varphi(t_2 - t_1),$$

which is equivalent to

$$F(x_1) + \ell(t_1) = F(x_2) + \ell(t_2) - \varphi(t_2 - t_1).$$

Together with the fact that $\varphi(t_2 - t_1) \in \mathbb{Z}^s$, this implies that π_{Ad} is well defined. Now we prove the continuity of π_{Ad}. Fix $y \in \Omega_\Lambda$ and suppose that $y = x - t$ for some $x \in \Xi_\Lambda$ and $t \in \mathbb{R}^d$. For every y' close to y there is x' in Ξ_Λ close to x and there is t' close to t such that $y' = x' - t'$. By the continuity of F and ℓ, the map $\widetilde{\pi}_{\mathrm{Ad}}$ defined in a sufficiently small neighborhood of y by $\widetilde{\pi}_{\mathrm{Ad}}(y') = F(x') + \ell(t')$ is continuous. By the continuity of the canonical projection of $\mathbb{R}^s$ onto $\mathbb{T}^s$ we conclude that π_{Ad} is continuous at y. It remains to check that for every y in Ω_Λ and every t in $\mathbb{R}^d$ we have $\pi_{\mathrm{Ad}}(y - t) = \pi_{\mathrm{Ad}}(y) + [\ell(t)]_{\mathbb{Z}^s}$. Fix y in Ω_Λ and fix t in $\mathbb{R}^d$. There are x_1 and x_2 in Ξ_Λ and t_1 and t_2 in $\mathbb{R}^d$ such that $y = x_1 - t_1$ and $y - t = x_2 - t_2$. Then $x_2 = x_1 - (t_1 - t_2 + t)$. Using this, (3.5) and the

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fact that $\varphi(t_1 - t_2 + t) \in \mathbb{Z}^s$, we get that

$$\begin{aligned}\pi_{\mathrm{Ad}}(y - t) &= [F(x_2)]_{\mathbb{Z}^s} + [\ell(t_2)]_{\mathbb{Z}^s} \\ &= [F(x_1) + \ell(t_1 - t_2 + t) - \varphi(t_1 - t_2 + t)]_{\mathbb{Z}^s} + [\ell(t_2)]_{\mathbb{Z}^s} \\ &= [F(x_1) + \ell(t_1)]_{\mathbb{Z}^s} + [\ell(t)]_{\mathbb{Z}^s} = \pi_{\mathrm{Ad}}(y) + [\ell(t)]_{\mathbb{Z}^s},\end{aligned}$$

which concludes the proof of the proposition. □

4. *Proof of Theorem A*

In this section we prove Theorem A. First, we prove a characterization of the maximal equicontinuous factor for a Euclidean CPS, and then we prove the necessary condition. Having done so, we use the address map to construct a Euclidean CPS that we use in the proof of the sufficient condition. Finally, we prove the sufficient condition assuming the main technical lemma. This lemma is stated in §4.2 and proved in §5.

4.1. *Necessary condition.*

Let Λ be an inter-model set for a Euclidean CPS over $\mathbb{R}^d$ with internal space $\mathbb{R}^n$, lattice L and window W. Denote by Ω_{MS} the hull of the non-singular model sets generated by these data. Repetitivity of Λ and Proposition 2.6 imply that $\Omega_{\mathrm{MS}} = \Omega_\Lambda$. By [**Auj16a**, Theorem 8.1], the associated dynamical system $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ is almost automorphic (see also [**Sch00**, **FHK02**]). The remaining part of the proof of the necessary condition follows directly from the following proposition.

PROPOSITION 4.1. *Let Ω_{MS} be the hull of the non-singular model sets generated by a Euclidean cut and project scheme $(\mathbb{R}^n, \Gamma, s_{\mathbb{R}^n})$ over $\mathbb{R}^d$ and a window W. Then, for every Λ in Ω_{MS}, we have that the group $\langle \Lambda - \Lambda \rangle$ is equal to Γ and its rank is $d + n$. Moreover, the maximal equicontinuous factor of $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ is topologically conjugate to an address system of Λ.*

Proof. Denote by p_1 and by p_2 the orthogonal projections from $\mathbb{R}^d \times \mathbb{R}^n$ onto $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively, and put $L := \mathcal{G}(s_{\mathbb{R}^n})$. Fix Λ in Ω_{MS}. By [**Moo97**, Proposition 2.6(ii)], for every w in $\mathbb{R}^n$ we have that

$$\langle \curlywedge(w + W) \rangle = \Gamma.$$

In particular, $\langle \curlywedge(w + W) - \curlywedge(w + W) \rangle = \Gamma$. By Proposition 2.6, there is w in NS such that $\curlywedge(w + W)$ is in Ω_{MS}, and thus by repetitivity

$$\langle \Lambda - \Lambda \rangle = \langle \curlywedge(w + W) - \curlywedge(w + W) \rangle = \Gamma.$$

We now prove that the maximal equicontinuous factor of $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ is topologically conjugate to the address system of Λ. Fix a basis $\mathcal{B} = \{\widetilde{v}_1, \ldots, \widetilde{v}_s\}$ of L. Let ℓ be the linear map given by the Proposition 1.1 applied to Λ with the basis $p_1(\mathcal{B})$ for Γ and let $(\mathbb{T}^s, \mathbb{R}^d)$ be the corresponding address system. Denote by $\psi : \mathbb{R}^s \to \mathbb{R}^d \times \mathbb{R}^n$ the linear isomorphism sending the canonical basis of $\mathbb{R}^s$ onto $\{\widetilde{v}_1, \ldots, \widetilde{v}_s\}$, that is,

$$\psi(u_1, \ldots, u_s) = u_1\widetilde{v}_1 + \cdots + u_s\widetilde{v}_s.$$

By (2.2), for every $t \in \mathbb{R}^d$ we have

$$p_1(\psi(\ell(t))) = t. \tag{4.1}$$

Define the map $\Psi : \mathbb{T}^s \to \mathbb{T}_{\mathcal{G}}$ by $\Psi([w]_{\mathbb{Z}^s}) = [\psi(w)]_L$. Note that Ψ is a homeomorphism. By (4.1), for all $t \in \mathbb{R}^d$ and $[w] \in \mathbb{T}^s$, we have

$$\begin{aligned}\Psi([w]_{\mathbb{Z}^s} + [\ell(t)]_{\mathbb{Z}^s}) &= \Psi([w + \ell(t)]_{\mathbb{Z}^s})\\ &= [\psi(w + \ell(t))]_L = [\psi(w)]_L + [\psi(\ell(t))]_L\\ &= [\psi(w)]_L + [(p_1(\psi(\ell(t))), p_2(\psi(\ell(t))))]_L\\ &= [\psi(w)]_L + [(t, p_2(\psi(\ell(t))))]_L.\end{aligned}$$

