ON SEMI INNER PRODUCT SPACES
OVER QUATERNIONS

DILIP KUMAR SEN

Abstract

In this note semi inner product spaces over quaternions are studied. We investigate the properties of complex strict convexity over C, left and right Q-complex strict convexity over Q, and left and right quaternion strict convexity. Finally we discuss the permanence properties with respect to formation of products.

1. Introduction

Lumer (1961) introduced the concept of semi inner product spaces over K ($K = R$ or C). Generalising this concept Torgashev (1974) introduced the concept of semi inner product spaces over quaternions and studied some of its properties. In this note, we obtain some further results on semi inner product spaces over quaternions.

Throp and Whitley (1967) introduced the concept of complex extreme points for studying the Banach space valued analytic functions. Later on, Istratescu (1979a, 1979b) studied complex extreme points and complex strictly convex spaces in detail. Here we give a sufficient condition for complex strict convexity over C or Q. We also introduce and study left and right quaternion strictly convex spaces.

In particular characterizations of left and right quaternion strict convexity of product space with l_p-norm are given. Here vector spaces over quaternions and normed spaces over quaternions are all in the sense of Torgashev (1974).

2. Left and right semi inner product spaces

We recalll the following from Torgashev (1974).

DEFINITION 2.1. Let X be a vector space over Q. Then a s.i.p on X is a functional $(,): X \times X \to Q$ satisfying the following properties:
  (a) $(x + x', y) = (x, y) + (x', y),$
  (b') $(\lambda x, y) = \lambda(x, y),$
  (b'') $(x, \lambda y) = (x, y)\bar{\lambda},$
  (b''') $(x\lambda, y) = (x, y\bar{\lambda}),$
  (c) $(x, x) \geqslant 0, = 0$ only if $x = 0,$
  (d) $|(x, y)|^2 \leqslant (x, x) \cdot (y, y)$
for each $x, x', y \in X$ and each $\lambda \in Q$.

NOTE 2.2. We shall call a functional $(,)$ as defined above a left s.i.p. A vector space X over Q with a left s.i.p. will be called a left s.i.p. space over Q.

We now introduce the notion of right s.i.p. in the following way.

DEFINITION 2.3. A right s.i.p. on a vector space X over Q is a functional $\langle , \rangle$: $X \times X \to Q$ satisfying the following properties:
  (a) $\langle x + x', y \rangle = \langle x, y \rangle + \langle x', y \rangle,$
  (b') $\langle x\lambda, y \rangle = \langle x, y \rangle\lambda,$
  (b'') $\langle x, y\lambda \rangle = \bar{\lambda}\langle x, y \rangle,$
  (b''') $\langle \lambda x, y \rangle = \langle x, \bar{\lambda} y \rangle,$
  (c) $\langle x, x \rangle \geqslant 0, = 0$ only if $x = 0,$
  (d) $|\langle x, y \rangle|^2 \leqslant \langle x, x \rangle \cdot \langle y, y \rangle$
for each $x, x', y \in X$ and each $\lambda \in Q$.

NOTE 2.4. When the field of scalars is R or C then the notion of left and right s.i.p. coincides with the notion of s.i.p. introduced by Lumer (1961) and Giles (1967).

We quote the following result which is straightforward from the generalised Hahn-Banach theorem.

THEOREM 2.5. *Let X be a normed space over Q. For each arbitrary $x_0 \in X$ ($x_0 \neq 0$) there exist functional $f_l \in X_l^*$ and $f_r \in X_r^*$ such that $f_l(x_0) = \|x_0\|, \|f_l\| = 1$ and $f_r(x_0) = \|x_0\|, \|f_r\| = 1$. Here X_l^* (X_r^*) is the space of all bounded left (right) linear functionals on X.*

It can be easily verified that every left and right s.i.p. space over Q is a normed space over Q.

Lumer (1961) has shown that every normed space over R or C can be represented as a s.i.p. space (in general in infinitely many different ways); we now generalise this result as follows.

