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Orthogonal Bundles and Skew-Hamiltonian Matrices

Roland Abuaf and Ada Boralevi

Abstract. Using properties of skew-Hamiltonian matrices and classic connectedness results, we prove that the moduli space $M^0_{\mathrm{ort}}(r,n)$ of stable rank r orthogonal vector bundles on $\mathbb{P}^2$, with Chern classes $(c_1,c_2)=(0,n)$ and trivial splitting on the general line, is smooth irreducible of dimension $(r-2)n-\binom{r}{2}$ for $r=n$ and $n\geq 4$, and $r=n-1$ and $n\geq 8$. We speculate that the result holds in greater generality.

1 Introduction

A holomorphic vector bundle on a projective variety is called *orthogonal* if it is isomorphic to its dual via a symmetric map. While there is a vast literature about orthogonal bundles on curves (let us quote at least [Hul81], [Ram83], [Bea06], and [Ser08]), almost nothing is known about the case of surfaces. To the best of our knowledge, the only existing references are [GS05] and [JMW14]; the former concerns the general problem of semistable principal sheaves, while the latter is a study of autodual instanton bundles via generalized ADHM equations. In this work we are interested in studying irreducible components of the moduli space of stable orthogonal bundles on $\mathbb{P}^2$ with fixed invariants.

In the celebrated paper [Hul80], the author described the moduli space $M(r,n)$ of Mumford–Takemoto (slope) stable rank r vector bundles on $\mathbb{P}^2$ with Chern classes $(c_1,c_2)=(0,n)$ and proved its smoothness and irreducibility for $2\leq r\leq n$. The case $r>n$ is easily dismissed. Indeed if $E\in M(r,n)$ is stable, then both E and its dual have no sections, and this entails that the cohomology group $\mathrm{H}^1(F)$ has dimension $n-r\geq 0$. In [Ott07], Ottaviani used Hulek's techniques to show that the same properties hold for $M_{sp}(r,n)$, the moduli space of symplectic bundles with the same invariants.

Symplectic bundles are the skew-symmetric counterparts of the orthogonal ones: they are isomorphic to their dual via a skew-symmetric map. The generalization to $M_{sp}(r,n)$ is quite straightforward, and the question of whether or not these same techniques could be applied to the orthogonal case of $M_{\mathrm{ort}}(r,n)$ arose naturally, *cf.* [Ott07, Problem 7.8]. As it turns out, it is definitely not the case.

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The smoothness of the moduli space $M_{\mathrm{ort}}(r,n)$ is very easy to prove, but the same cannot be said about its (potential) irreducibility. A first obstacle is caused by the fact that, when deformed on a line, orthogonal bundles behave very differently from their symplectic equivalent and from the unstructured case, that is, elements of $M(r,n)$ with no additional structure.

Indeed, while in the latter two cases the only rigid bundle is the trivial bundle, in the orthogonal case there are two rigid bundles, namely the trivial one $\mathcal{O}_{\mathbb{P}^1}^r$ and $\mathcal{O}_{\mathbb{P}^1}(1)\oplus\mathcal{O}_{\mathbb{P}^1}^{r-2}\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$ and bundles $\bigoplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$, whose value of $\sum_i a_i \bmod 2$ (known as the Mumford invariant) is different, do not deform into each other. This behavior is somehow expected and is connected with the contrasting properties of the group $\mathrm{SO}(r)$ for even and odd values of r.

On the one hand, this result forces us to restrict our attention to the moduli space of orthogonal bundles having the same invariants as above and trivial splitting on the general line, that we denote by $M^0_{\mathrm{ort}}(r,n)$. On the other hand, a careful analysis of this case allows us to extend the notion of Mumford invariant to the case of $\mathbb{P}^2$.

In the attempt to study irreducibility properties of $M^0_{\mathrm{ort}}(r,n)$, we apply techniques that are similar to [Hul80] and [Ott07]. Using standard fibration arguments, we are able to reduce ourselves to the proof of the irreducibility of the space $\{(A,B)\in\Lambda^2V\times\Lambda^2V \mid \mathrm{rk}(AJB-BJA)=r\}$, where V is a complex vector space of even dimension n.

The technical difficulties that this task presents for values of r smaller than n are a lot higher than one could expect. By combining a description of the commutator of two skew-Hamiltonian matrices together with a strong connectedness result, we further reduce the problem to the estimate of the dimension of the singular locus of highly non-general hyperplane sections of the determinantal variety $S^2V_{\le r}$ of symmetric matrices of rank at most r. (The secant variety to the Veronese variety if one prefers this terminology.) This estimate is possible for the cases $r=n$ and $r=n-1$, and gives, respectively, bounds $n\ge 4$ and $n\ge 8$. Our results can be summarized in the following.

Main Theorem *Let n be an even integer. The moduli space $M^0_{\mathrm{ort}}(r,n)$ of rank r stable orthogonal vector bundles on $\mathbb{P}^2$, with Chern classes $(c_1,c_2)=(0,n)$, and trivial splitting on the general line, is smooth irreducible of dimension $(r-2)n-\binom{r}{2}$ for $r=n$ and $n\ge 4$, and $r=n-1$ and $n\ge 8$.*

For general values of r and n, the technical difficulties that we mentioned above seem impossible to overcome. For this reason we are forced to state the following statement as a conjecture, even though we firmly believe in its veracity, being backed up by good computational evidence.

Conjecture *Let n and $3\le r\le n$ be two positive integers, n even. The moduli space $M^0_{\mathrm{ort}}(r,n)$ of rank r stable orthogonal vector bundles on $\mathbb{P}^2$, with Chern classes $(c_1,c_2)=(0,n)$, and trivial splitting on the general line, is smooth irreducible of dimension $(r-2)n-\binom{r}{2}$ for any $6r-5n\ge 2$.*

For small values of n the behavior is even less predictable, but not less interesting, as it is explained in Section 5.

The paper is structured as follows: in Section 2 we introduce the moduli spaces $M(r,n)$, $M_{sp}(r,n)$ and $M_{\mathrm{ort}}(r,n)$ of unstructured, symplectic and orthogonal stable rank r vector bundles on $\mathbb{P}^2$ with Chern classes $(c_1,c_2)=(0,n)$ and $2\le r\le n$. We give the monad construction and prove that $M_{\mathrm{ort}}(r,n)$ is smooth of dimension $(r-2)n-\binom{r}{2}$. In Section 3 we concentrate on the case of those bundles with trivial splitting on the general line, that we denote by $M^0_{\mathrm{ort}}(r,n)$: we deduce some interesting consequences of the trivial splitting assumption, and proceed to give a proof of irreducibility that relies on a key lemma. In Section 4 skew-Hamiltonian matrices and their properties are introduced, and the key lemma is proved in some specific case. Section 5 contains a detailed description of the image of the map sending a pair of skew-symmetric matrices (A,B) to the symmetric matrix $AJB-BJA$, where J is the standard symplectic form. Section 6 is devoted to some explicit examples, remarks, and more details on the stronger result that we conjecture.

2 The Moduli Space of Orthogonal Bundles on $\mathbb{P}^2$

2.1 Notation

We work over the field $\mathbb{C}$ of complex numbers. Given a 3-dimensional vector space U over $\mathbb{C}$, we denote by $U^*=\mathrm{Hom}(U,\mathbb{C})$ its dual, and we fix a determinant form so that $U\simeq U^*$. The projective space $\mathbb{P}^2=\mathbb{P}(U)$ is the space of lines through 0, thus $\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}(U)}(1)\big)=U^*$.

Given a vector bundle E on $\mathbb{P}^2$ we denote by $E(t)$ the tensor product $E\otimes\mathcal{O}_{\mathbb{P}^2}(t)$, for any integer t.

We use lowercase letters to denote the dimension of a cohomology group; for any vector bundle E on $\mathbb{P}^2$, $\mathrm{h}^i(E):=\dim\mathrm{H}^i(\mathbb{P}^2,E)$.

2.2 The Monad Construction

Let $M(r,n)$ denote the moduli space of (slope) stable vector bundles E on $\mathbb{P}^2$, with rank $r\ge 2$ and Chern classes $\big(c_1(E),c_2(E)\big)=(0,n)$. In [Hul80, Section 2.1] the author proved that $M(r,n)$ is non-empty if and only if $r\le n$, hence we will always restrict our discussion to this case.

Lemma 2.1 ([Hul80, Lemma 1.1.2]) *If E is an element of $M(r,n)$, then $\chi\big(E(-i)\big)=-\mathrm{h}^1\big(E(-i)\big)=-n$ for $i=1$ and 2. In particular its value is independent from r.*

Lemma 2.2 ([Hul80]) *Let E be an element of $M(r,n)$, and set $\mathbb{P}^2=\mathbb{P}(U)$. Denote by $V:=\mathrm{H}^1\big(E(-1)\big)$, which is a vector space of dimension n. Then E is the cohomology bundle of the following monad:*

$$I\otimes\mathcal{O}_{\mathbb{P}^2}\xrightarrow{g}V^*\otimes\Omega^1_{\mathbb{P}^2}(2)\xrightarrow{f}V\otimes\mathcal{O}_{\mathbb{P}^2}(1),\tag{2.1}$$

where $f\in U\otimes V\otimes V$ is the natural multiplication map and $I:=\mathrm{H}^1\big(E(-3)\big)$ has dimension $n-r$.

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Proof The proof is a standard application of Beilinson Theorem; we give a sketch for the reader's convenience. After Lemma 2.1, we can write down the Beilinson table of E:

$$\begin{array}{r|ccc}
\mathrm{H}^2\big(E(t)\big) & 0 & 0 & 0 \\
\mathrm{H}^1\big(E(t)\big) & I & V^* & V \\
\mathrm{H}^0\big(E(t)\big) & 0 & 0 & 0 \\
\hline
t & -3 & -2 & -1
\end{array}$$

Thus the monad (2.1) is the spectral sequence entailed by Beilinson's result, whose cohomology abutts to E.

The map f is an element of the vector space

$$\begin{aligned}
\operatorname{Hom}\big(V^* \otimes \Omega^1_{\mathbb{P}^2}(2), V \otimes \mathcal{O}_{\mathbb{P}^2}(1)\big) &= V \otimes V \otimes \operatorname{Hom}\big(\Omega^1_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(1)\big) \\
&= V \otimes V \otimes U.
\end{aligned}$$

■

As remarked in [Ott07, Proposition 7.3], two simple bundles $E(f)$ and $E(f')$ as in Lemma 2.2 are isomorphic if and only if f and f' are $\mathrm{SL}(V)$-equivalent.

Using the monad (2.1) we compute that

$$\mathrm{H}^0(f) \colon V^* \otimes \mathrm{H}^0\big(\Omega^1_{\mathbb{P}^2}(2)\big) \to V \otimes \mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^2}(1)\big).$$

Now $\mathrm{H}^0\big(\Omega^1_{\mathbb{P}^2}(2)\big) = \Lambda^2 U^* \simeq U$, once the determinant form is fixed, and we have $\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^2}(1)\big) = U^*$, so the map $\mathrm{H}^0(f)$ is in fact

$$\mathrm{H}^0(f) \colon V^* \otimes U \to V \otimes U^*,$$

and it can be identified with the contraction operator that from an element $f \in U \otimes V \otimes V$ induces an element

$$S_f \colon V^* \otimes U \to \Lambda^2 U \otimes V \simeq U^* \otimes V$$

through the following steps:

$$\begin{tikzcd}
U \otimes V^* \arrow[d, "\otimes f"] \arrow[dddd, bend right=70, "s_f"'] \\
U \otimes V^* \otimes U \otimes V \otimes V \arrow[d, "\text{reordering}"] \\
(U \otimes U) \otimes (V^* \otimes V) \otimes V \arrow[d, "\text{projection}"] \\
\Lambda^2 U \otimes (V^* \otimes V) \otimes V \arrow[d, "\text{contraction}"] \\
\Lambda^2 U \otimes V.
\end{tikzcd}$$

In particular, if P, Q and R are the three $n \times n$ "slices" of f, then $S_f = \mathrm{H}^0(f)$ can be written as

$$\mathrm{H}^0(f) = \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix}. \tag{2.2}$$

Lemma 2.3 *With the same notation as above,* $\operatorname{rk} \mathrm{H}^0(f) = 2n + r$.