To prove that Ψ conjugates the address system with the maximal equicontinuous factor $(\mathbb{R}^d \times \mathbb{R}^n/L, \mathbb{R}^d)$, we need to show that for every $t \in \mathbb{R}^d$,

$$p_2(\psi(\ell(t))) = 0.$$

By Remark 2.5, Proposition 2.6 and the fact that the window W has non-empty interior, there is w in NS such that $0 \in w + W$ and the set $\curlywedge(w + W)$ is in Ω_{MS}. Put $\Lambda_0 := \curlywedge(w + W)$. We have that Λ_0 is in Ξ_Λ. Observe that φ is also the address map for Λ_0 associated to the basis $p_1(\mathcal{B})$. By Proposition 1.1, there is a constant $\widehat{C} > 0$ such that for every $t \in \Lambda_0$ we have

$$\|p_2(\psi(\varphi(t))) - p_2(\psi(\ell(t)))\|_d \leq \widehat{C}.$$

Together with the fact that $p_2(\psi(\varphi(\Lambda_0))) = p_2(s_{\mathbb{R}^n}(\Lambda_0)) \subseteq w + W$, this implies that the map $p_2 \circ \psi \circ \ell$ is uniformly bounded on Λ_0. Using that Λ_0 is relatively dense in $\mathbb{R}^d$ and that $p_2 \circ \psi \circ \ell$ is linear, we get that $p_2(\psi(\ell(\mathbb{R}^d)))$ is bounded, which implies that $p_2(\psi(\ell(\mathbb{R}^d))) = 0$. We conclude that $(\mathbb{T}^s, \mathbb{R}^d)$ and $(\mathbb{T}_{\mathcal{G}}, \mathbb{R}^d)$ are topologically conjugated, finishing the proof of the lemma. □

4.2. *Sufficient condition*

4.2.1. *The Lagarias cut and project scheme.* Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and suppose that $\langle \Lambda - \Lambda \rangle$ has rank $s > d$. Let $\mathcal{B}$ be a basis of $\langle \Lambda - \Lambda \rangle$ formed by vectors $\{v_1, \ldots, v_s\} \subseteq \mathbb{R}^d$ and let $\varphi : \langle \Lambda - \Lambda \rangle \to \mathbb{Z}^d$ be the coordinate map with respect to the basis $\mathcal{B}$. Fix Λ_0 in Ξ_Λ. Remember that since $0 \in \Lambda_0$, we have $\langle \Lambda - \Lambda \rangle = \langle \Lambda_0 - \Lambda_0 \rangle = \langle \Lambda_0 \rangle$ and that φ is also the address map for Λ_0. Let $\ell : \mathbb{R}^d \to \mathbb{R}^s$ be the linear map given by Proposition 1.1. Define $\phi : \mathbb{R}^s \to \mathbb{R}^d$ by $\phi(u_1, \ldots, u_s) = u_1 v_1 + \cdots + u_s v_s$. By (2.2), for every $t \in \mathbb{R}^d$ we have

$$\phi \circ \ell(t) = t. \tag{4.2}$$

In particular,

$$\mathrm{Ker}(\ell) = \{0\} \quad \text{and} \quad \mathrm{Im}(\phi) = \mathbb{R}^d. \tag{4.3}$$

Put $n := s - d$ and note that the dimension of $\mathrm{Ker}(\phi)$ is n. Let $\mathcal{B}' := \{k_1, \ldots, k_n\}$ be an orthonormal basis for $\mathrm{Ker}(\phi)$. Notice that for every $1 \leq j \leq s$ we have that the vector

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$w_j := \ell(v_j) - e_j$ belongs to $\mathrm{Ker}(\phi)$, where e_j is the jth canonical coordinate vector. For every $j \in \{1, \ldots, s\}$ denote by $(\alpha_{j,1}, \ldots \alpha_{j,n})$ the coordinates of w_j in the basis $\mathcal{B}'$, and define for every $j \in \{1, \ldots, s\}$ the vectors

$$v_j^\star := (\alpha_{j,1}, \ldots \alpha_{j,n})^t \quad \text{and} \quad \widetilde{v}_j := (v_j, v_j^\star).$$

In the proof of [**Lag99**, Theorem 3.1], Lagarias proved that the set $\widetilde{\mathcal{B}} := \{\widetilde{v}_1 \ldots, \widetilde{v}_s\}$ is $\mathbb{Z}$-linearly independent in $\mathbb{R}^d \times \mathbb{R}^n$ and generates a full-rank lattice. Denote by $\widetilde{L}$ the lattice generated by $\widetilde{\mathcal{B}}$. Denote by p_1 and p_2 the orthogonal projections of $\mathbb{R}^d \times \mathbb{R}^n$ onto $\mathbb{R}^d$ and $\mathbb{R}^n$, respectively. By construction, p_1 is injective on $\widetilde{L}$ and its image is $\langle \Lambda - \Lambda \rangle$. Denote by $\psi : \mathbb{R}^s \to \mathbb{R}^d \times \mathbb{R}^n$ the linear isomorphism sending the canonical basis of $\mathbb{R}^s$ onto $\{\widetilde{v}_1, \ldots, \widetilde{v}_s\}$, that is,

$$\psi(u_1, \ldots, u_s) = u_1\widetilde{v}_1 + \cdots + u_s\widetilde{v}_s.$$

In the proof of [**Lag99**, Theorem 3.1], it was proved that for every t in $\langle \Lambda - \Lambda \rangle$ we have

$$\|p_2(\psi(\varphi(t)))\|_n = \|\varphi(t) - \ell(t)\|_s. \tag{4.4}$$

LEMMA 4.2. *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$. If the address system of Λ associated with $\mathcal{B}$ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$, then $p_2(\widetilde{L})$ is dense in $\mathbb{R}^n$.*