THEOREM 2.6. *Every normed space over Q can be represented as a left s.i.p. space over Q (in general in infinitely many different ways).*

PROOF. Let us introduce on the unit sphere S of X an equivalence relation $\sim$ by $s_1 \sim s_2$ if and only if there exist $\alpha_1, \alpha_2 \in Q$ ($|\alpha_1| = |\alpha_2| = 1$) such that $s_1 = \alpha_1 s_2 \alpha_2$ (α_1 and α_2 are unique). Let S_0 be any equivalence class. By the axiom of choice, we can select an element $s_0 \in S_0$. Now by the generalised Hahn Banach theorem, there exists an $f_{S_0} \in X_l^*$ such that $f_{S_0}(s_0) = 1$ and $\|f_{S_0}\| = 1$. For each $s \in S_0$, $s = \alpha_1 s_0 \alpha_2$ ($|\alpha_1| = |\alpha_2| = 1$). We define $f_S = f_{\alpha_1 S_0 \alpha_2} = \bar{\alpha}_2 f_{S_0} \bar{\alpha}_1$. Since f_{S_0} is a left linear functional then so also is f_S.

Now $f_S(s) = (\bar{\alpha}_2 f_{S_0} \bar{\alpha}_1)(\alpha_1 s_0 \alpha_2) = (\bar{\alpha}_2 f_{S_0})(\alpha_1 s_0 \alpha_2)\bar{\alpha}_1 = f_{S_0}(\alpha_1 s_0 \alpha_2 \bar{\alpha}_2)\bar{\alpha}_1 = 1$. For arbitrary $y \in X$ ($y \neq 0$),

$$y = \|y\| \cdot s = \|y\|\alpha_1 s_0 \alpha_2 \qquad (|\alpha_1| = |\alpha_2| = 1, s_0 \in S_0).$$

We define $f_y = \|y\| f_S = \|y\|\bar{\alpha}_2 f_{S_0}\bar{\alpha}_1$ and $f_0 = 0$. Now let $\beta \in Q$ and $\beta = |\beta|\beta_0$ ($|\beta_0| = 1$). Then $f_{\beta y} = f_{|\beta|\|y\|\beta_0\alpha_1 S_0 \alpha_2} = |\beta| \cdot \|y\|\bar{\alpha}_2 f_{S_0}\bar{\alpha}_1\bar{\beta}_0 = f_y\bar{\beta}$.

Similarly, $f_{y\beta} = \bar{\beta}f_y$.

We define a transformation $W: X \to X_l^*$, $W(y) = f_y$. Then from above, this transformation has the following properties:

1) $f_y(y) = \|y\|^2$ ($\forall y \in X$),
2) $f_{\beta y} = f_y\bar{\beta}$ ($\beta \in Q$),
3) $f_{y\beta} = \bar{\beta}f_y$ ($\beta \in Q$),
4) $\|f_y\| = \|y\|$.

For each $x, y \in X$, we define $(x, y) = f_y(x)$. Then it can be easily verified that this defines a left s.i.p. on X. Hence X is a left s.i.p. space over Q. In general there are many such mappings W from X to X_l^*. Therefore X can be represented as a left s.i.p. space over Q in infinitely many different ways.

By symmetry, the following result also holds.

THEOREM 2.7. *Every normed space over Q can be represented as a right s.i.p. space over Q (in general in infinitely many different ways).*

NOTE 2.8. Torgashev (1974), in Theorem 1, has proved that every normed space over Q can be represented as a left s.i.p. with properties (b′) and (b″) of Definition 2.1.

NOTE 2.9. Let X be a normed space over Q. Then X is also a normed space over R, which will be denoted by X_R. If f_0 is any real linear functional on X_R we define for $x \in X$,

(1) $$f_l(x) = f_0(x) - if_0(ix) - jf_0(jx) - kf_0(kx)$$

and

(2) $$f_r(x) = f_0(x) - if_0(xi) - jf_0(xj) - kf_0(xk),$$

Then $f_l \in X_l^*$ and $f_r \in X_r^*$.

Conversely, if f_l and f_r are elements of X_l^* and X_r^* respectively such that $\mathrm{Re}(f_l) = \mathrm{Re}(f_r) = f_0$ (say), then f_0 is a real linear functional on X_R, and further f_l and f_r satisfy (1) and (2). Now we define the map $T: X_l^* \to X_r^*$ by $T(f_l) = f_r$ where f_l and f_r are given by (1) and (2). Then it can be easily verified that $\|f_l\| = \|f_0\| = \|T(f_l)\| = \|f_r\|$ and hence T is a real linear isometry from X_l^* onto X_r^*.