Proof The same proof as in [Ott07] and [Hul80, Lemma 1.3] applies. Again, we give a sketch for the reader's convenience. From the monad (2.1) the kernel of $\mathrm{H}^0(f)$ contains the vector space I of dimension $n - r$, hence,

$$\operatorname{rk} \mathrm{H}^0(f) \leq 3n - (n - r) = 2n + r.$$

Looking at the display associated to (2.1),

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & I \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d, equal] & \operatorname{Ker}(f) \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r, "g"] & V^* \otimes \Omega^1_{\mathbb{P}^2}(2) \arrow[r] \arrow[d, "f"] & \operatorname{Coker}(g) \arrow[r] \arrow[d] & 0 \\
 & & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \arrow[r, equal] \arrow[d] & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \arrow[d] & \\
 & & 0 & 0, &
\end{tikzcd}$$

we get the cohomology sequence for the bundle E. If $\operatorname{rk} \mathrm{H}^0(f) < 2n + r$, then we have $\dim\big(\mathrm{H}^0(\operatorname{Ker} f)\big) > \dim(I) = n - r$ and thus $\mathrm{H}^0(E) \neq 0$, which is a contradiction because E is stable. ■

2.3 Unstructured, Symplectic, and Orthogonal Bundles

Definition 2.1 An *orthogonal vector bundle* is a pair (E, α) consisting of a vector bundle E and an isomorphism $\alpha \to E \to E^*$ such that ${}^t\alpha = \alpha$. If ${}^t\alpha = -\alpha$, then the pair (E, α) is called a *symplectic vector bundle*. A bundle E with no additional structure is said to be *unstructured*.

Remark 2.1 If E is orthogonal, then S^2E contains $\mathcal{O}_{\mathbb{P}^2}$ as a direct summand. If, moreover, E is stable, then it is simple and this forces $\mathrm{H}^0(S^2E) = \mathbb{C}$, so the isomorphism α is unique up to scalar. The same remark holds for symplectic bundles, once we substitute the symmetric power S^2E with the skew-symmetric Λ^2E. In particular, notice that a stable bundle cannot be both orthogonal and symplectic at the same time.

Theorem 2.4 *The bundle $E(f)$ cohomology of the monad* (2.1) *is*
- *orthogonal if and only if the map $f \in U \otimes \Lambda^2 V$;*
- *symplectic if and only if the map $f \in U \otimes S^2 V$.*

Proof The symplectic case is proved in [Ott07, Theorem 7.2]. For the orthogonal one, we simply generalize the argument. A similar statement can also be found (without proof) in [Hul80, 1.7.3].

Suppose first that the bundle is orthogonal, and that we have an isomorphism $\alpha \to E \to E^*$ such that $\alpha = {}^t\alpha$. Then we define the pairing

$$\mathrm{H}^1\big(E(-1)\big) \otimes \mathrm{H}^1\big(E(-2)\big) \to \mathbb{C}, \quad \phi \otimes \psi \mapsto (\phi, \psi)_{E(-1)} \tag{2.3}$$

as follows. First we recall Serre duality

$$\mathrm{H}^1\big(E(-1)\big) \otimes \mathrm{H}^1\big(E^*(-2)\big) \to \mathbb{C}, \quad \phi \otimes \psi^* \mapsto \langle \phi, \psi^* \rangle_{E(-1)}$$

which is induced by cup product. Since cup product is skew-commutative in odd dimension, we have that $\langle \phi, \psi^* \rangle_{E(-1)} = -\langle \psi^*, \phi \rangle_{E(-2)}$. (For details, see [Bar77, Proposition 1].)

Now define the pairing (2.3) by setting, with obvious notation

$$(\phi, \psi)_{E(-1)} := \langle \phi, \alpha(-2)\psi \rangle_{E(-1)}. \tag{2.4}$$

Note that the natural multiplication map f is its own adjoint with respect to the pairing (2.4), so if α is symmetric, then f is skew-symmetric (which is our orthogonal case) and conversely in the symplectic case [Ott07].

The converse uses a similar argument and we omit it. ∎

In [Hul80, Section 2.1] it is shown that, when non-empty, $M(r,n)$ is a smooth irreducible variety of dimension $2rn - r^2 + 1$.

Denote by $M_{\text{ort}}(r,n)$ (respectively $M_{sp}(r,n)$) the moduli space of orthogonal (resp. symplectic) elements of $M(r,n)$.

In [Ott07] the author proved that, when non-empty (in particular when r is even), the space $M_{sp}(r,n)$ is a smooth irreducible variety of dimension $(r+2)n - \binom{r+1}{2}$.

In this work we wish to investigate smoothness and irreducibility properties of the moduli space $M_{\text{ort}}(r,n)$.

2.4 Smoothness Results, Degeneration Arguments

We start with the following smoothness result.

Lemma 2.5 *When non-empty, the moduli space $M_{\text{ort}}(r,n)$ is smooth of dimension $(r-2)n - \binom{r}{2}$.*

Proof An orthogonal bundle E of rank r may be regarded as a principal bundle with structural group $\mathrm{SO}(r)$. The tangent space at a point E to the moduli space of such principal bundles with assigned invariants is isomorphic to the cohomology group $\mathrm{H}^1(\mathrm{ad}:E)$, where $\mathrm{ad}:E$ is the adjoint bundle defined by the adjoint representation $\mathrm{SO}(r) \to \mathrm{ad}:\mathrm{SO}(r)$, see [Ram75]. Recall that such adjoint representation is isomorphic to the wedge power $\Lambda^2\mathbb{C}^r$, so $\mathrm{ad}:E = \Lambda^2 E$.

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So let us compute $h^1(\Lambda^2 E)$: any $E \in M_{\mathrm{ort}}(r,n)$ is simple, and we have seen in Remark 2.1 that S^2E contains $\mathcal{O}_{\mathbb{P}^2}$ as a direct summand, therefore we must have $h^0(\Lambda^2 E) = 0$. By Serre duality we also have $h^2(\Lambda^2 E) = h^0\big(\Lambda^2 E(-3)\big) = 0$.

Hence $h^1(\Lambda^2 E) = -\chi(\Lambda^2 E)$, and applying the Hirzebruch–Riemann–Roch formula we have

$$h^1(\Lambda^2 E) = -\chi(\Lambda^2 E) = c_2(\Lambda^2 E) - \binom{r}{2}.$$

By the splitting principle we compute that $c_2(\Lambda^2 E) = (r-2)n$, and the statement follows. ∎

Define the moduli of bundles with trivial splitting on a line by

$$M^0_\star(r,n) := \{E \in M_\star(r,n) \mid E_{|\ell} = \mathcal{O}^r_{\mathbb{P}^1} \text{ for some line } \ell\},$$

for $\star = \varnothing, \mathrm{sp}, \mathrm{ort}$, and where $M_\varnothing(r,n) = M(r,n)$.

Notice that by semicontinuity, if $E_{|\ell}$ is trivial on a line ℓ, then it is trivial on the general line ℓ.

Remark 2.2 It is important to underline here that orthogonal bundles behave quite differently from their symplectic and unstructured counterparts. In those cases—with obvious notation—one can use a deformation argument due to Hirschowitz to prove that $\overline{M^0(r,n)} = M(r,n)$ [Hul80, 2.4.1] and $\overline{M^0_{sp}(r,n)} = M_{sp}(r,n)$ [Ott07, Proposition 7.4].

Hence the fact that $M^0(r,n)$ and $M^0_{sp}(r,n)$ are irreducible implies that the same is true for $M(r,n)$ and $M_{sp}(r,n)$. Proving the former statement turns out to be easier than proving the latter, see Section 3.1 below, and as such it is the strategy chosen by both [Hul80] and [Ott07]. Indeed when we restrict a symplectic bundle on $\mathbb{P}^1$, the only rigid bundle is the trivial one [Ram83, Section 9.7].

In the orthogonal case the situation is more involved. There is no restriction on the parity of the rank, hence we can consider both the orthogonal group $\mathrm{SO}(2l+1)$ (type B_l) and the group $\mathrm{SO}(2l)$ (type D_l). A B_l-type orthogonal bundle on $\mathbb{P}^1$ is of the form $\mathcal{O} \oplus \bigoplus_{i=1}^l \mathcal{O}(a_i) \oplus \mathcal{O}(-a_i)$, while B_l-type ones are $\bigoplus_{i=1}^l \mathcal{O}(a_i) \oplus \mathcal{O}(-a_i)$.

In both cases the rigid bundles are the trivial bundle and the bundle $\mathcal{O}(1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}^{2l-1}$ for B_l-type, or the trivial bundle and $\mathcal{O}(1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}^{2l-2}$ if we are in the D_l-type case. (We refer the reader to [Ram83, Section 9.5] for details.)

3 Irreducibility Results

3.1 Consequences of Trivial Splitting

We have claimed above in Remark 2.2 that dealing with the moduli spaces $M^0_\star(r,n)$ of bundles with trivial splitting type on some line is easier than dealing directly with $M_\star(r,n)$, again for $\star = \varnothing, \mathrm{sp}, \mathrm{ort}$. This is mainly due to the fact that under this assumption, the map $H^0(f)$ coming from the monad and represented in (2.2) has a more explicit description.

To see this, we start from the following straightforward lemma.

Lemma 3.1 *Let P, Q, and R be $n \times n$ matrices, and let Q be invertible. Then*

$$\begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & -PQ^{-1} \\ 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & -Q^{-1}P & I_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & Q \\ 0 & Z & R \\ -Q & -R & 0 \end{bmatrix},$$

where $Z := PQ^{-1}R - RQ^{-1}P$, hence

$$\operatorname{rk} \begin{bmatrix} 0 & P & Q \\ -P & 0 & R \\ -Q & -R & 0 \end{bmatrix} = 2n + \operatorname{rk} Z.$$

Lemma 3.2 *E is an element of $M^0_\star(r,n)$ for $\star = \varnothing, \mathrm{sp}, \mathrm{ort}$, if and only if the matrix Q in the representation* (2.2) *of the map $\mathrm{H}^0(f)$ is invertible.*

Proof We need to introduce the discriminant of a morphism from [Hul80, 1.7.1]. In fact the statement of Lemma 3.2 can be found in both [Hul80] and [Ott07], but we report it for the sake of completeness.