Proof. The proof is by contradiction. Assume that $p_2(\widetilde{L})$ is not dense. Then there is a non-empty closed ball $V \subseteq \mathbb{R}^n$ such that $p_2(\widetilde{L}) \cap V = \{\emptyset\}$. In particular,

$$\widetilde{L} \cap (\mathbb{R}^d \times V) = \{\emptyset\}. \tag{4.5}$$

By Proposition 1.1 and (4.4), there is a constant $\widehat{C} > 0$ such that for every $t \in \Lambda_0$ we have

$$\max\{\|p_2(\psi(\varphi(t)))\|_n, \|p_2(\psi(\varphi(t))) - p_2(\psi(\ell(t)))\|_n\} \leq \widehat{C}.$$

Therefore the linear map $p_2 \circ \psi \circ \ell$ is uniformly bounded on Λ_0, which is relatively dense. Then, for all $t \in \mathbb{R}^d$, we have

$$p_2 \circ \psi \circ \ell(t) = 0. \tag{4.6}$$

Consider the dynamical system defined on the space $(\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}$ with the following $\mathbb{R}^d$-action: for every $t \in \mathbb{R}^d$ and every $w \in (\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}$,

$$w \cdot t := w + [(t, 0)]_{\widetilde{L}}.$$

Define the map $\Psi : \mathbb{T}^s \to (\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}$ by $\Psi([w]_{\mathbb{Z}^s}) = [\psi(w)]_{\widetilde{L}}$ for every $[w]_{\mathbb{Z}^s}$ in $(\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}$. By (4.6), the map Ψ is a topological conjugacy between the address system of Λ and the dynamical system just defined $((\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^d)$. Let π_{Ad} be the address homomorphism defined in Proposition 1.1. Since we are assuming that π_{Ad} is a factor of $(\Omega_\Lambda, \mathbb{R}^d)$, we have that the map $\Psi \circ \pi_{\mathrm{Ad}}$ is also a factor from $(\Omega_\Lambda, \mathbb{R}^d)$ to $((\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^d)$. By the repetitivity of Λ we have that $(\Omega_\Lambda, \mathbb{R}^d)$ is minimal, and then the factor $((\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^d)$ is also minimal. But the set $[\mathbb{R}^d \times V]_{\widetilde{L}}$ is closed and $\mathbb{R}^d$-invariant, and by (4.5), it is strictly contained in $(\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}$, which is a contradiction to the minimality of $((\mathbb{R}^d \times \mathbb{R}^n)/\widetilde{L}, \mathbb{R}^d)$. □

Put $s_L := p_2 \circ \psi \circ \varphi$ on $\langle \Lambda - \Lambda \rangle$. By Lemma 4.2 if the address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$ the triple $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$ is a CPS and we call it the *Lagarias CPS for* Λ.

Recall that a window is irredundant if its redundancies group is trivial (see §2.4). By compactness, every window in $\mathbb{R}^n$ is irredundant. By Theorem 2.1 and (4.4), the set $\overline{s_L(\Lambda_0)} \subseteq \mathbb{R}^n$ is a compact. Together with Proposition 5.2 we obtain the following result.

LEMMA 4.3. *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let $\langle \Lambda - \Lambda \rangle$ be the subgroup of $\mathbb{R}^d$ generated by $\Lambda - \Lambda$. Put $n = \mathrm{rank}(\langle \Lambda - \Lambda \rangle) - d$ and assume that $n > 0$. Also assume that some address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$. Let $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$ be the Lagarias CPS for Λ. For every Λ_0 in Ξ_Λ the set $\overline{s_L(\Lambda_0)}$ is an irredundant window.*

From the proof of Lemma 4.2 and by Lemma 4.3 we obtain the following lemma.

LEMMA 4.4. *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let $\langle \Lambda - \Lambda \rangle$ be the subgroup of $\mathbb{R}^d$ generated by $\Lambda - \Lambda$. Put $n = \mathrm{rank}(\langle \Lambda - \Lambda \rangle) - d$ and assume that $n > 0$. Also assume that some address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$. Let $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$ be the Lagarias CPS for Λ. For every Λ_0 in Ξ_Λ, let Ω_{MS} be the hull of the generic inter-model sets generated by $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$ and the window $\overline{s_L(\Lambda_0)}$. Then the maximal equicontinuous factor of $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ is topologically conjugated to each address system of Λ.*

4.2.2. *Proof of sufficient condition.* The main technical step in the proof of the sufficient condition is the following lemma, proved in §5.

MAIN TECHNICAL LEMMA. *Let $\Lambda \subseteq \mathbb{R}^d$ be a repetitive Meyer set and let Γ be the subgroup of $\mathbb{R}^d$ generated by Λ. Let $(H', \Gamma, s_{H'})$ be a CPS and suppose that $W' = \overline{s_{H'}(\Lambda)}$ is a window. Let Ω_{MS} be the hull of the generic model sets generated by $(H', \Gamma, s_{H'})$ and W'. Then there is a factor map*

$$\widetilde{\pi} : \Omega_\Lambda \to \Omega_{\mathrm{MS,me}},$$

such that if $(\Omega_\Lambda, \mathbb{R}^d)$ is an almost automorphic extension of $(\Omega_{\mathrm{MS,me}}, \mathbb{R}^d)$ for $\widetilde{\pi}$, then there are Λ_0 in Ω_Λ and a non-singular inter-model set Λ_1 in Ω_{MS} such that $\Lambda_0 = \Lambda_1$.