We now show that left and right s.i.p. products are equivalent in a certain sense.

THEOREM 2.10. *Let $(\ ,\ )$ be a left s.i.p. on a normed space X over Q. Then a right s.i.p. $\langle\ ,\ \rangle$ can be derived, and conversely.*

PROOF. Let $(\ ,\ )$ be a left s.i.p. on X. Suppose $[x, y] = \mathrm{Re}(x, y)$ for all $x, y \in X$. Then $[\ ,\ ]$ is a real s.i.p. on X_R and $(x, y) = [x, y] - i[ix, y] - j[jx, y] - k[kx, y]$. From this, $[x, x] = (x, x) = \|x\|^2$ and $[ix, x] = [jx, x] = [kx, x] = 0$ for all $x \in X$.

Let $\lambda \in Q$ and $\lambda = a_1 + ia_2 + ja_3 + ka_4$ say. Now

(i) $$[\lambda x, y] = a_1[x, y] + a_2[ix, y] + a_3[jx, y] + a_4[kx, y].$$

Again,

(ii) $$[x, \lambda y] = \mathrm{Re}(x, \lambda y) = \mathrm{Re}\{(x, y)\bar{\lambda}\} = [\bar{\lambda}x, y]$$

and

(iii) $\qquad [x, y\lambda] = \text{Re}(x, y\lambda) = \text{Re}(x\bar{\lambda}, y)$

$$= a_1[x, y] - a_2[xi, y] - a_3[xj, y] - a_4[xk, y].$$

Let us define a functional $\langle \, , \, \rangle$ on X by

$$\langle x, y \rangle = [x, y] - i[xi, y] - j[xj, y] - k[xk, y].$$

Then we have

(a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$,

(b') $\langle x\lambda, y \rangle = [x\lambda, y] - i[x\lambda i, y] - j[x\lambda j, y] - k[x\lambda k, y] = \langle x, y \rangle \lambda$,

(b'') $\langle x, y\lambda \rangle = [x, y\lambda] - i[xi, y\lambda] - j[xj, y\lambda] - k[xk, y\lambda] = \bar{\lambda}\langle x, y \rangle$, by the relation (iii),

(b''') $\langle \lambda x, y \rangle = [\lambda x, y] - i[\lambda xi, y] - j[\lambda xj, y] - k[\lambda xk, y] = \langle x, \hat{\lambda} y \rangle$, by the relation (ii),

(c) $\langle x, x \rangle \geqslant 0$, $= 0$ only if $x = 0$,

(d) by Note 2.9 and the Cauchy-Schwarz inequality for the real s.i.p. [,], we have $|\langle x, y \rangle| \leqslant \|x\| \cdot \|y\|$.

Therefore $\langle \, , \, \rangle$ defines a right s.i.p. on X. The converse statement is proved similarly.

NOTE 2.11. Though left and right s.i.p. products are equivalent in a certain sense, they are needed for studying the interconnection between left and right complex strictly convex, Q-complex strictly convex and quaternion strictly convex spaces.

Further, let us recall that Torgashev (1976) defined the numerical range of a right linear operator by using left linear functionals. But a left s.i.p. is nothing but a set of left linear functionals.

We now give a characterization of Hilbert spaces over Q.

THEOREM 2.12. *Let X be a Banach space over Q. If there exists a left s.i.p. on X such that $(x, y) + (y, x)$ is real for all $x, y \in X$, then X is a Hilbert space.*

PROOF. Suppose (,) is a left s.i.p. on X which satisfies the condition that, for all $x, y \in X$,

(i) $(x, y) + (y, x)$ is real. Now in (i), by replacing x by ix we have

(ii) $i(x, y) - (y, x)i$ is real. From the conditions (i) and (ii) we have $(x, y) = (y, x)$. Thus (,) is an inner product on X in the sense of Torgashev (1974). Hence X is a Hilbert space.

3. Strict convexity

We now introduce the concept of strict convexity in normed spaces over Q.