Consider the map $f\colon \mathrm{H}^1(E(-2)) \otimes \Omega^1_{\mathbb{P}^2}(2) \to \mathrm{H}^1(E(-1)) \otimes \mathcal{O}_{\mathbb{P}^2}(1)$ from the defining monad (2.1). Recall that f can be seen as an element of $V \otimes V \otimes U$, hence $f\colon U^* \otimes V^* \to V$ and for every $z \in U^*$ we can define a map

$$f(z)\colon V^* \to V, \quad f(z) := f(- \otimes z).$$

Define the discriminant of $f(z)$ as $\Delta(f) := \det(f(z))$. Then

$$\{z \in U^* \mid \Delta(f) = 0\} \simeq \{\ell \in \mathbb{P}^{2^*} \mid E_{|\ell} \neq \mathcal{O}^r_{\mathbb{P}^1}\}.$$

To see this, take a line with equation $\{z = 0\}$ and tensor its hyperplane sequence by $E(-1)$:

$$0 \to E(-2) \xrightarrow{z} E(-1) \to E(-1)_{|\ell} \to 0.$$

Taking cohomology, since $\mathrm{h}^0(E(-2)) = \mathrm{h}^0(E(-1)) = 0$, we get

$$0 \to \mathrm{H}^0(E(-1)_{|\ell}) \to \mathrm{H}^1(E(-2)) \xrightarrow{f(z)} \mathrm{H}^1(E(-1)) \to \cdots. \tag{3.1}$$

Hence $\det(f(z)) = 0$ if and only if $\mathrm{h}^0(E(-1)_{|\ell}) \neq 0$, and this condition is equivalent to $E_{|\ell} \neq \mathcal{O}^r_{\mathbb{P}^1}$.

From this observation it follows that if E has trivial splitting type on some line, then without loss of generality we can assume that any one of the three skew-symmetric slices P, Q and R of the map f is invertible. Just notice that in (3.1) the map $f(z)$ can be explicitly written as $z_0P + z_1Q + z_2R$. Now if the general line has trivial splitting type, taking coordinate lines the map still has to have nonzero determinant, and we can assume that the slice Q is invertible. ∎

By combining the results of Lemma 3.1, Lemma 3.2, and Lemma 2.3 we obtain the following.

Proposition 3.3 *Let $E = E(f)$ be an element of $M^0_\star(r,n)$ for $\star = \varnothing, \mathrm{sp}, \mathrm{ort}$, defined as cohomology of the monad* (2.1)*. Let P, Q, and R be the three $n \times n$ matrices slices of $f \in U \otimes V \otimes V$. Let $Z = PQ^{-1}R - RQ^{-1}P$. Then $\operatorname{rk} Z = r$.*

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Proposition 3.4 *If $E \in M^0_{\mathrm{ort}}(r,n)$, then $n = c_2(E)$ is even.*

Proof Consider the map $f(z)$ and its discriminant from the proof of Lemma 3.2. If E is orthogonal, then $f(z)$ is skew-symmetric, and we can consider its Pfaffian instead of the determinant. In order for it to be non-zero, one needs $c_2(E) = \mathrm{h}^1\big(E(-2)\big) = \mathrm{h}^1\big(E(-1)\big)$ even, and this concludes the proof. ∎

Remark 3.1 The discriminant $\Delta(f) := \mathrm{Pfaff}\big(f(z)\big) \in \mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^{2*}}(\frac{n}{2})\big)$ is a homogeneous polynomial of degree $\frac{n}{2}$ (up to a scalar it is uniquely determined by the class $[f]$ in the $\mathrm{SL}(V)$ equivalence). Its zero set is a curve of degree $\frac{n}{2}$ in the plane and the proof of Proposition 3.4 shows how this curve is related to the splitting behavior of E.

Remark 3.2 From a result by Mumford [Mum71, p. 184], it follows that if E is an orthogonal bundle on the projective line, then $\mathrm{h}^0\big(E(-1)\big) \bmod 2$ is invariant under deformations. In [Hul81] the author proved that orthogonal rank 2 bundles on $\mathbb{P}^1$ are rigid, while for higher rank the Mumford invariant is the only one. More precisely, two orthogonal bundles on $\mathbb{P}^1$ can be deformed into each other if and only if they have the same Mumford invariant. In what follows, one could define $n \bmod 2$ (that is, $\mathrm{h}^1\big(E(-1)\big) \bmod 2$) to be the "Mumford invariant" for the case of $\mathbb{P}^2$. (Notice that by Serre duality, on $\mathbb{P}^1$ one has that $\mathrm{h}^0\big(E(-1)\big) = \mathrm{h}^1\big(E(-1)\big)$.) Proposition 3.4 tells us that the parity of n is indeed connected with the splitting behavior of E on the general line.

3.2 Irreducibility of $M^0_{\mathrm{ort}}(r,n)$

We will now prove the irreducibility of the moduli space $M^0_{\mathrm{ort}}(r,n)$.

Recall that in our setting $n = 2p$ is even. There is no loss in generality if we assume that the general invertible skew-symmetric matrix Q is the standard symplectic form $J := \left[\begin{smallmatrix} 0 & I_p \\ -I_p & 0 \end{smallmatrix}\right]$.

Then the matrix Z from Proposition 3.3 is $Z = PJR - RJP$, where again both P and R are skew-symmetric $n \times n$ matrices.

Theorem 3.5 *Let n and $3 \le r \le n$ be two positive integers, n even. Let V be a complex vector space of dimension n. If the variety*

$$\hat{\mathcal{C}}_{r,n} := \{(A,B) \in \Lambda^2 V \times \Lambda^2 V \mid \mathrm{rk}(AJB - BJA) = r\} \tag{3.2}$$

is irreducible, then the same is true for the moduli space $M^0_{\mathrm{ort}}(r,n)$ of rank r stable orthogonal vector bundles on $\mathbb{P}^2$, with Chern classes $(c_1,c_2) = (0,n)$, and trivial splitting on the general line.

Proof Following [Hul80, Theorem 1.5.2] and [Ott07, Theorem 7.7], we start by giving a necessary and sufficient condition for an element $f \in \mathbb{P}(U \otimes \Lambda^2 V)$ to give a bundle $E(f)$. Define:

$$K_{r,n} := \big\{ f \in \mathbb{P}(U \otimes \Lambda^2 V) \mid \mathrm{rk}\big(\mathrm{H}^0(f)\big) = 2n + r \big\}.$$

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$K_{r,n}$ is quasi-affine, and any $f \in K_{r,n}$ defines $E(f)$ as cohomology bundle of the corresponding monad, once we impose the extra condition that the morphism

$$V^* \otimes \Omega^1(2) \xrightarrow{f} V \otimes \mathcal{O}(1)$$

is surjective.

Such maps f form an open subvariety $\tilde{K}_{r,n} \subseteq K_{r,n}$. There is a universal bundle $\mathcal{E}$ over $\mathbb{P}^2 \times \tilde{K}_{r,n}$ such that the fiber $\mathcal{E}_{\mathbb{P}^2 \times \{f\}}$ is exactly the bundle $E(f)$, see [Hul80, Proposition 1.6.1]. Moreover we have an open subvariety $\tilde{K}^s_{r,n} \subseteq \tilde{K}_{r,n}$ consisting of those f giving rise to a stable $E(f)$. By the universal property of the moduli space, we have a surjection

$$\tilde{K}^s_{r,n} \longrightarrow\!\!\!\!\!\rightarrow M_{\text{ort}}(r,n),$$

and in particular a surjection

$$\tilde{K}^s_{r,n} \xrightarrow{\pi}\!\!\!\!\!\rightarrow M^0_{\text{ort}}(r,n).$$

To prove the theorem it is enough to show that $\pi^{-1}\big(M^0_{\text{ort}}(r,n)\big)$ is irreducible. One has that $\pi^{-1}\big(M^0_{\text{ort}}(r,n)\big) = \tilde{K}^s_{r,n} \smallsetminus Z(\Delta)$, and

$$\tilde{K}^s_{r,n} \smallsetminus Z(\Delta) = \bigcup_{x \in \mathbb{P}^2} \{f \in \tilde{K}_{r,n} \mid \Delta(f)(x) \neq 0\} = \bigcup_{x \in \mathbb{P}^2} \tilde{K}_{r,n,x}.$$

Since any two $\tilde{K}_{r,n,x}$ and $\tilde{K}_{r,n,y}$ have non-empty intersection, we can take advantage of the $\mathrm{SL}(U)$-action: it is enough to prove that $\tilde{K}_{r,n,\overline{x}}$ is irreducible for $\overline{x} = (0,1,0)$. Finally, notice that we have a fibration

$$\tilde{K}_{r,n,\overline{x}} \to \Lambda^2 V \tag{3.3}$$

sending f to the invertible slice Q of the matrix representation (2.2), which is $\mathrm{SL}(V)$ invariant with fibers isomorphic to $\hat{\mathcal{C}}_{r,n}$. Irreducibility then follows from the key Lemma 4.1.

To conclude the proof, we use the fibration (3.3) to compute that

$$\begin{aligned}\dim M^0_{\text{ort}}(r,n) &= \dim \Lambda^2 V + \dim \hat{\mathcal{C}}_{r,n} - \dim \mathrm{GL}(V)\\ &= \binom{n}{2} + \Big[2\binom{n}{2} - \binom{n-r+1}{2}\Big] - n^2\\ &= (r-2)n - \binom{r}{2},\end{aligned}$$

which agrees with the estimate that we made in Lemma 2.5. ∎

Theorem 3.5 shows that the irreducibility of the variety (3.2) $\hat{\mathcal{C}}_{r,n}$ defined above is the key result needed to prove the irreducibility of the moduli space $M^0_{\text{ort}}(r,n)$. The next section, which constitutes the heart of this paper, is devoted to the study of irreducibility of $\hat{\mathcal{C}}_{r,n}$.

4 The Key Lemma

The aim of this section is to prove the following key result.

Lemma 4.1 *Let V be a complex vector space of even dimension n. The subvariety $\hat{\mathcal{C}}_{r,n}$ from* (3.2) *is irreducible of codimension $\binom{n-r+1}{2}$ in $\Lambda^2 V \times \Lambda^2 V$ for $r = n$ and $n \geq 4$, and for $r = n-1$ and $n \geq 8$.*

The reasoning is somewhat similar to what is done in the unstructured case treated in [Hul80]. There one reduces to prove the irreducibility of pairs of $n \times n$ matrices (A, B) whose commutator $[A, B]$ has constant rank r. This result also has a symmetric analogue proved in [Bas00, BPV90], which is used in [Ott07] to show irreducibility in the symplectic case.

The technical difficulty of the skew-symmetric case is however considerably higher than the other two cases. In particular the proof of Lemma 4.6 requires the use of non-trivial connectedness results.

Here are the steps leading to the proof of Lemma 4.1.

(1) We work on the variety

$$\mathcal{C}_{r,n} := \{(A, B) \in \Lambda^2 V \times \Lambda^2 V \mid \mathrm{rk}(AJB - BJA) \leq r\},$$

and prove its irreducibility. The irreducibility of $\hat{\mathcal{C}}_{r,n}$ then follows from the fact that $\hat{\mathcal{C}}_{r,n}$ is a Zariski open subset of $\mathcal{C}_{r,n}$.

(2) We give a definition of *regular matrix* that works in our setting. We first remark that $\mathrm{rk}(AJB - BJA) = \mathrm{rk}[JA, JB]$, with $[-,-]$ the usual commutator of matrices. This leads us to introduce in the picture skew-Hamiltonian matrices, *i.e.*, matrices of the form JB, B skew-symmetric. This is done in Definition 4.2. We then prove in Proposition 4.4 that for a regular matrix JB the kernel of the homomorphism $\phi^B \colon \Lambda^2 V \to S^2 V$ sending a skew-symmetric A to the symmetric matrix $AJB - BJA$ has the smallest possible dimension, namely $\frac{n}{2}$.

(3) Fixing a skew-symmetric matrix B, we define

$$\mathcal{S}^B_{r,n} := \{S \in S^2 V \mid S = AJB - BJA \text{ for some } A \in \Lambda^2 V, \mathrm{rk}\, S \leq r\},$$

and in Lemma 4.6 we show that if JB is regular, then $\mathcal{S}^B_{r,n}$ is irreducible of dimension $nr - \frac{3}{2}n - \binom{r}{2}$ for $r = n \geq 4$ and for $r = n-1$ and $n \geq 8$.