Proof of sufficient condition in Theorem A. Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let $\langle \Lambda - \Lambda \rangle$ be the subgroup of $\mathbb{R}^d$ generated by $\Lambda - \Lambda$. Assume that $\mathrm{rank}(\langle \Lambda - \Lambda \rangle) = s > d$, that some address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$ and that $(\Omega_\Lambda, \mathbb{R}^d)$ is an almost automorphic extension of this address system. Since the address systems are topologically conjugated among them we have that every address system of Λ is a topological factor of $(\Omega_\Lambda, \mathbb{R}^d)$ and that $(\Omega_\Lambda, \mathbb{R}^d)$ is an almost automorphic extension of every address system of Λ.

Let $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$ be the Lagarias CPS for Λ where $n = s - d$. Fix Λ_* in Ξ_Λ and recall that by the repetitivity of Λ we have that $\Omega_\Lambda = \Omega_{\Lambda_*}$. By Lemma 4.3, the set $W' = \overline{s_L(\Lambda_*)}$ is an irredundant window. Denote by Ω_{MS} the hull of the generic inter-model sets generated by $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$ and W'. By Lemma 4.4, the maximal equicontinuous factor of $(\Omega_{\mathrm{MS}}, \mathbb{R}^d)$ is topologically conjugated to every address system of Λ which

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agrees with the address systems of Λ_* by Proposition 1.1. By hypothesis, the dynamical system $(\Omega_{\Lambda_*}, \mathbb{R}^d)$ is an almost automorphic extension of every address system of Λ_*, and then it is also an almost automorphic extension of $(\Omega_{\mathrm{MS,me}}, \mathbb{R}^d)$. By the main technical lemma applied to Λ_* and $(\mathbb{R}^n, \langle \Lambda - \Lambda \rangle, s_L)$, there are $\Lambda_0 \in \Omega_{\Lambda_*}$ and $\Lambda_1 \in \Omega_{\mathrm{MS}}$ such that $\Lambda_0 = \Lambda_1$. By the minimality of $(\Omega_{\Lambda_*}, \mathbb{R}^d)$ we have that Ω_{Λ_*} is equal to the hull of Λ_0 which is equal to the hull of the generic model sets generated by a Euclidean CPS. Since $\Omega_\Lambda = \Omega_{\Lambda_*}$ by Proposition 2.6 we conclude that Λ is an inter-model set generated by a CPS with Euclidean internal space, finishing the proof of the sufficient condition. $\square$

5. *Proof of main technical lemma*

In this section we prove the main technical lemma used in the proof of Theorem A. Indeed, we prove a more detailed version of the main technical lemma for future reference.

MAIN TECHNICAL LEMMA$'$. *Let $\Lambda \subseteq \mathbb{R}^d$ be a repetitive Meyer set and let Γ the subgroup of $\mathbb{R}^d$ generated by Λ. Let $(H', \Gamma, s_{H'})$ be a CPS and suppose that $W' = \overline{s_{H'}(\Lambda)}$ is a compact, irredundant window in H'.*

Let Ω_{MS} be the hull of the generic inter-model sets for the CPS $(H', \Gamma, s_{H'})$ and window W'. Let π_0 be the maximal equicontinuous factor map from Ω_{MS} to $\Omega_{\mathrm{MS,me}}$, and denote by $R_{\pi_0}(\Omega_{\mathrm{MS}})$ the set of non-singular points in Ω_{MS} for π_0. Then there is a factor map

$$\widetilde{\pi} : \Omega_\Lambda \to \Omega_{\mathrm{MS,me}}.$$

Put $\Omega_\Lambda^0 := \widetilde{\pi}^{-1}(\pi_0(R_{\pi_0}(\Omega_{\mathrm{MS}})))$. There is a continuous map

$$\pi_1 : \Omega_\Lambda^0 \to R_{\pi_0}(\Omega_{\mathrm{MS}})$$

such that for every $\Lambda_0 \in \Omega_\Lambda^0$ we have

$$\pi_1(\Lambda_0 - t) = \pi_1(\Lambda_0) - t \quad \text{and} \quad \widetilde{\pi}(\Lambda_0) = \pi_0 \circ \pi_1(\Lambda_0).$$

Moreover, for every Λ_1 in $R_{\pi_0}(\Omega_{\mathrm{MS}})$ we have

$$\Lambda_1 = \bigcup_{\Lambda' \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1))} \Lambda'. \tag{5.1}$$

In addition, if $\widetilde{\pi} : \Omega_\Lambda \to \Omega_{\mathrm{MS,me}}$ is an almost automorphic extension then

$$\pi_0(R_{\pi_0}(\Omega_{\mathrm{MS}})) \cap \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$$

is a residual set in $\Omega_{\mathrm{MS,me}}$, and for every Λ_1 in $R_{\pi_0}(\Omega_{\mathrm{MS}})$ such that $\pi_0(\Lambda_1) \in \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$ we have that Λ_1 is in Ω_Λ^0.

The proof of the lemma will be given in §5.2 after recalling the definition of the optimal CPS of a Meyer set introduced in [**Auj16a**].

5.1. *The optimal CPS and the optimal window.*

Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let Γ be the subgroup of $\mathbb{R}^d$ generated by Λ. Define Ξ^Γ as the collection of all $\Lambda' \in \Omega_\Lambda$ having support in Γ:

$$\Xi^\Gamma := \{\Lambda' \in \Omega_\Lambda \mid \Lambda' \subseteq \Gamma\}.$$

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Observe that $\Xi_\Lambda \subseteq \Xi^\Gamma$. We consider the *combinatorial topology* on Ω_Λ, which is obtained from the distance

$$\mathrm{dist}(\Lambda', \Lambda'') = \left\{ \frac{1}{R+1} \middle| \Lambda' \cap \overline{B(0, R)} = \Lambda'' \cap \overline{B(0, R)} \right\}.$$