DEFINITION 3.1. Let X be a normed space over Q. We say X is left (right) strictly convex if and only if every non-zero continuous left (right) linear functional attains its maximum on the unit sphere at most at one point.

THEOREM 3.2. *A left (right) linear functional attains its maximum on the unit sphere at a unique point if and only if it's real part attains its maximum on the unit sphere at a unique point.*

PROOF. The result follows from Note 2.9.

THEOREM 3.3. *Let T be the map defined in Note 2.9. Then a left linear functional f_l attains its maximum on the unit sphere at a unique point if and only if $T(f_l)$ attains its maximum on the unit sphere at a unique point.*

PROOF. The result follows from Note 2.9 and Theorem 3.2.

NOTE 3.4. From Theorem 3.3 we conclude that left strict convexity implies right strict convexity and conversely. Hence we simply use the phrase strictly convex (in short, s.c.).

NOTE 3.5. Hilbert spaces over Q are s.c. spaces.

NOTE 3.6. From Theorem 3.2 we conclude that X is s.c. if and only if X_R is s.c.

We now generalise Theorem 1 of Beesack, Hughes and Ortel (1979).

THEOREM 3.7. *A normed space X over Q is s.c. if and only if whenever $x, y \in X$ with $x \neq y$ and $\|x\| = 1 = \|y\|$, we have $\|qx + (1-q)y\| < 1$ for some $q \in Q$.*

PROOF. Suppose X is s.c. Then X_R is s.c. and the result holds with $q = \frac{1}{2}$.
Conversely suppose the given condition is satisfied. If X is not s.c. then there exist x, y and a bounded left linear functional f_l, such that $f_l(x) = f_l(y) = \|x\| = \|y\| = 1 = \|f_l\|$. Hence for any $q \in Q$,

$$\|qx + (1-q)y\| \geq \|f_l(qx + (1-q)y)\| = 1$$

and so the given condition fails.

Similarly the following result holds.

THEOREM 3.8. *A normed space X over Q is s.c. if and only if whenever $x, y \in X$ with $x \neq y$ and $\|x\| = 1 = \|y\|$, we have $\|xq + y(1 - q)\| < 1$ for some $q \in Q$.*

Results of Köethe (1969) (Theorem 3), Torrance (1970), Gudder and Strawther (1976) and Sen (1981) are also true in this setting.

4. C-strict convexity and Q-strict convexity

We recall the following definitions from Istratescu (1979a).

DEFINITION 4.1. Let K be a convex subset of a normed space X over C and $x_0 \in K$. The element x_0 is a complex extreme point of K if and only if $\{x + zy: |z| \leqslant 1, z \in C\} \subset K$ for $y \in X$, implies $y = 0$.

DEFINITION 4.2. A normed space X over C is called complex strictly convex (c.s.c.) if and only if each point of the unit sphere is a complex extreme point of the unit ball.

Istratescu (1979b), Theorem 2.7, gave a sufficient condition for a Banach space to be c.s.c. An easy application of Theorem 1 of Holub (1975) shows that it is in reality a sufficient condition for a space to be s.c.

We now give a sufficient condition for a normed space over C to be c.s.c.

DEFINITION 4.3. For every normalized pair $\{x_1, x_2\}$ of linearly independent elements of a normed space X over C, let $V[x_1, x_2]$ be the complex vector space spanned by x_1 and x_2 and let $R[x_1, x_2]$ be the set of all those elements $y = a_1 x_1 + a_2 x_2 \in V[x_1, x_2]$ with $a_1 a_2$ real.

THEOREM 4.4. *Let X be a normed space over C. If for every normalized pair $\{x_1, x_2\}$ of linearly independent elements in X, the points in $V[x_1, x_2]$ equidistant from x_1 and x_2 are in $R[x_1, x_2]$ then X is c.s.c.*

PROOF. Suppose that X is not a c.s.c. space. Then there exist an $x \in X, \|x\| = 1$ and $ay \neq 0$ such that $\|x + zy\| = 1$ for all $|z| \leqslant 1$, z complex. Let $x_1 = ix + y$, $x_2 = x + y$. Then $\|x_1\| = \|x_2\| = 1$.