(4) Lemma 4.9 is the second to last step. We define

$$\mathcal{C}^0_{r,n} := \{(A, B) \in \Lambda^2 V \times \Lambda^2 V \mid JB \text{ is regular}, \mathrm{rk}(AJB - BJA) \leq r\},$$

and we use a fibration argument to deduce the irreducibility of $\mathcal{C}^0_{r,n}$ from the irreducibility of $\mathcal{S}^B_{r,n}$.

(5) The last step consists in showing that $\mathcal{C}_{r,n}$ is the closure of $\mathcal{C}^0_{r,n}$. This concludes the proof of Lemma 4.1, as well as Section 4.

4.1 Regular Skew-Hamiltonians

For any pair of (skew-symmetric) matrices (A, B) we make the trivial observation that

$$\mathrm{rk}(AJB - BJA) = \mathrm{rk}[JA, JB],$$

where $[-,-]$ is the usual commutator of matrices. Hence studying symmetric matrices of the form $AJB - BJA$ and fixed (or bounded) rank is equivalent to studying the commutator of matrices of the form JB, where $B \in \Lambda^2 V$. Such matrices are called *skew-Hamiltonian* (or *anti-Hamiltonian*) in literature. Let us recall the following.

Definition-Proposition 4.1 Let W and H be elements of $V \otimes V$.

- W is called *skew-Hamiltonian* if $JW = {}^tWJ$. W is skew-Hamiltonian if and only $W = JB$, with $B \in \Lambda^2 V$ skew-symmetric. We indicate the space of skew-Hamiltonian matrices by $\mathcal{W}$.
- H is called *Hamiltonian* if $JH = -{}^tHJ$. H is Hamiltonian if and only $H = JS$, with $S \in S^2 V$ skew-symmetric. We indicate the space of Hamiltonian matrices by $\mathcal{H}$.

$\mathcal{W}$ and $\mathcal{H}$ correspond, respectively, to the Jordan algebra and to the Lie algebra of the symplectic group $\mathrm{Sp}(n)$.

We mentioned above that Lemma 4.1 has an unstructured as well as a symmetric analogue, proven respectively in [Hul80, Proposition 2.3.6] and in [Bas00, Theorem 2.6] and [BPV90, Corollary 3.6]. Both arguments make use of regular matrices: a *regular matrix* is a regular element of the Lie algebra, and in particular it is an element whose commutator has minimal dimension.

The proof in the symplectic case is particularly easy: given a symmetric matrix B, one defines the linear morphism $\phi^B\colon S^2 V \to \Lambda^2 V$, mapping any A to the commutator $[A, B]$. If B is regular the kernel of the morphism is n-dimensional, which means that ϕ^B is surjective. A standard fibration argument then allows one to conclude irreducibility of the space of symmetric matrices whose commutator has fixed rank.

Unfortunately the notion of regular element is meaningless for the Jordan algebra $\mathcal{W}$; therefore we give an "ad hoc" definition of regularity for skew-Hamiltonian matrices and justify our choice by proving that the dimension of the commutator of regular matrices is indeed minimal. In theory of structured matrices our definition corresponds to that of *non-derogatory* matrices, but we prefer to adopt the terminology "regular" for consistency with the unstructured and symplectic cases.

Remark 4.1 The symplectic group $\mathrm{Sp}(n)$ acts on skew-Hamiltonians $\mathcal{W}$ by conjugation. For $M \in \mathrm{Sp}(n)$ and $W \in \mathcal{W}$ one defines

$$M * W := M^{-1}WM, \tag{4.1}$$

and this action preserves sums and products.

By definition, $W \in \mathcal{W}$ if and only if ${}^tWJ = JW$. If $M \in \mathrm{Sp}(n)$ then ${}^tMJ = JM^{-1}$ and $JM = {}^tM^{-1}J$, and thus if W is skew-Hamiltonian, so is $M^{-1}WM$, because:

$$\begin{aligned} {}^t(M^{-1}WM)J &= {}^tM^tW({}^tM^{-1}J) = {}^tM({}^tWJ)M \\ &= ({}^tMJ)WM = J(M^{-1}WM). \end{aligned}$$

The fact that this action preserves sums and products is an immediate check.

The symplectic group $\mathrm{Sp}(n)$ also acts on skew-symmetric matrices $\Lambda^2 V$ by congruence. For $M \in \mathrm{Sp}(n)$ and $B \in \Lambda^2 V$:

$$M \star B := {}^t M B M, \tag{4.2}$$

and this action preserves sums and products.

Lemma 4.2 *The two actions* (4.1) *and* (4.2) *are consistent with each other.*

Proof Write a skew-Hamiltonian W as JB, with B skew-symmetric. Then, given M symplectic, its inverse M^{-1} is symplectic as well, and we have that

$$M * (JB) = M^{-1}(JB)M = J({}^t MBM) = J(M \star B). \qquad \blacksquare$$

Lemma 4.3 ([Wat05, Theorem 3]) *Let $W \in \mathcal{W}$ be a skew-Hamiltonian matrix of even size n. Up to symplectic conjugation W is of the form $\left[\begin{smallmatrix} P & 0 \\ 0 & {}^t P \end{smallmatrix}\right]$ for some $\frac{n}{2} \times \frac{n}{2}$ matrix $P \in \mathrm{Mat}(\frac{n}{2}, \mathbb{C})$.*

Definition 4.2 Let $W \in \mathcal{W}$ be a skew-Hamiltonian matrix. We call W *regular* if the minimal polynomial of P in Lemma 4.3 has degree $\frac{n}{2}$. We denote the set of regular skew-Hamiltonians by $\mathcal{W}_{\mathrm{reg}}$.

If W is regular then P is a regular element of the Lie algebra $\mathfrak{gl}_{\frac{n}{2}}$. In particular, for each of its distinct eigenvalues there is only one corresponding Jordan block in its Jordan normal form. This is equivalent to asking for the minimal polynomial of P to coincide (up to sign) with the characteristic polynomial.

Proposition 4.4 *Let $JB \in \mathcal{W}_{\mathrm{reg}}$ be a regular skew-Hamiltonian. Then*

$$\{JA \in \mathcal{W} \mid [JA, JB] = 0\} = \Big\langle (JB)^k \mid k = 0, \dots, \Big(\frac{n}{2} - 1\Big) \Big\rangle. \tag{4.3}$$

In particular, the centralizer of JB has minimal dimension $\frac{n}{2}$.

Proof The inclusion $\supseteq$ is immediate. Equality follows for dimensional reasons. Indeed, by Lemma 4.3 there is a symplectic matrix $M \in \mathrm{Sp}(n)$ such that $M^{-1}(JB)M = \left[\begin{smallmatrix} P & 0 \\ 0 & {}^t P \end{smallmatrix}\right]$ for some regular P. Let us look at all matrices $C = \left[\begin{smallmatrix} C_1 & C_2 \\ C_3 & C_4 \end{smallmatrix}\right]$ that commute with JB. Imposing that

$$[C, JB] = \begin{bmatrix} [C_1, P] & C_2{}^t P - PC_2 \\ C_3 P - {}^t P C_3 & [C_4, {}^t P] \end{bmatrix} = 0$$

means that $C_2 = C_3 = 0$, while, since P is regular, C_1 and C_4 are parametrized by $\frac{n}{2}$ degrees of freedom each. Imposing to C the extra condition of being skew-Hamiltonian means imposing to $JC = \left[\begin{smallmatrix} 0 & C_4 \\ -C_1 & 0 \end{smallmatrix}\right]$ to be skew-symmetric, and this halves the degrees of freedom to $\frac{n}{2}$. Hence the dimension of the left-hand side in (4.3) equals the dimension of the right-hand side, and they are both equal to $\frac{n}{2}$. $\blacksquare$

It is worthwhile recalling part of [Bas00, Proposition 2.2], where it is proved that $(\Lambda^2 V)_{\mathrm{reg}} = \Lambda^2 V \cap \{\text{regular matrices}\}$ is the open subset of all elements having centralizer of minimal dimension $\frac{n}{2}$.

4.2 Irreducibility of $\mathcal{S}^B_{r,n}$ and Diamond Matrices

For any skew-symmetric matrix $B \in \Lambda^2 V$, consider the vector space homomorphism

$$\begin{aligned} \Lambda^2 V &\xrightarrow{\phi^B} S^2 V \\ A &\longmapsto AJB - BJA, \end{aligned}$$

and define

$$\mathcal{S}^B_n := \operatorname{Im} \phi^B = \{ S \in S^2 V \mid S = AJB - BJA \text{ for some } A \in \Lambda^2 V \}.$$

If JB is regular, since $AJB - BJA = -J[JA, JB]$, then by Lemma 4.4 the defect of the map ϕ^B is $\frac{n}{2}$, and $\mathcal{S}^B_n$ is a linear subspace of symmetric matrices $S^2 V$ of codimension

$$\binom{n+1}{2} - \left[\binom{n}{2} - \frac{n}{2} \right] = \frac{3}{2} n.$$

We can give explicit equations for $\mathcal{S}^B_n$, for which we need the following definition from [Nof13].

Definition 4.3 Let $M = (m_{ij})$ be a $d \times d$ square matrix, and let $k \in \mathbb{Z}$, $-d < k < d$.

(i) The *k-th trace* of M is the sum $\sum_{i=1}^{d-k} m_{i,i+k}$ if $k \geq 0$, or $\sum_{j=1}^{d+k} m_{j-k,j}$ if $k \leq 0$. The usual trace of a matrix corresponds to the 0-th trace.
If the k-th trace is zero for all $k \geq 0$ (respectively $k \leq 0$), M is called *supertraceless* (resp. *subtraceless*).

(ii) The *k-th antitrace* of M is the sum $\sum_{i+j=d+1-k} m_{i,j}$.
If the k-th antitrace is zero for all $k \geq 0$ (respectively $k \leq 0$), M is called *superantitraceless* (resp. *subantitraceless*).

Definition 4.4 Given any partition $\underline{d} = (d_1, \ldots, d_m)$ of $\frac{n}{2}$, the *$\underline{d}$-block partition* of a $\frac{n}{2} \times \frac{n}{2}$ matrix X is the set of blocks X_{ij}, for $i, j = 1, \ldots, m$, such that X_{ij} is a $d_i \times d_j$ submatrix of X and

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \cdots & X_{mm} \end{bmatrix}.$$

Definition 4.5 An $n \times n$ matrix $Y = \left[\begin{smallmatrix} Y_1 & Y_2 \\ Y_4 & Y_3 \end{smallmatrix} \right] \in V \otimes V$ is called a *diamond matrix* if there exists a partition $\underline{d}$ of $\frac{n}{2}$ such that

- the diagonal blocks in the $\underline{d}$-block partition of Y_1 are superantitraceless,
- those of Y_2 are supertraceless,
- those of Y_3 are subantitraceless, and
- those of Y_4 are subtraceless.

The n linear conditions imposing the vanishing of traces and antitraces are called the *diamond conditions* ($\diamond$-conditions), and each of them vanishes on a *$\diamond$-hyperplane*.

To understand the origin of the terminology "diamond", the reader should look at Figures 1 and 2, where diamond matrices corresponding respectively to the partition

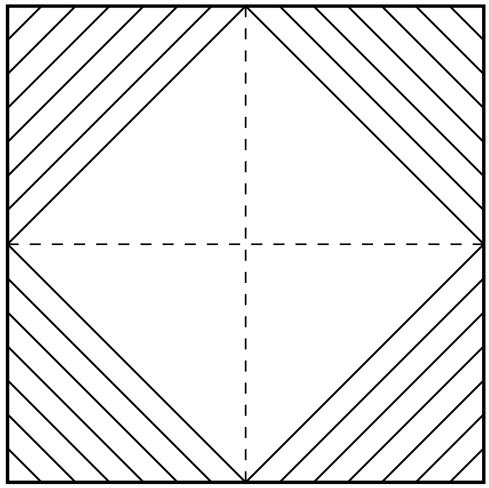

Figure 1: A diamond matrix corresponding to the partition $\underline{d} = (\frac{n}{2})$.