The combinatorial topology is always strictly finer than the usual topology on Ω_Λ, and on the transversal Ξ_Λ both topologies coincide. We endow $\Xi^{\Gamma'}$ with the combinatorial topology. We say that Λ' and Λ'' in Ω_Λ are *strongly regionally proximal*, denoted $\Lambda' \sim_{\mathrm{srp}} \Lambda''$, if for each $R > 0$ there are $\Lambda_1, \Lambda_2 \in \Omega_\Lambda$ and $t \in \mathbb{R}^d$ such that

$$\begin{aligned} \Lambda' \cap \overline{B(0, R)} &= \Lambda_1 \cap \overline{B(0, R)}, \\ \Lambda'' \cap \overline{B(0, R)} &= \Lambda_2 \cap \overline{B(0, R)}, \\ (\Lambda_1 - t) \cap \overline{B(0, R)} &= (\Lambda_2 - t) \cap \overline{B(0, R)}. \end{aligned}$$

Since Λ is a repetitive Meyer set we have that the strongly regionally proximal relation is a closed $\mathbb{R}^d$-invariant equivalent relation on Ω_Λ, and moreover, it agrees with the equicontinuous relation; see [**BK13**]. In particular, the quotient $\Omega_\Lambda / \sim_{\mathrm{srp}}$ gives the maximal equicontinuous factor.

In the following proposition we recall some results in [**Auj16a**] which allow us to introduce the optimal CPS and optimal window for a Meyer set. More precisely, part (1) is deduced by [**Auj16a**, Proposition 4.4 and Lemma 4.5], part (2) comes from [**Auj16a**, Proposition 6.1 and Definition 6.2], and part (3) is in [**Auj16a**, Theorem 7.1].

PROPOSITION 5.1. *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let Γ the subgroup of $\mathbb{R}^d$ generated by Λ.*

(1) *If $\Lambda' \in \Xi^\Gamma$ then its equivalence class $[\Lambda']_{\mathrm{srp}}$ is contained into Ξ^Γ.*
(2) *The set $H := \Xi^\Gamma / \sim_{\mathrm{srp}}$ with the quotient topology admits a locally compact abelian group structure such that $[\Lambda]_{\mathrm{srp}}$ is the identity element, the map $s_H : \Gamma \to H$ defined by $s_H(\gamma) = [\Lambda - \gamma]_{\mathrm{srp}}$ is a group morphism and $\overline{s_H(\Gamma)} = H$.*

We remark that Aujogue defined s_H in [**Auj16a**] with a plus sign instead of a minus as we do. So some results that we use from [**Auj16a**, **Auj16b**] look slightly different since we need to make a sign correction. From Proposition 5.1, the triple (H, Γ, s_H) is a CPS. Moreover, by [**Auj16a**, Theorem 6.3], the set $[\Xi_\Lambda]_{\mathrm{srp}}$ is a window for (H, Γ, s_H). The CPS (H, Γ, s_H) and the window $[\Xi_\Lambda]_{\mathrm{srp}}$ are called the *optimal CPS* and the *optimal window* for Λ, respectively. Indeed, in [**Auj16b**], the author proved that the model set that it defines,

$$\underline{\Lambda} := \{\gamma \in \mathbb{R}^d \mid s_H(\gamma) \in [\Xi_\Lambda]_{\mathrm{srp}}\},$$

satisfies that for every model set M that includes Λ we have $\Lambda \subseteq \underline{\Lambda} \subseteq M$.

Finally, we recall some results in [**Auj16b**] that we use in the proof of the main technical lemma′. The first result allows us to prove that a compact and irredundant set is a window.

PROPOSITION 5.2. [**Auj16b**, Proposition 3.3] *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let Γ the subgroup of $\mathbb{R}^d$ generated by Λ. Let (H, Γ, s_H) and W be the optimal CPS and window for Λ, respectively. Suppose that $(H', \Gamma, s_{H'})$ is a CPS such that the closure W'*

of the set $s_{H'}(\Lambda)$ is compact and irredundant in H'. Then there is a continuous open and onto morphism

$$\theta : H \to H'$$

such that $s_{H'} = \theta \circ s_H$ on Γ. Moreover, the set W' is a window in H' and $W' = \theta([\Xi_\Lambda]_{\mathrm{srp}})$.

In the following result we recall the definition of a map that we use to construct the maps π_1 and $\widetilde{\pi}$ in the statement of the main technical lemma$'$.

LEMMA 5.3. [Auj16b, Lemmas 3.4, 3.5, 3.6] *Let Λ be a repetitive Meyer set in $\mathbb{R}^d$ and let Γ be the subgroup of $\mathbb{R}^d$ generated by Λ. Suppose that $(H', \Gamma, s_{H'})$ is a CPS such that the closure W' of the set $s_{H'}(\Lambda)$ is compact and irredundant in H'. We have that each Λ' in Ξ^Γ defines a unique element w'_Λ through*

$$\{w'_\Lambda\} = \bigcap_{\gamma \in \Lambda'} s_{H'}(\gamma) - W'.$$

Define the map

$$\begin{aligned} \omega : \Xi^\Gamma &\to H' \\ \Lambda' &\mapsto w_{\Lambda'}. \end{aligned}$$

We have that ω is uniformly continuous for the combinatorial topology, and for all $\Lambda' \in \Xi^\Gamma$ and $\gamma \in \Gamma$ we have:

(1) $\omega(\Lambda' - \gamma) = \omega(\Lambda') - s_{H'}(\gamma)$;

(2) $\omega(\Lambda') = -\theta([\Lambda']_{\mathrm{srp}})$, *where θ is the morphism in Proposition 5.2.*

5.2. *Proof of main technical lemma$'$.* Let $\Lambda \subseteq \mathbb{R}^d$ be a repetitive Meyer set and let Γ be the subgroup of $\mathbb{R}^d$ generated by Λ. Let $(H', \Gamma, s_{H'})$ be a CPS and assume that $W' = \overline{s_{H'}(\Lambda)}$ is a compact and irredundant window in H'. Let Ω_{MS} be the hull of inter-model sets generated by $(H', \Gamma, s_{H'})$ and W'. Recall that the maximal equicontinuous factor $\Omega_{\mathrm{MS,me}}$ can be obtained by the quotient $(\mathbb{R}^d \times H')/\mathcal{G}(s_{H'})$ and denote by π_0 be the maximal equicontinuous factor map from Ω_{MS} to $\Omega_{\mathrm{MS,me}}$.