Suppose $u = ix_1 + x_2$. Then $u \in V[x_1, x_2]$. Now $u - x_1 = -i(x - y)$ and $u - x_2 = -x + iy$. As $\|u - x_1\| = \|u - x_2\| = 1$, u is equidistant from x_1 and x_2. Since u is not in $R[x_1, x_2]$, we have a contradiction.

NOTE 4.5. Let X be a complex normed space. Then X can be considered as a normed space over the reals. If x_1, x_2 is a normalized pair of linearly independent elements of X, then $V[x_1, x_2] = V_R[x_1, x_2, ix_1, ix_2]$, where $V_R[x_1, x_2, ix_1, ix_2]$ is the real linear span of x_1, x_2, ix_1, ix_2.

Thus $V[x_1, x_2]$ can be considered as a real subspace of X and $R[x_1, x_2]$ is a subset of $V[x_1, x_2]$.

Therefore it is relevant to ask whether in Theorem 4.4, c.s.c. can be replaced by s.c. But the author has not been able to settle this problem. The author has however observed that the condition in Theorem 4.4 is not necessary for s.c. For let $X = l_2$ and e_1 and e_2 be two mutuallyorthogonal unit vectors of X. Then the vector $(1 + i)(e_1 + e_2) \in V[e_1, e_2]$ and is equidistant from e_1 and e_2 but not in $R[e_1, e_2]$.

We now define left and right complex extreme points on a normed space over Q.

DEFINITION 4.6. A quaternion $a_0 + ia_1 + ja_2 + ka_3$ is said to be Q-complex (complex) if and only if at least one (two) of a_1, a_2, a_3 is (are) zero.

DEFINITION 4.7. Let C be a convex set of a normed space X over Q. An element $x_0 \in C$ is said to be a left complex extreme point of C if and only if $\{x_0 + zy: |z| \leqslant 1, z \text{ complex}\} \subset C$ for y in X, implies $y = 0$.

Similarly a right complex extreme point can be defined in the same manner.

DEFINITION 4.8. A normed space X over Q is said to be left (right) complex strictly convex if and only if every element of the unit sphere is a left (right) complex extreme point of the unit ball. Similarly, we can define left and right Q-complex strict convexity and quaternion strict convexity.

NOTE 4.9. It would be interesting to know whether the left and right concepts are equivalent.

EXAMPLE 4.10. Let l_1 be the set of all infinite sequences of quaternions $x = \{x_i: i = 1, 2, \dots\}$ such that $\sum_{i=1}^{\infty} |x_i| < \infty$. Addition and left and right scalar multiplications on l_1 are defined as usual; l_1 forms a Banach space over Q with the norm $\|x\| = \sum_{i=1}^{\infty} |x_i|$. The space l_1 is both a left and right c.s.c. space; this follows from Corollary 4.23.

NOTE 4.11. Henceforth we only consider left properties as the corresponding right ones can be discussed similarly.

NOTE 4.12. Every left c.s.c. space is a left Q-c.s.c. space and evry left Q-c.s.c. space is a left q.s.c. space.

NOTE 4.13. It would be interesting to find an example of a left Q-c.s.c. space which is not left c.s.c. and another example of a left q.s.c. space which is not a left Q-c.s.c. space.

We now give a sufficient condition for a normed space over Q to be left c.s.c.

DEFINITION 4.14. For every normalized pair $\{x_1, x_2\}$ of left linearly indepen-dent elements of a normed space X over Q, let $V_l[x_1, x_2]$ be the left complex vector space spanned by x_1, x_2 and let $R_l[x_1, x_2]$ be the set of all $y = a_1 x_1 + a_2 x_2 \in V_l[x_1, x_2]$ with $a_1 \cdot a_2$ real.

THEOREM 4.15. *Let X be a normed space over Q. If for every normalized pair $\{x_1, x_2\}$ of left linearly independent elements in X, the points in $V_l[x_1, x_2]$ equidis-tant from x_1 and x_2 are in $R[x_1, x_2]$, then X is a left c.s.c. space.*

PROOF. The proof is the same as for Theorem 4.4.

We now give a sufficient condition for a normed space over Q to be left Q-c.s.c.