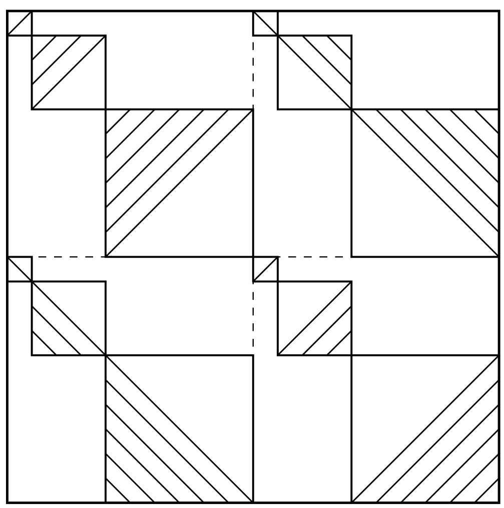

Figure 2: A diamond matrix corresponding to a partition $\underline{d} = (d_1, d_2, d_3)$.

$\underline{d} = (\frac{n}{2})$ and to $\underline{d} = (d_1, d_2, d_3)$ are shown. The diagonal lines represent the traces and antitraces that are zero.

Proposition 4.5 ([Nof13, Cor. 3.13]) *Let $B \in \Lambda^2 V$ be such that $JB \in \mathcal{W}_{\mathrm{reg}}$ is a regular skew-Hamiltonian, and let $A \in \Lambda^2 V$ be any skew-symmetric matrix. Then $AJB - BJA$ is symplectically congruent to a diamond matrix.*

Proof We start by making the observation that the morphism ϕ^B is $\mathrm{Sp}(n)$-equivariant under the congruence action. (It is easy to check that the group $\mathrm{Sp}(n)$ acts by congruence not only on skew-symmetric, but also on symmetric matrices.)

Given any skew-symmetric matrix A one has

$$M \star \phi^B(A) = {}^tM\big(\phi^B(A)\big)M = \phi^{{}^tMBM}({}^tMAM) = \phi^{M\star B}(M \star A).$$

Now put JB in its Jordan normal form via symplectic conjugation. The Jordan normal form of JB will consist in m Jordan blocks, each of dimension d_i with $\sum_{i=1}^m d_i = \frac{n}{2}$. Set $\underline{d} = (d_1, \dots, d_m)$ as before. For any $A \in \Lambda^2 V$, consider the $\underline{d}$-block partitions of the four quadrants of $S = \phi^B(A)$. A direct computation now shows that S is a diamond matrix. ∎

Remark 4.2 Notice that for a symmetric matrix the $\diamond$-conditions reduce to $\frac{3}{2}n$ conditions, simply because if the blocks of S_2 are supertraceless, then the ones of ${}^tS_2$ will automatically be subtraceless.

Denote by $S^2V_{\le r}$ the determinantal variety

$$S^2V_{\le r} := \{S \in S^2V \mid \operatorname{rk} S \le r\},$$

which is irreducible of dimension $nr - \binom{r}{2}$. There are several proofs of this fact, we briefly recall here the one due to Kempf [Kem76].

Let $\mathbb{G}(n-r, n)$ be the Grassmannian of $n-r$-dimensional subspaces of the n-dimensional vector space V, and consider the incidence variety

$$\tilde{X} := \{(L, S) \in \mathbb{G}(n-r, n) \times S^2V \mid L \subseteq \operatorname{Ker} S\} \tag{4.4}$$

that comes with the two standard projections:

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{G}(n-r, n) & & S^2V. \end{tikzcd}$$

It is an affine bundle over the Grassmannian, whose fibers are vector spaces of dimension equal to $\dim S^2(V/L) = \binom{r+1}{2}$. The canonical projection of $\tilde{X}$ on S^2V is surjective and the restriction of it to the inverse image of the open subset of all symmetric matrices of rank r is injective; hence $S^2V_{\le r}$ is irreducible of dimension:

$$\dim S^2V_{\le r} = \dim \tilde{X} = \dim \mathbb{G}(n-r, n) + \binom{r+1}{2} = nr - \binom{r}{2}.$$

Intersecting the linear space $\mathcal{S}_n^B$ with the determinantal variety $S^2V_{\le r}$ we define

$$\mathcal{S}_{r,n}^B := \mathcal{S}_n^B \cap S^2V_{\le r} = \{S \in S^2V \mid S \in \mathcal{S}_n^B, \operatorname{rk} S \le r\}.$$

Lemma 4.6 *If $B \in \Lambda^2 V$ is such that $JB \in \mathcal{W}_{\mathrm{reg}}$ is regular, then $\mathcal{S}^B_{r,n}$ is irreducible of dimension $nr - \frac{3}{2}n - \binom{r}{2}$ for $r = n$ and $n \geq 4$, and for $r = n-1$ and $n \geq 8$.*

Proof The proof constitutes the remainder of this section. The case $r = n$ is trivial, hence in what follows we will concentrate on the case $r = n-1$.

Similarly to what is done above, define the incidence variety

$$\tilde{X}_r = \{(L, S) \in \mathbb{G}(n-r, n) \times S^2 V_{\leq r} \mid L \subseteq \operatorname{Ker} S\}.$$

Again, we have the two standard projections:

$$\begin{tikzcd} & \tilde{X}_r \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{G}(n-r, n) & & S^2 V_{\leq r} & \mathcal{S}^B_{r,n}. \arrow[l, hook'] \end{tikzcd}$$

In order to prove Lemma 4.6, it is enough to show that $p_2^{-1}(\mathcal{S}^B_{r,n})$ is irreducible. We will do so for $r = n-1$.

By definition, the variety $\mathcal{S}^B_{r,n}$ is the intersection of $S^2 V_{\leq r}$ with $\mathcal{S}^B_n$, which is a vector space of codimension $\frac{3}{2}n$. Let $K \subset S^2 V$ be a general vector space of codimension $\frac{3}{2}n$. Since $\frac{3}{2}n < nr - \binom{r}{2}$, we can apply Bertini's theorem for a proper morphism, see [Laz04, Theorem 3.3.1], and we obtain that $p_2^{-1}(K)$ is irreducible of dimension $nr - \binom{r}{2} - \frac{3}{2}n$. Then, using [Gro68, Exposé XIII, Corollary 2.2], we deduce that $p_2^{-1}(\mathcal{S}^B_n) = p_2^{-1}(\mathcal{S}^B_{r,n})$ is connected in dimension $nr - \binom{r}{2} - \frac{3}{2}n - 1$.

This connectedness result allow us to reduce the proof of the irreducibility of $p_2^{-1}(\mathcal{S}^B_{r,n})$ to showing that it has the expected dimension $\dim S^2 V_{\leq r} - \frac{3}{2}n$, and that it is smooth in codimension 1, that is, its singular locus is in codimension at least 2.

To prove that this is indeed true, we study the projection

$$p_1|_{p_2^{-1}(\mathcal{S}^B_{r,n})} \colon p_2^{-1}(\mathcal{S}^B_{r,n}) \to \mathbb{G}(n-r, n).$$

Recall that $p_2^{-1}(\mathcal{S}^B_{r,n})$ is defined as

$$p_2^{-1}(\mathcal{S}^B_{r,n}) = \{(L, S) \in \tilde{X}_r \mid S: \text{ satisfies the } \diamond\text{-conditions}\}.$$

Hence given an element L of the Grassmannian $\mathbb{G}(n-r, n)$, the fiber of $p_1|_{p_2^{-1}(\mathcal{S}^B_{r,n})}$ over L is

$${p_1}^{-1}|_{p_2^{-1}(\mathcal{S}^B_n)}(L) = \{S \in S^2 V_{\leq r} \mid L \subseteq \operatorname{Ker} S, S \text{ satisfies the } \diamond\text{-conditions}\}.$$

We wish to identify the elements L in the Grassmannian whose fiber

$${p_1}^{-1}|_{p_2^{-1}(\mathcal{S}^B_{r,n})}(L)$$

is not a linear space of the expected dimension $\binom{r+1}{2} - \frac{3}{2}n$. In other words, we are looking for all L in $\mathbb{G}(n-r, n)$ for which the conditions "$L \subseteq \operatorname{Ker} S$" are not independent from the $\diamond$-conditions.

Let us denote by $\{e_1, \dots, e_n\}$ the basis of V with respect to which the $\diamond$-conditions are represented. We recall that it is the basis in which the regular skew-Hamiltonian JB is in its Jordan normal form.

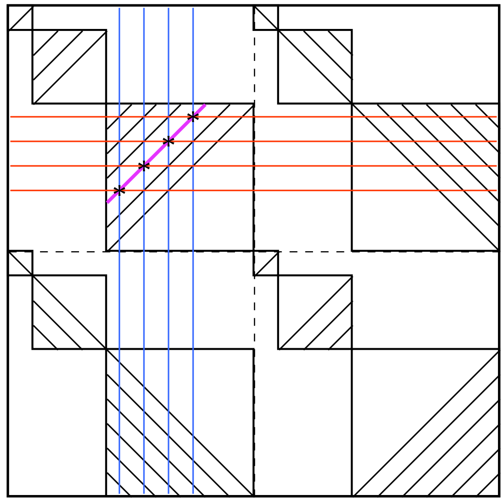

Figure 3: Representation of a diagonal condition generated by vertical and horizontal ones.

From now on, we set $r = n - 1$**.** Let $L \in \mathbb{G}(1, n)$ be generated by $L = \langle \ell \rangle$, with $\ell = \sum_{k=1}^{n} \lambda_k e_k$. Then if $S = (s_{pq})$, the condition $L \subseteq \operatorname{Ker} S$ translates into the n conditions $\sum_{k=1}^{n} \lambda_k s_{kj} = 0$, one for each $j = 1, \dots, n$, that we denote by $\sharp^{j}$.

Notice that each $\sharp^{j}$ involves the n entries of the j-th column of the matrix S. For this reason, we will think of them as *vertical* conditions.

The symmetry of the matrix S yields that for any vertical condition $\sharp^{j}$ there is a *horizontal* one, involving the j-th row of S. We denote such horizontal condition by $\sharp_{j}$, with obvious notation.

In the same mindset, the $\diamond$-conditions, involving traces and anti-traces, are thought of as *diagonal* conditions. The number of entries of S that are involved in a diagonal condition varies from 1 to $n/2$.

The notation has been chosen in order to remind the reader whether a particular condition is diagonal ($\diamond$), or vertical and horizontal ($\sharp$).

Consider now a $\diamond$-condition d. Notice that it involves entries of the matrix S that are all in one of the four quadrants. We say that d *is generated* by $\sharp$-conditions if there exist some vertical and horizontal conditions v_i and h_j such that it is possible to write $d = \sum \alpha_i v_i + \sum \beta_j h_j + \delta$, where δ is a linear form that only involves entries of S that are not in the same quadrant of d.

Lemma 4.7 *To generate a diagonal $\diamond$-condition involving z entries, one needs x vertical conditions of type $\sharp^{i}$ and y horizontal conditions of type $\sharp\ j$, with $z = x + y$.*

Proof In light of the remarks above, the proof is almost immediate.