5.2.1. *Construction of $\widetilde{\pi}$.* We now construct the map $\widetilde{\pi} : \Omega_\Lambda \to \Omega_{\mathrm{MS,me}}$. For every (t, w) in $\mathbb{R}^d \times H'$ we denote by $[(t, w)]$ its equivalent class in $\Omega_{\mathrm{MS,me}}$. For every $\widetilde{\Lambda}$ in Ω_Λ there is $t \in \mathbb{R}^d$ such that $\widetilde{\Lambda} - t$ is in Ξ^Γ, and we define $\widetilde{\pi}(\widetilde{\Lambda})$ by

$$\widetilde{\pi}(\widetilde{\Lambda}) := [(-t, \omega(\widetilde{\Lambda} - t))] \in \Omega_{\mathrm{MS,me}}.$$

We verify that $\widetilde{\pi}$ is well defined. Assume that there is s in $\mathbb{R}^d$ such that $\widetilde{\Lambda} - s$ is in Ξ^Γ. Observe that $t - s$ is in Γ. By part (1) in Lemma 5.3, we have that

$$\begin{aligned} (-t, \omega(\widetilde{\Lambda} - t)) &= (-t + s - s, \omega(\widetilde{\Lambda} - (t + s - s))) \\ &= (-s - (t - s), \omega(\widetilde{\Lambda} - s) - s_{H'}(t - s)) \\ &= (-s, \omega(\widetilde{\Lambda} - s)) - (t - s, s_{H'}(t - s)). \end{aligned}$$

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Since $(t-s, s_{H'}(t-s))$ belongs to $\mathcal{G}(s_{H'})$, we have that

$$[(-t, \omega(\widetilde{\Lambda}-t))] = [(-s, \omega(\widetilde{\Lambda}-s))],$$

and hence $\widetilde{\pi}$ is well defined.

We now check that $\widetilde{\pi}$ commutes with the $\mathbb{R}^d$ action on Ω_Λ and on $\Omega_{\mathrm{MS,me}}$. Let $\widetilde{\Lambda}$ be in Ω_Λ and t be in $\mathbb{R}$. There are s and s' in $\mathbb{R}^d$ such that $\widetilde{\Lambda}-s$ and $(\widetilde{\Lambda}-t)-s' = \widetilde{\Lambda}-(t+s')$ are in Ξ^Γ. Notice that $t+s'-s$ belongs to Γ. Again, by part (1) in Lemma 5.3 we have

$$\begin{aligned}(-s', \omega((\widetilde{\Lambda}-t)-s')) &= (-s', \omega((\widetilde{\Lambda}-s)-(t+s'-s)))\\ &= (-s', \omega(\widetilde{\Lambda}-s) - s_{H'}(t+s'-s))\\ &= (-s'+(t+s'-s), \omega(\widetilde{\Lambda}-s)) - (t+s'-s, s_{H'}(t+s'-s))\\ &= (t-s, \omega(\widetilde{\Lambda}-s)) - (t+s'-s, s_{H'}(t+s'-s)).\end{aligned}$$

Since $(t+s'-s, s_{H'}(t+s'-s))$ is in $\mathcal{G}(s_{H'})$ we have

$$\widetilde{\pi}(\widetilde{\Lambda}-t) = [(-s', \omega((\widetilde{\Lambda}-t)-s'))] = [(-s, \omega(\widetilde{\Lambda}-s))] + [(t,0)] = \widetilde{\pi}(\widetilde{\Lambda}) + [(t,0)].$$

Now we prove that $\widetilde{\pi}$ is continuous. Let Λ' be Ω_{MS} and let U be a neighborhood of 0 in $\Omega_{\mathrm{MS,me}}$. We can assume that $U = [B(0,r_0)\times U_{H'}]$ where $r_0>0$ and $U_{H'}$ is a neighborhood of 0 in H'. There exists $t'\in\Lambda'$ such that $\Lambda'-t'\in\Xi_\Lambda\subseteq\Xi^\Gamma$. For $r>0$, denote

$$C_r = \bigcap_{\gamma\in(\Lambda'-t')\cap\overline{B(0,r)}} s_{H'}(\gamma) - W',$$

and observe that for $r>r'$ we have $C_r\subseteq C_{r'}$. By Lemma 5.3,

$$\bigcap_{r>0} C_r = \{\omega(\Lambda'-t')\}. \tag{5.2}$$

Now we prove that there is $r'>0$ such that for every $r\geq r'$,

$$C_r \subseteq \omega(\Lambda'-t') + U_{H'}. \tag{5.3}$$

By contradiction, suppose that there is an increasing sequence $(r_i)_{i\in\mathbb{N}}$ of positive real numbers converging to infinity as i goes to infinity, such that $(C_{r_i}-\omega(\Lambda'-t'))\cap U_{H'}^c\neq\emptyset$. Then, for every $i\in\mathbb{N}$, there is

$$x_i \in (C_{r_i}-\omega(\Lambda'-t'))\cap U_{H'}^c$$

since for every i, j in $\mathbb{N}$ with $j\geq i$ we have $C_{r_j}\subseteq C_{r_i}$. By compactness of C_{r_1} there is an accumulation point $\tilde{x}$ of $(x_i)_{i\in\mathbb{N}}$ in $U_{H'}^c$, and thus $\tilde{x}\neq 0$. But $\tilde{x}$ also belongs to $\bigcap_{r>0} C_r - \omega(\Lambda'-t')$ which is $\{0\}$ by (5.2), giving the desired contradiction.