DEFINITION 4.16. For every normalized pair $\{x_1, x_2\}$ of left linearly indepen-dent elements of a normed space X over Q, let $V_l[x_1, x_2]$ be the left Q-complex vector space spanned by x_1, x_2 and let $C_l[x_1, x_2]$ be the set of all $y = a_1 x_1 + a_2 x_2 \in V_l[x_1, x_2]$ with $a_1 \cdot a_2$ complex.

THEOREM 4.17. *Let X be a normed space over Q. If for every normalized pair $\{x_1, x_2\}$ of left linearly independent elements in X, the points in $V_l[x_1, x_2]$ equidis-tant from x_1 from x_2 are in $C_l[x_1, x_2]$, then X is left Q-c.s.c.*

PROOF. Suppose that X is not a left Q-c.s.c. space. Then there exist an $x \in X$, $\|x\| = 1$, and $y \neq 0$ such that $\|x + qy\| = 1$ for all Q-complex q with $|q| \leqslant 1$. Let $x_1 = x$ and $x_2 = x - ky$. Then $\|x_1\| = \|x_2\| = 1$.
  Suppose $u = (i + j)x_1 + x_2$. Then $u \in V_l[x_1, x_2]$. Now $u - x_1 = (i + j - 1)x + x - ky = (i + j)x - ky$ and $u - x_2 = (i + j)x$.

From this, $\|u - x_1\| = |i + j| \cdot \|x - (i + j)^{-1}ky\| = \sqrt{2}$, as $(i + j)^{-1}k$ is Q-complex and $|(i + j)^{-1}k| < 1$ and $\|u - x_2\| = \sqrt{2}$. Therefore u is equidistant from x_1 and x_2. Since u is not in $C_l[x_1, x_2]$, we have a contradiction.

We now give a sufficient condition for a normed space over Q to be left q.s.c.

DEFINITION 4.18. For every normalized pair $\{x_1, x_2\}$ of left linearly independent elements of a normed space X over Q, let $V_l[x_1, x_2]$ be the left quaternion vector space spanned by x_1, x_2 and let $Q_l[x_1, x_2]$ be the set of all $y = a_1 x_1 + a_2 x_2 \in V_l[x_1, x_2]$ with $a_1 \cdot a_2 Q$-complex.

THEOREM 4.19. *Let X be a normed space over Q. If for every normalized pair $\{x_1, x_2\}$ of left linearly independent elements in X, the points in $V_l[x_1, x_2]$ equidistant from x_1 and x_2 are in $Q_l[x_1, x_2]$, then X is left q.s.c.*

PROOF. Suppose that X is not a left q.s.c. space. Then there exist an $x \in X$, $\|x\| = 1$, and a $y \neq 0$ such that $\|x + qy\| = 1$ for all $|q| \leq 1$, q quaternion. Let $x_1 = x + y$ and $x_2 = x - y$. Then $\|x_1\| = \|x_2\| = 1$. Suppose $u = (i + 1)x_1 + (j + i)x_2$. Then it can be easily verified that u is equidistant from x_1 and x_2 but not in $Q_l[x_1, x_2]$. Hence we have a contradiction.

We now give a characterization of left q.s.c. product spaces with l_p-norm. Before proving our required characterization we prove a lemma.

LEMMA 4.20. *Let X be a normed space over Q. If $\|u\| = 1$ and $\|u \pm v\| = 1$ then $(v, u) = 0 = \langle v, u \rangle$, where $(,)$ and $\langle , \rangle$ denote left and right s.i. products on X respectively.*

PROOF. We have $|1 \pm (v, u)| = |(u \pm v, u)| \leq \|u \pm v\| \cdot \|u\| = 1$ or $|1 \pm (v, u)| \leq 1$. Since the space Q of quaternions with usual norm is strictly convex this implies that $(v, u) = 0$. Similarly $\langle v, u \rangle = 0$.

THEOREM 4.21. *Let X be the product space $X_1 \times X_2$ with l_p-norm $(1 \leq p \leq \infty)$. For $p = \infty$, X is never left q.s.c. For $1 \leq p < \infty$, X is left q.s.c. if and only if each X_i $(i = 1; 2)$ is left q.s.c.*

PROOF. Let X be left q.s.c. and $1 \leq p < \infty$. If the result does not hold then there exists an X_i (say X_1) such that X_1 is not left q.s.c. Since X_1 is not left q.s.c. then there exist a u, $\|u\| = 1$ and a $v \neq 0$ such that $\|u + qv\| = 1$ for all $|q| \leq 1$. We choose $x = (u, 0)$, $y = (v, 0)$.