We fix any integer $1 \le z \le \frac{n}{2}$. A $\diamond$-condition involving z entries of S is either a trace or an antitrace in one of the m blocks in which S is block-partitioned. More in detail, there is an index $1 \le \iota \le m$ such that this $\diamondsuit$-condition is either a $\pm(d_\iota - z)$-th trace or a $\pm(d_\iota - z)$-th antitrace. We only look at the case when it is a $(d_\iota - z)$-th antitrace, since the proof goes through verbatim in the other cases. In order for this antitrace condition to be generated by the $\sharp$-conditions, we need to see where its z entries appear. Recall from Definition 4.3(2) that they are entries of type s_{pq}, with $p + q = z + 1$.

Each of these z entries s_{pq} can show up in at most one vertical condition, namely $\sharp^q$, and at most one horizontal condition, namely $\sharp_p$. And in each of these the entry s_{pq} appears exactly once, that is, for $k = z + 1 - q$ and $k = z + 1 - p$ respectively. Hence the statement follows. ∎

We now make the following claims.

Claim 1. Let the Jordan normal form of JB consist in m Jordan blocks, each of dimension d_i with $\sum_{i=1}^m d_i = n/2$. For every $i = 1, \dots, m$ define $\delta_i := \sum_{j=1}^{i-1} d_j$. Let $L = \langle \ell \rangle$ be an element in the Grassmannian $\mathbb{G}(1, n)$. Then the condition $L \subseteq \operatorname{Ker} S$ is non-transverse with the $\diamondsuit$-conditions if and only if

$$L \subset \bigcup_{i=1}^{m} \langle e_{\delta_i+1}, e_{\delta_{i+1}+\frac{n}{2}} \rangle. \tag{4.5}$$

Proof of Claim 1 The implication $(\Leftarrow)$ is immediate. If $L \subseteq \bigcup_{i=1}^m \langle e_{\delta_i+1}, e_{\delta_{i+1}+\frac{n}{2}} \rangle$, then there is an index $1 \le \alpha \le m$ such that the generator $\ell \in \langle e_{\delta_\alpha+1}, e_{\delta_{\alpha+1}+\frac{n}{2}} \rangle$. This means that the condition $\sharp^{\delta_\alpha+1}$ reads

$$\lambda_{\delta_\alpha+1} s_{\delta_\alpha+1,\delta_\alpha+1} + \lambda_{\delta_{\alpha+1}+\frac{n}{2}} s_{\delta_{\alpha+1}+\frac{n}{2},\delta_\alpha+1} = 0.$$

Divide $S = \begin{bmatrix} S_1 & S_2 \\ {}^tS_2 & S_3 \end{bmatrix}$ into four quadrants as usual. Among the $\diamondsuit$-conditions, the vanishing of the d_α−1-th antitrace of the block $(S_1)_{\alpha\alpha}$ entails that the entry $s_{\delta_\alpha+1,\delta_\alpha+1} = 0$, while the vanishing of the $-d_\alpha + 1$-th trace of the block $({}^tS_2)_{\alpha\alpha}$ entails that the entry $s_{\delta_{\alpha+1}+\frac{n}{2},\delta_\alpha+1} = 0$. Hence $\sharp^{\delta_\alpha+1}$ is trivially satisfied, and the $\diamondsuit$-conditions are not independent from the conditions $L \subset \operatorname{Ker} S$.

To prove $(\Rightarrow)$, we show that if L is not included in $\bigcup_{i=1}^m \langle e_{\delta_i+1}, e_{\delta_{i+1}+\frac{n}{2}} \rangle$, then none of the conditions generating $L \subseteq \operatorname{Ker} S$ can be obtained from a combination of $\diamondsuit$-conditions. Indeed, assume that L is not included in $\bigcup_{i=1}^m \langle e_{\delta_i+1}, e_{\delta_{i+1}+\frac{n}{2}} \rangle$, then there exist indices $1 \le \beta < \gamma \le n$ such that:

(i) either β is different from all $\delta_i + 1$ for $i = 1, \dots, m$, and the coefficient $\lambda_\beta \ne 0$,

(ii) or if $\beta = \delta_\alpha + 1$ for an index $1 \le \alpha \le m$, then $\gamma \ne \delta_{\alpha+1} + \frac{n}{2}$, and both coefficients λ_β and λ_γ are non-zero.

In both instances, in every $\sharp^j$ (and in every $\sharp_j$) the entries $s_{\beta j}$ (and respectively $s_{j\beta}$) will show up with non-zero coefficients. Recall that these same entries do not show up in any other $\sharp$-conditions.

Moreover in both instances the β-row of the matrix S will cut one of the blocks in which S is partitioned, say the ι-th block. Then as soon as j is either smaller than δ_ι, or bigger than $\delta_{\iota+1}$, none of the indices $s_{\beta j}$ will appear in a $\diamondsuit$-condition, and this concludes the proof. ∎

Claim 1 can be re-interpreted as saying that transversality fails whenever there is a whole line contained in the intersection (4.5). We call these lines *bad lines*. Notice that the bad lines form a subvariety of dimension 1.

Claim 2. Let L be an element of the Grassmannian $\mathbb{G}(1,n)$. Then the dimension of the fiber ${p_1}^{-1}|_{p_2^{-1}(\mathcal{S}^B_{n-1,n})}(L)$ is bounded above by

$$\dim\big({p_1}^{-1}|_{p_2^{-1}(\mathcal{S}^B_{n-1,n})}(L)\big) \le \binom{n}{2} - \Big(\frac{3}{2}n - 3\Big),$$

where $\binom{n}{2} = \dim S^2(V/L)$ is the dimension of the fiber of the affine bundle $\tilde{X} \to \mathbb{G}(1,n)$ defined in (4.4).

Proof of Claim 2 To prove this inequality, we have to evaluate the number of $\diamond$-conditions which can be generated by the condition $L \subset \operatorname{Ker} S$. A close look at Figure 4.2 should convince the reader that either two or three among the $\diamond$-conditions are generated by the condition $L \subset \operatorname{Ker} S$ if and only if

$$L \subset \bigcup_{i=1}^{m} \langle e_{\delta_i+1}, e_{\delta_{i+1}+\frac{n}{2}} \rangle.$$

Indeed, if L is not included in $\bigcup_{i=1}^{m} \langle e_{\delta_i+1}, e_{\delta_{i+1}+\frac{n}{2}} \rangle$, then by using Claim 1 we deduce that the $\diamond$-conditions and the $L \subset \operatorname{Ker} S$ are independent.

Conversely, assume that L is generated by $\mu e_{\delta_i+1} + \lambda e_{\delta_{i+1}+\frac{n}{2}}$ for some $1 \le i \le m$ and $\mu, \lambda \in \mathbb{C}$ nonsimultaneously zero.

Suppose that $d_i = 1$. Then the linear space $M_{L\subset \operatorname{Ker} S}$ generated by the conditions

$$\begin{gathered} \mu s_{1,\delta_i+1} + \lambda s_{1,\delta_{i+1}+\frac{n}{2}} = 0 \\ \vdots \\ \mu s_{n,\delta_i+1} + \lambda s_{n,\delta_{i+1}+\frac{n}{2}} = 0 \end{gathered}$$

intersects the linear space generated by the conditions

$$\begin{aligned} s_{\delta_i+1,\delta_i+1} &= 0 \\ s_{\delta_i+1,\delta_{i+1}+\frac{n}{2}} &= 0 \\ s_{\delta_{i+1}+\frac{n}{2},\delta_{i+1}+\frac{n}{2}} &= 0 \end{aligned}$$

in dimension 2, and no other $\diamond$-conditions are included in $M_{L\subset \operatorname{Ker} S}$.

If instead $d_i > 1$, then that same linear space $M_{L\subset \operatorname{Ker} S}$ intersects the linear space generated by

$$\begin{aligned} s_{\delta_i+1,\delta_i+1} &= 0 \\ s_{\delta_i+2,\delta_i+1} + s_{\delta_i+1,\delta_i+2} &= 0 \\ s_{\delta_i+1,\delta_{i+1}+\frac{n}{2}} &= 0 \\ s_{\delta_i+1,\delta_{i+1}+\frac{n}{2}-1} + s_{\delta_i+2,\delta_{i+1}+\frac{n}{2}} &= 0 \\ s_{\delta_{i+1}+\frac{n}{2},\delta_{i+1}+\frac{n}{2}} &= 0 \\ s_{\delta_{i+1}+\frac{n}{2},\delta_{i+1}+\frac{n}{2}-1} + s_{\delta_{i+1}+\frac{n}{2}-1,\delta_{i+1}+\frac{n}{2}} &= 0 \end{aligned}$$

in dimension 3, and no other $\diamond$-conditions are included in $M_{L\subset \operatorname{Ker} S}$. This concludes the proof of Claim 2. ∎

The two claims allow us to give an estimate of the dimension of the locus $\mathcal{B}_{n-1,n} \subset p_2^{-1}(\mathcal{S}^B_{n-1,n})$ where the intersection of the $\diamond$-conditions with the $\sharp$-conditions is non-transverse. Such dimension is bounded above by the sum of the dimension of the variety of bad lines and the dimension of the fiber. So altogether

$$\dim \mathcal{B}_{n-1,n} \leq 1 + \binom{n}{2} - \Big(\frac{3}{2}n - 3\Big). \tag{4.6}$$

On the other hand

$$\dim p_2^{-1}(\mathcal{S}^B_{n-1,n}) = \dim S^2 V_{\leq n-1} - \frac{3}{2}n = n(n-1) - \binom{n-1}{2} - \frac{3}{2}n, \tag{4.7}$$

and combining (4.6) and (4.7) we get

$$\operatorname{codim}_{p_2^{-1}(\mathcal{S}^B_{n-1,n})}(\mathcal{B}_{n-1,n}) \geq n - 5.$$

We can thus conclude that the codimension of $\mathcal{B}_{n-1,n}$ in $p_2^{-1}(\mathcal{S}^B_{n-1,n})$ is strictly greater than 2 as soon as $n \geq 8$. In this range we obtain the irreducibility of $p_2^{-1}(\mathcal{S}^B_{n-1,n})$, and hence that of $\mathcal{S}^B_{n-1,n}$, which is what we wanted. This completes the proof of Lemma 4.6. ■

4.3 Proof of the Key Lemma

We need the following well-known irreducibility criterion.

Lemma 4.8 *Let $f\colon X \to Y$ be a surjective morphism of algebraic varieties. Suppose that Y is irreducible, and that all the fibers of f are irreducible of the same dimension d. If X is pure-dimensional, then it is irreducible of dimension* $\dim Y + d$.

Lemma 4.9 *Let V be a complex vector space of even dimension n. The subvariety*

$$\mathcal{C}^0_{r,n} := \{(A, B) \in \Lambda^2 V \times \Lambda^2 V \mid \operatorname{rk}(AJB - BJA) \leq r \text{ and } JB \text{ is regular}\}$$

is irreducible of dimension $nr - n + \binom{n}{2} - \binom{r}{2}$, for $r = n \geq 4$ and for $r = n-1$, and $n \geq 8$.