Put $R := \|t'\|_d + r' + r_0$ and consider the set

$$T := \{\widetilde{\Lambda}\in\Omega_\Lambda \mid \Lambda'\cap\overline{B(0,R)} = \widetilde{\Lambda}\cap\overline{B(0,R)}\}.$$

For every $\varepsilon>0$ sufficiently small the set

$$V_\varepsilon := \{\Lambda''\in\Omega_\Lambda \mid \text{there exists } \widetilde{\Lambda}\in T,\ \text{there exists } t\in B(0,\varepsilon),\ \Lambda'' = \widetilde{\Lambda}-t\}$$

is an open neighborhood of Λ'. Fix $\varepsilon < r_0$. By the definition of R, for every Λ'' in V_ε there are t in $B(0, \varepsilon)$ and $\widetilde{\Lambda}$ in T such that

$$(\Lambda'' - (t' - t)) \cap \overline{B(0, r')} = (\widetilde{\Lambda} - t') \cap \overline{B(0, r')} = (\Lambda' - t') \cap \overline{B(0, r')}.$$

Put $t'' := t' - t$. We have $\|t' - t''\|_d < r_0$ and since $\Lambda' - t'$ is in Ξ_Λ we also have that $\Lambda'' - t''$ is in $\Xi_\Lambda \subseteq \Xi^\Gamma$. Then

$$\bigcap_{\gamma \in (\Lambda'' - t'') \cap \overline{B(0,r')}} s_{H'}(\gamma) - W' = \bigcap_{\gamma \in (\Lambda' - t') \cap \overline{B(0,r')}} s_{H'}(\gamma) - W'.$$

Together with (5.3), this implies $\omega(\Lambda'' - t'') \in \omega(\Lambda' - t') + U_{H'}$. Therefore, $\widetilde{\pi}(\Lambda'') = [-t'', \omega(\Lambda'' - t'')]$ is included in

$$\begin{aligned}[-t' + (t' - t''), \omega(\Lambda' - t') + U_{H'}] &\subseteq [-t' + B(0, \delta), \omega(\Lambda' - t') + U_{H'}] \\ &= [-t', \omega(\Lambda' - t')] + [B(0, \delta), U_{H'}],\end{aligned}$$

showing the continuity of $\widetilde{\pi}$ at Λ' in Ω_{MS}.

Finally, since the $\mathbb{R}^d$-action on $\Omega_{\mathrm{MS,me}}$ is minimal we have that $\widetilde{\pi}$ is surjective, which concludes the proof that $\widetilde{\pi}$ is a factor map.

5.2.2. *Definition of* π_1. Recall that $R(\Omega_{\mathrm{MS}})$ denotes the set of non-singular points of Ω_{MS} for π_0 as defined in §2.5. By definition, all sections of π_0 agree on $\pi_0(R(\Omega_{\mathrm{MS}}))$. Let $\widetilde{s} : \Omega_{\mathrm{MS,me}} \to \Omega_{\mathrm{MS}}$ be a section of π_0. Put $\Omega^0_\Lambda := \widetilde{\pi}^{-1}(\pi_0(R(\Omega_{\mathrm{MS}})))$, and define the surjective map $\pi_1 : \Omega^0_\Lambda \to R(\Omega_{\mathrm{MS}})$ by $\pi_1 := \widetilde{s} \circ \widetilde{\pi}$.

By the continuity of $\widetilde{\pi}$ and Proposition 2.7, the map π_1 is also continuous. Since $\widetilde{s}$ is a section of π_0, for every Λ' in Ω^0_Λ we have

$$\widetilde{\pi}(\Lambda') = \pi_0 \circ \pi_1(\Lambda'). \tag{5.4}$$

Since $\widetilde{s}$ commutes with the action of $\mathbb{R}^d$ on the set $\pi_0(R(\Omega_{\mathrm{MS}}))$, we get that for every Λ' in Ω^0_Λ and t in $\mathbb{R}^d$,

$$\pi_1(\Lambda' - t) = \pi_1(\Lambda') - t.$$

5.2.3. *Proof of (5.1).* Fix Λ_1 in $R_{\pi_0}(\Omega_{\mathrm{MS}})$. We prove that (5.1) holds. First, we assume that Λ_1 is in $\pi_1(\Omega^0_\Lambda \cap \Xi^\Gamma)$. By Theorem 2.2, if $\pi_0(\Lambda_1) = [(t, w)]$ then

$$\curlywedge(w + \mathrm{int}(W')) = \Lambda_1 + t = \curlywedge(w + W'). \tag{5.5}$$

Observe that, by definition of $\widetilde{\pi}$, for every Λ' in $\Omega^0_\Lambda \cap \Xi^\Gamma$ we have $\widetilde{\pi}(\Lambda') = [(0, \omega(\Lambda')]$. In addition, if Λ' satisfies $\pi_1(\Lambda') = \Lambda_1$ then, using (5.4), we get

$$\pi_0(\Lambda_1) = \pi_0 \circ \pi_1(\Lambda') = \widetilde{\pi}(\Lambda') = [(0, \omega(\Lambda'))].$$

Together with (5.5), this implies that for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ such that $\pi_1(\Lambda') = \Lambda_1$, we have

$$\Lambda_1 = \{\gamma \in \Gamma \mid s_{H'}(\gamma) \in \omega(\Lambda') + W'\}. \tag{5.6}$$

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By Proposition 5.2 and part (2) of Lemma 5.3, we have

$$-\omega(\Xi_\Lambda) = \theta([\Xi_\Lambda]_{\mathrm{srp}}) = W'. \tag{5.7}$$

Since $\Lambda' \in \Xi^\Gamma$, and for every $\gamma \in \Lambda'$ we have $\Lambda' - \gamma \in \Xi_\Lambda$, using part (1) of Lemma 5.3, we get $\omega(\Lambda' - \gamma) = \omega(\Lambda') - s_{H'}(\gamma)$. Together with (5.6) and (5.7), this implies that for every γ in Λ' we have

$$\begin{aligned}\omega(\Lambda' - \gamma) \in \omega(\Xi_\Lambda) &\iff s_{H'}(\gamma) \in \omega(\Lambda') - \omega(\Xi_\Lambda)\\ &\iff s_{H'}(\gamma) \in \omega(\Lambda') + W' \iff \gamma \in \Lambda_1.\end{aligned}$$