Proof Let $\mathcal{W}_{\mathrm{reg}}$ be the open subset of all regular skew-Hamiltonians. We have a map

$$\mathcal{C}^0_{r,n} \xrightarrow{p} \mathcal{W}_{\mathrm{reg}},$$

$$(A, B) \longmapsto JB$$

By the affine Bezout theorem, $\mathcal{S}^B_{r,n}$ is non-empty, and p is surjective. Consider now a fiber $F^B = p^{-1}(B) = \{A \in \Lambda^2 V \mid \operatorname{rk}[JA, JB] \leq r\}$. There is an epimorphism

$$F^B \xrightarrow{q} \mathcal{S}^B_{r,n},$$

$$A \longmapsto -J[JA, JB]$$

Since JB is regular, by Lemma 4.4 F^B is an affine $\mathbb{C}^{n/2}$-bundle over $\mathcal{S}^B_{r,n}$, and hence it is irreducible of dimension

$$\dim F^B = \dim \mathcal{S}^B_{r,n} + \frac{n}{2} = nr - n - \binom{r}{2}.$$

Thus for every irreducible component C of $\mathcal{C}^0_{r,n}$,

$$\dim C \le \binom{n}{2} + \dim F^B = \binom{n}{2} + nr - n - \binom{r}{2}.$$

Counting equations one sees that the dimension of such components must be at least

$$2\binom{n}{2} - \left[\binom{n+1}{2} - \left(nr - \binom{r}{2}\right)\right] = \binom{n}{2} + nr - n - \binom{r}{2}.$$

All in all $\mathcal{C}^0_{r,n}$ fibers over the irreducible variety $\mathcal{W}_{\mathrm{reg}}$, all the fibers are irreducible of the same dimension, and $\mathcal{C}^0_{r,n}$ is pure-dimensional. Hence Lemma 4.8 applies, and the statement follows. ∎

Proof of Lemma 4.1 We are left to prove that $\mathcal{C}_{r,n} = \overline{\mathcal{C}^0_{r,n}}$. Take a pair (A, B) with $\mathrm{rk}[JA, JB] = r$ where JB is not regular. It is possible to find a regular skew-Hamiltonian JB' such that $[JA, JB'] = 0$. Then $JB + t B'$ will be regular for all but finitely many t and $[JA, JB + t\, JB'] = [JA, JB]$. This concludes the proof. ∎

5 Dominant Maps and Low-dimensional Cases

The study of the varieties $\mathcal{C}_{r,n}$ has led us to prove a somewhat curious result, that we have not been able to find anywhere in literature, and that we therefore report here. Similar results were recently obtained by [Nof13] using completely different techniques.

Proposition 5.1 *Let V be a complex vector space of even dimension n. Define ϕ as*

$$\mathbb{P}(\Lambda^2 V \times \Lambda^2 V) \overset{\phi}{\dashrightarrow} \mathbb{P}(S^2 V),$$

$$([A],[B]) \longmapsto [AJB - BJA].$$

(i) *For $n = 2$, ϕ is not defined.*
(ii) *For $n = 4$, $\mathrm{Im}\,\phi$ is the Grassmannian $\mathbb{G}(2,5)$ in $\mathbb{P}^9 = \mathbb{P}(S^2\mathbb{C}^4)$.*
(iii) *For $n = 6$, $\mathrm{Im}\,\phi$ is a hypersurface of degree 4 in $\mathbb{P}^{20} = \mathbb{P}(S^2\mathbb{C}^6)$.*
(iv) *For $n \ge 8$, ϕ is locally of maximal rank. In particular, ϕ is dominant.*

Proof (i) By direct computation, $AJB - BJA = 0$ for all $A, B \in \Lambda^2 V$.

(ii) We look at the bilinear map ϕ as a tensor map:

$$\mathbb{P}(\Lambda^2\mathbb{C}^4) \times \mathbb{P}(\Lambda^2\mathbb{C}^4) \hookrightarrow \mathbb{P}(\Lambda^2\mathbb{C}^4 \otimes \Lambda^2\mathbb{C}^4) \dashrightarrow \mathbb{P}(S^2\mathbb{C}^4).$$

Being skew-symmetric, this map factorizes through

$$\mathbb{P}\big(\Lambda^2(\Lambda^2\mathbb{C}^4)\big) \overset{\Phi}{\dashrightarrow} \mathbb{P}(S^2\mathbb{C}^4).$$

The image of ϕ is now the image through Φ of the Grassmannian of 2-planes in $\mathbb{C}^6 \simeq \Lambda^2\mathbb{C}^4$: $\mathbb{G}(2,6) \hookrightarrow \mathbb{P}\big(\Lambda^2(\Lambda^2\mathbb{C}^4)\big) = \mathbb{P}(\Lambda^2\mathbb{C}^6)$.

The same argument that we used in the proof of Proposition 4.5 shows that map is $\mathrm{Sp}(4)$-equivariant. We can thus look at the decomposition of $\Lambda^2(\Lambda^2\mathbb{C}^4)$ and

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$S^2\mathbb{C}^4$ as $\mathrm{Sp}(4)$-modules. We see that $S^2\mathbb{C}^4$ is irreducible, while $\Lambda^2\mathbb{C}^4$ decomposes as $\Lambda_0^2\mathbb{C}^4 \oplus \mathbb{C} \simeq \mathbb{C}^5 \oplus \mathbb{C}$, so that

$$\Lambda^2(\Lambda^2\mathbb{C}^4) = \Lambda^2(\Lambda_0^2\mathbb{C}^4 \oplus \mathbb{C}) = \Lambda^2(\Lambda_0^2\mathbb{C}^4) \oplus \Lambda_0^2\mathbb{C}^4 \simeq \mathbb{C}^{10} \oplus \mathbb{C}^5. \tag{5.1}$$

Schur's lemma now tells us that Φ maps $\mathbb{P}\big(\Lambda^2(\Lambda_0^2\mathbb{C}^4)\big)$ isomorphically on $\mathbb{P}S^2\mathbb{C}^4$. Moreover notice that the two summands in (5.1) above correspond to a $\mathbb{P}^9$ and a $\mathbb{P}^4$ disjoint in $\mathbb{P}^{14} = \mathbb{P}\big(\Lambda^2(\Lambda^2\mathbb{C}^4)\big)$.

We have a Grassmannian $\mathbb{G}(2,5) \hookrightarrow \mathbb{P}\big(\Lambda^2(\Lambda_0^2\mathbb{C}^4)\big) = \mathbb{P}(\Lambda^2\mathbb{C}^5)$. Moreover remark that the $\mathbb{P}^4 = \mathbb{P}(\Lambda_0^2\mathbb{C}^4)$ is entirely contained in $\mathbb{G}(2,6)$, simply because all elements of $\Lambda_0^2\mathbb{C}^4$ will be rank 2 tensors in $\mathbb{P}\big(\Lambda^2(\Lambda^2\mathbb{C}^4)\big)$.

This means that we can identify ϕ with the projection $\mathbb{G}(2,6) \dashrightarrow \mathbb{P}^9 = \mathbb{P}(\Lambda_0^2\mathbb{C}^4)$ from $\mathbb{P}^4 = \mathbb{P}(\Lambda_0^2\mathbb{C}^4)$. But since $\mathbb{P}^4 \subset \mathbb{G}(2,6)$, the projection is induced by the projection $\mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ and necessarily maps $\mathbb{G}(2,6)$ in the Grassmannian $\mathbb{G}(2,5)$, concluding our proof.

In [Nof13, Proposition 4.1] it is shown that for $n = 4$ the $\operatorname{Im}\phi$ coincides with the set of Hamiltonian square roots of the identity times a scalar.

As an addendum, we point out that the image of ϕ the union of the orbits C, G and H listed in Table I of [OSM94].

(iii) Let $S \in \mathbb{P}(S^2\mathbb{C}^6)$ be a symmetric matrix. The determinant of $S - tJ$ has the form

$$\det(S - tJ) = t^6 + \gamma_4(S)t^4 + \gamma_2(S)t^2 + \det(S),$$

with only even terms. The image of ϕ is a hypersurface of degree 4 with equation $\gamma_4^2 + 4\gamma_2 = 0$.

This can be proved either by direct computation, computing all possible normal form of the matrices in $\operatorname{Im}\phi$, as is done in [Nof13, Proposition 4.2], or else checked with the computer algebra system Macaulay2 [GS]. To understand where does the equation come from, notice that Lemma 4.3 together with the equivariancy of the map ϕ guarantees that, up to the symplectic action, we can take the matrix B to be of the form $B = \left[\begin{smallmatrix} 0 & \beta \\ -{}^t\beta & 0 \end{smallmatrix}\right]$. Let us suppose now that A is also in the block form $A = \left[\begin{smallmatrix} 0 & \alpha \\ -\alpha^T & 0 \end{smallmatrix}\right]$.

Then $AJB - BJA = \left[\begin{smallmatrix} 0 & [\beta,\alpha] \\ [\beta,\alpha]^T & 0 \end{smallmatrix}\right]$ and we can compute

$$\begin{aligned} \det\big((AJB - BJA) - tJ\big) &= \det([\beta,\alpha] - tI)\det([\beta,\alpha]^T + tI) \\ &= (-t^3 + at + b)(t^3 - at + b) \\ &= -t^6 + 2at^4 - a^2x^2 + b^2, \end{aligned}$$

for some coefficients a and b.

(iv) We perform a local computation and then induction on $p = n/2$. A different proof of the fact that ϕ is dominant can be found in [Nof13, Proposition 4.4], where the author proves that for $n \geq 8$ the set of symmetric matrices with all distinct eigenvalues is inside $\operatorname{Im}\phi$.

So let us start our induction. For the low values of p that are the starting point of our induction, namely $p = 4$ and 5, the statement can be checked directly with the aid of the computer algebra system Macaulay2 [GS].

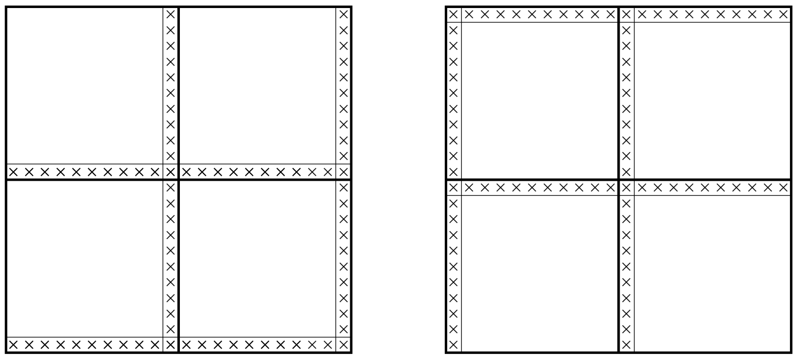

Figure 4: Shape of the matrices M^1 (left) and M_2 (right).

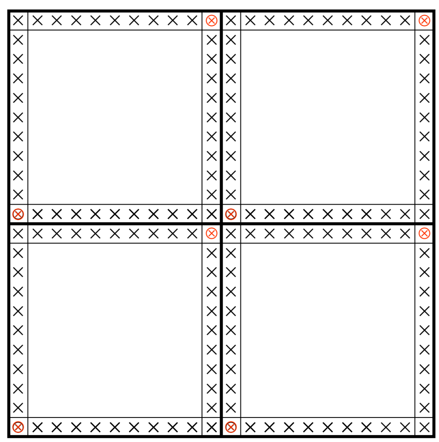

Figure 5: Shape of the matrices M^1_2.

Assume now $p \geq 6$. Let $M \in \Lambda^2 V$ and consider the submatrices M^1 and M_2 obtained by removing respectively the p-th and the $2p$-th rows and columns and the first and $(p+1)$-th rows and columns, as illustrated in Figure 5.

Notice that M^1 and M_2 are $(n-1)$–dimensional skew-symmetric matrices and that the symplectic form J is "preserved" under this cropping. Moreover the submatrix obtained by removing from M all eight rows and columns combined is an $(n-2)$–dimensional skew-symmetric matrix as well and the symplectic form J is "preserved" in this case, too. Call this submatrix M^1_2. It has the form illustrated in Figure 5.