Therefore, for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ such that $\pi_1(\Lambda') = \Lambda_1$ we have

$$\Lambda' \subseteq \Lambda_1. \tag{5.8}$$

On the other hand, fix γ in Λ_1. By (5.6), for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ such that $\pi_1(\Lambda') = \Lambda_1$ we have

$$s_{H'}(\gamma) \in \omega(\Lambda') + W' \Leftrightarrow \omega(\Lambda') \in \omega(\Xi_\Lambda + \gamma).$$

Thus, there is Λ'' in $\Xi_\Lambda + \gamma \subseteq \Xi^\Gamma$ such that $\omega(\Lambda'') = \omega(\Lambda')$. Then $\Lambda'' - \gamma$ is in Ξ_Λ, and thus γ is in Λ''. Therefore,

$$\Lambda_1 \subseteq \bigcup_{\Lambda'' \in \Omega^0_\Lambda \cap \Xi^\Gamma \text{ s.t. } \omega(\Lambda'') = \omega(\Lambda')} \Lambda''. \tag{5.9}$$

Observe that for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ and every $\Lambda'' \in \Xi^\Gamma$ such that $\omega(\Lambda') = \omega(\Lambda'')$ we have that $\widetilde{\pi}(\Lambda') = \widetilde{\pi}(\Lambda'')$, and thus $\Lambda'' \in \Omega^0_\Lambda \cap \Xi^\Gamma$. In particular, $\pi_1(\Lambda') = \pi_1(\Lambda'')$, which, together with (5.9), implies

$$\Lambda_1 \subseteq \bigcup_{\pi_1(\Lambda'') = \Lambda_1} \Lambda''. \tag{5.10}$$

We now prove that for every $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ and every $\Lambda'' \in \Omega^0_\Lambda$ such that $\pi_1(\Lambda') = \pi_1(\Lambda'')$, we have that

$$\Lambda'' \in \Xi^\Gamma. \tag{5.11}$$

First, observe that for all Λ' and Λ'' in Ω^0_Λ we have that $\pi_1(\Lambda') = \pi_1(\Lambda'') \Leftrightarrow \widetilde{\pi}(\Lambda'') = \widetilde{\pi}(\Lambda')$. Now let $\Lambda' \in \Omega^0_\Lambda \cap \Xi^\Gamma$ and $\Lambda'' \in \Omega^0_\Lambda$ be such that $\widetilde{\pi}(\Lambda'') = \widetilde{\pi}(\Lambda')$. By definition of $\widetilde{\pi}$, this holds if and only if there exists t in $\mathbb{R}^d$ such that $\Lambda'' - t \in \Xi^\Gamma$ and $[(-t, \omega(\Lambda'' - t))] = [(0, \omega(\Lambda'))]$, which is equivalent to the existence of γ in Γ such that

$$(-t, \omega(\Lambda'' - t)) - (0, \omega(\Lambda')) = (\gamma, s_{H'}(\gamma)).$$

Then $-t = \gamma \in \Gamma$ and we get $\Lambda'' \subseteq \Gamma - \gamma = \Gamma$, which proves (5.11). By (5.8), (5.10) and (5.11), we conclude that

$$\Lambda_1 = \bigcup_{\pi_1(\Lambda'') = \Lambda_1} \Lambda'',$$

which is equivalent to

$$\Lambda_1 = \bigcup_{\Lambda' \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1))} \Lambda'. \tag{5.12}$$

If Λ_1 is not in $\pi_1(\Omega^0_\Lambda \cap \Xi^\Gamma)$ then there is t in $\mathbb{R}^d$ such that $\Lambda_1 - t$ is in $\pi_1(\Omega^0_\Lambda \cap \Xi^\Gamma)$. By (5.12), we have that

$$\Lambda_1 - t = \bigcup_{\widetilde{\Lambda} \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1 - t))} \widetilde{\Lambda}.$$

Since $\widetilde{\pi}(\widetilde{\Lambda}) = \pi_0(\Lambda_1 - t)$ if and only if $\widetilde{\pi}(\widetilde{\Lambda} - (-t)) = \pi_0(\Lambda_1)$, we conclude that

$$\Lambda_1 = \bigcup_{\widetilde{\Lambda} \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1 - t))} \widetilde{\Lambda} - (-t) = \bigcup_{\Lambda' \in \widetilde{\pi}^{-1}(\pi_0(\Lambda_1))} \Lambda', \tag{5.13}$$

which finishes the proof of (5.1).

5.2.4. $(\Omega_\Lambda, \mathbb{R}^d)$ *almost automorphic extension of* $(\Omega_{\mathrm{MS,me}}, \mathbb{R}^d)$. Finally, suppose that $\widetilde{\pi}$ is an almost automorphic extension of $(\Omega_{\mathrm{MS,me}}, \mathbb{R}^d)$. By [**Vee70**, Lemma 4.1], we have that the set

$$\widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda)) = \{x \in \Omega_{\mathrm{MS,me}} \mid \widetilde{\pi}^{-1}(x) \text{ is a singleton}\}$$

is a residual set in $\Omega_{\mathrm{MS,me}}$, and by [**Auj16a**], the set $\pi_0(R_{\pi_0}(\Omega_{\mathrm{MS}}))$ is also a residual set in $\Omega_{\mathrm{MS,me}}$. Then

$$\pi_0(R_{\pi_0}(\Omega_{\mathrm{MS}})) \cap \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$$

is also a residual set in $\Omega_{\mathrm{MS,me}}$. By (5.13), for every Λ_1 in $R_{\pi_0}(\Omega_{\mathrm{MS}})$ such that $\pi_0(\Lambda_1) \in \widetilde{\pi}(R_{\widetilde{\pi}}(\Omega_\Lambda))$ we have that Λ_1 is in Ω^0_Λ, which concludes the proof of the main technical lemma′.

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