If we consider the sum M^1+M_2, it gives back all the original matrix M with the exception of four elements, namely $m_{1,p}$, $m_{1,n}$, $m_{p,p+1}$ and $m_{p+1,n}$ (and their four symmetric). We call the submatrix consisting of these four elements M'. The four elements of M', together with their four symmetric ones, are identified with a different symbol and color in Figure 5.

Let us now consider the derivative of the map ϕ, which is a linear map. Take A and B to be two general elements in $\Lambda^2 V \times \Lambda^2 V$. The Jacobian maps from $T_{A,B} \to S^2 V$ and is a $2 \cdot \binom{n}{2} \times \binom{n+1}{2}$ matrix. One has that

$$\operatorname{Im} T_{A,B} = \operatorname{Im} T_{A^1,B^1} + \operatorname{Im} T_{A_2,B_2} + \operatorname{Im} T_{A',B'}. \tag{5.2}$$

On the other hand,

$$\dim(\operatorname{Im} T_{A^1,B^1} + \operatorname{Im} T_{A_2,B_2}) = \dim(\operatorname{Im} T_{A^1,B^1}) + \dim(\operatorname{Im} T_{A_2,B_2}) - \dim(\operatorname{Im} T_{A_2^1,B_2^1}).$$

Now the fact that our cropping preserves the form J implies that the map ϕ doesn't change the form of the matrices, meaning that for example $A^1JB^1 - B^1JA^1$ is a symmetric matrix of the same form of A^1 and B^1 and the same holds for the other two types. Hence applying our inductive hypothesis,

$$\dim(\operatorname{Im} T_{A^1,B^1} + \operatorname{Im} T_{A_2,B_2}) = \binom{n-1}{2} + \binom{n-1}{2} - \binom{n-3}{2},$$

which once simplified becomes

$$\dim(\operatorname{Im} T_{A^1,B^1} + \operatorname{Im} T_{A_2,B_2}) = \binom{n+1}{2} - 4. \tag{5.3}$$

Combining (5.2) and (5.3), if we can prove that the four rows of the Jacobian matrix of ϕ corresponding to the four elements of A' and B' are independent from the rest, then the result will follow. But this is again a consequence of the fact that our cropping preserves the form J. Once cleverly ordered the rows of the Jacobian matrix corresponding to T_{A^1,B^1} and T_{A_2,B_2} will have all zeros in the entries of the columns corresponding to $T_{A_2^1,B_2^1}$. ■

Remark 5.1 In light of the results of Proposition 5.1, it is quite natural to ask whether for big values of n the map ϕ is in fact surjective and not only dominant. Even more interesting would be finding out whether the map ϕ composed with the projection $\mathbb{P}(S^2V) \twoheadrightarrow \mathbb{P}(S^2V_{\leq r})$ is surjective for some value of the rank r. In other words, is there an r for which all symmetric matrices of rank r are of the form $AJB - BJA$ for a pair of skew-symmetric matrices (A, B)? Noferini [Nof13] has shown that the statement is false for $r = 2$, any n. The question remains open for higher values of r.

It is worth remarking that for any even rank $4 \leq r \leq n$ it is possible to exhibit regular skew-Hamiltonian matrices JA and JB whose commutator has rank r.

6 Concluding remarks

6.1 Linear monads

Any element E of $M(r,n)$ can be expressed as the cohomology bundle of a *linear monad*, of type

$$V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \xrightarrow{\alpha} K \otimes \mathcal{O}_{\mathbb{P}^2} \xrightarrow{\beta} V \otimes \mathcal{O}_{\mathbb{P}^2}(1), \tag{6.1}$$

where $V = \mathrm{H}^1\big(E(-1)\big)$ as usual and $K := \mathrm{H}^1(E \otimes \Omega^1_{\mathbb{P}^2})$ is a vector space of dimension $2n + r$, as proved in [Hul80, Lemma 1.4.2]. To see this one only needs to repeat the proof of Lemma 2.2 applying the "dual Beilinson" theorem, that is decomposing the bundle in $D^b(\mathbb{P}^2)$ with respect to the exceptional collection $\langle \mathcal{O}_{\mathbb{P}^2}(-2), \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}\rangle$ instead of the collection $\langle \mathcal{O}_{\mathbb{P}^2}(-1), \Omega^1_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}\rangle$.

The unstructured case is treated by Hulek using monads of type (6.1).

This also means that, once we fix a framing, all elements of $M^0(r,n)$ are generalized instanton bundles in the sense of [HJM12].

Remark that by combining (6.1) and (n copies of) the Euler sequence in a diagram, as it is done in (6.3), we get the monad

$$V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \to V \otimes \Omega^1_{\mathbb{P}^2}(1) \to I^* \otimes \mathcal{O}_{\mathbb{P}^2} \tag{6.2}$$

which is nothing but the dual (2.1).

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & V^* \otimes \mathcal{O}_{\mathbb{P}^2}(-1) \arrow[r] \arrow[d, "\alpha"] & V \otimes \Omega^1_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & I^* \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[d, equal] & \\
0 \arrow[r] & K \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d, "\beta"] & V \otimes U \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & I^* \otimes \mathcal{O}_{\mathbb{P}^2} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \arrow[r, equal] & V \otimes \mathcal{O}_{\mathbb{P}^2}(1) \arrow[r] \arrow[d] & 0 & \\
 & & 0. & &
\end{tikzcd} \tag{6.3}$$

The second row comes from the cohomology of the Euler sequence tensored by $E(-1)$,

$$0 \to E \otimes \Omega^1_{\mathbb{P}^2} \to U \otimes E(-1) \to E \to 0,$$

that at the H^1 level reads

$$0 \to K \to U \otimes V \to I^* \to 0.$$

On the converse, to go from (2.1) to (6.1) it is enough for the bundle E to be stable.

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6.2 Explicit examples

According to our construction, in order to obtain explicit examples of elements of $M^0_{\mathrm{ort}}(r,n)$ we simply need to exhibit the morphism $f \in U \otimes \wedge^2 V$, given by its three skew-symmetric $n\times n$ slices. In other words we need two skew-symmetric matrices A and B such that $\operatorname{rk}(AJB - BJA) = r$. It seems interesting and useful to give such an explicit construction for the linear monad. For other explicit examples, see [JMW14].

The case when the rank r equals the second Chern class n is the easiest, because the monad (2.1) simplifies, and its dual (6.2) gives us a resolution of the bundle E:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \otimes V^* \xrightarrow{{}^t f} \Omega^1_{\mathbb{P}^2}(1) \otimes V \to E \to 0.$$

Keeping the same notations as above, let $f \in U \otimes \wedge^2 V$ be given by its three skew-symmetric $n \times n$ slices A, J and B. We want to construct explicitly the linear monad having E as its cohomology. We can again combine everything in a diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^n \arrow[r, "{}^t f"] \arrow[d, equal] & \Omega^1_{\mathbb{P}^2}(1)^n \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^2}(-1)^n \arrow[r, "\alpha"] & \mathcal{O}^{3n}_{\mathbb{P}^2} \arrow[r] \arrow[d, "\beta"] & \operatorname{Ker}\beta \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathbb{P}^2}(1)^n \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^2}(1)^n \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

To construct explicitly α and β we start from the morphism f. Let $\mathbb{C}[x_0, x_1, x_2]$ be the coordinate ring. Denote by $\underline{x_i} := x_i I_n$ the $n \times n$ scalar matrix, and set

$$X := \begin{bmatrix} \underline{x_0} \\ \underline{x_1} \\ \underline{x_2} \end{bmatrix}.$$

Recall that the matrix

$$\mathrm{H}^0(f) = \begin{bmatrix} 0 & J & A \\ -J & 0 & B \\ -A & -B & 0 \end{bmatrix}$$

is symmetric, and in this case it is also of maximal rank $\operatorname{rk}\mathrm{H}^0(f) = 2n + r = 3n$. Considering the associated complex quadratic form, it factorizes as $\mathrm{H}^0(f) = {}^tQQ$ for some matrix Q. Now set $\alpha := QX$ and $\beta := {}^t\alpha$, so that

$$\beta\alpha = ({}^tX\,{}^tQ)(QX) = {}^tX({}^tQQ)X = {}^tX\,\mathrm{H}^0(f)X = 0.$$

6.3 Conjectural Bound

The work presented in this paper leaves many open questions. First of all, we firmly believe that the following statement is true.

Conjecture 6.4 (Strong Lemma 4.1) *Let n and $3 \le r \le n$ be two positive integers, n even. Let V be a complex vector space of dimension n. The subvariety $\hat{\mathcal{C}}_{r,n}$ is irreducible of codimension $\binom{n-r+1}{2}$ in $\Lambda^2 V \times \Lambda^2 V$ for any $6r - 5n \ge 2$.*

Remark that the linear bound $6r - 5n \ge 2$ can be re-written as $n \ge 6(n-r) + 2$; once this is done it becomes apparent that Conjecture 6.4 reduces to Lemma 4.1 when $r = n$ and $r = n - 1$.

In virtue of Theorem 3.5, the above conjecture would of course imply the irreducibility of the moduli space $M^0_{\mathrm{ort}}(r,n)$ in the same range $6r - 5n \ge 2$.

We have good computational evidence for Conjecture 6.4, nevertheless in its possible proof a generalization of the two claims, 1 and 2, to any co-rank appears to be a technical impasse. With the same notation used in Section 4, we believe that the following two results hold.

Claim 1: conjectural strong version. Let the Jordan normal form of JB consist in m Jordan blocks, each of dimension d_i with $\sum_{i=1}^m d_i = \frac{n}{2}$. For every $i = 1, \dots, m$ define $\delta_i := \sum_{j=1}^{i-1} d_j$, and $\rho_i := \min\{d_i, n - r\}$. Let L be an element in the Grassmannian $\mathbb{G}(n-r, n)$. If the condition $L \subseteq \ker S$ is non-transverse with the $\diamond$-conditions, then

$$L \cap \bigcup_{i=1}^{m} \langle e_{\delta_i+1}, \dots, e_{\delta_i+\rho_i}, e_{\delta_i+\frac{n}{2}+1}, \dots, e_{\delta_i+\frac{n}{2}+\rho_i} \rangle \neq 0.$$

Claim 2: conjectural strong version. Let L be an element of the Grassmann variety $\mathbb{G}(n-r, n)$. Then the dimension of the fiber $p_1{}^{-1}|_{p_2^{-1}(\mathcal{S}^B_{r,n})}(L)$ is bounded above by

$$\dim\big(p_1{}^{-1}|_{p_2^{-1}(\mathcal{S}^B_{r,n})}(L)\big) \le \binom{r+1}{2} - \Big(\frac{3}{2}n - 3(n-r)\Big),$$

where $\binom{r+1}{2} = \dim S^2(V/L)$, the dimension of the fiber of the affine bundle $\tilde{X} \to \mathbb{G}(n-r, n)$ defined in (4.4).

6.4 The Case of Odd c_2

The case c_2 odd is very interesting, especially because it appears that (almost) none of the techniques used in this work apply. We know that orthogonal bundles with odd c_2 cannot have trivial splitting on the general line, and that they do not deform to ones that do. How can we study their moduli space? Remark that the second symmetric power of (a twist of) the tangent bundle, namely $(S^2\,\mathrm{T}\,\mathbb{P}^2)(-3)$, is an example of a stable rank 3 orthogonal bundle on $\mathbb{P}^2$ with Chern classes $(c_1, c_2) = (0, 3)$, and that its splitting type on the general line ℓ is $\mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell \oplus \mathcal{O}_\ell(1)$.

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Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK
e-mail: r.abuaf@imperial.ac.uk

Scuola Internazionale Superiore di Studi Avanzati, via Bonomea 265, 34136 Trieste, Italy
e-mail: ada.boralevi@sissa.it