Then $\|x\| = 1$ and $\|x + qy\| = 1$, $|q| \leqslant 1$. Since X is left q.s.c., $y = 0$, a contradiction.

We now prove the converse.

Case I, $1 < p < \infty$. Let each X_i $(i = 1, 2)$ be left q.s.c. Suppose $[\ ,\ ]_{X_i}$ is any consistent left s.i.p. on X_i $(i = 1, 2)$. Then a left s.i.p. on X is given by

$$[(x_1, x_2), (y_1, y_2)] = \frac{\sum_{i=1}^{2} [x_i, y_i]_{X_i} \cdot \|y_i\|^{p-2}}{\|(y_1, y_2)\|^{p-2}}, \qquad 1 < p < \infty.$$

Let $u, v \in X$, $\|u\| = 1$ and $\|u + qv\| = 1$, $|q| \leqslant 1$. Then by Lemma 4.20, $[v, u] = 0$. If $u = (x_1, x_2)$, $v = (y_1, y_2)$, then

$$1 = \|x_1\|^p + \|x_2\|^p = [x_1 + qy_1, x_1]_{X_1} \cdot \|x_1\|^{p-2} + [x_2 + qy_2, x_2]_{X_2} \cdot \|x_2\|^{p-2}$$

$$\leqslant \|x_1 + qy_1\| \cdot \|x_1\|^{p-1} + \|x_2 + qy_2\| \cdot \|x_2\|^{p-1}$$

$$\leqslant \left(\|x_1 + qy_1\|^p + \|x_2 + qy_2\|^p\right)^{1/p} \cdot \left(\|x_1\|^{(p-1)q} + \|x_2\|^{(p-1)q}\right)^{1/q} = 1,$$

Therefore by Hölder's inequality (note that $1/p + 1/q = 1$).

(i) $\quad \|x_1 + qy_1\| \cdot \|x_1\|^{p-1} + \|x_2 + qy_2\| \cdot \|x_2\|^{p-1}$

$$= \left(\|x_1 + qy_1\|^p + \|x_2 + qy_2\|^p\right)^{1/p} \cdot \left(\|x_1\|^p + \|x_2\|^p\right)^{1/q} = 1.$$

Since equality holds in Hölder's inequality there exists a non-zero positive number K_q such that

(ii) $\qquad \|x_1 + qy_1\|^p = K_q \cdot \|x_1\|^p, \qquad \|x_2 + qy_2\|^p = K_q \cdot \|x_2\|^p$

for $|q| \leqslant 1$.

Now from (i) and (ii), $K_q = 1$. Again since each X_i is left q.s.c. then from (ii), $y_i = 0$ for $i = 1, 2$.

Hence $v = 0$. Therefore X is left q.s.c.

Case II, $p = 1$. Suppose each X_i $(i = 1, 2)$ is left q.s.c. Let $u, v \in X$, $\|u\| = 1$ and $\|u + qv\| = 1$ for all $|q| \leqslant 1$. Suppose $u = (x_1, x_2)$, $v = (y_1, y_2)$. Then for $|q| \leqslant 1$,

(i) $\qquad \|x_1\| + \|x_2\| = 1, \qquad \|x_1 + qy_1\| + \|x_2 + qy_2\| = 1.$

Without loss of generality we may assume that $\|x_1\|, \|x_2\| \neq 0$. Let $G_1 = \{q: |q| \leqslant 1, \|x_1 + qy_1\| \leqslant \|x_1\|\}$ and $G_2 = \{q: |q| \leqslant 1, \|x_2 + qy_2\| \leqslant \|x_2\|\}$. Then $G_1 \cup G_2 = B$, the closed unit ball of Q, and G_1 and G_2 are closed sets.

Suppose $I = G_1 \cap G_2$. Then $0 \in I$.

We now show that I contains more than one element. If possible let $I = \{0\}$. Then $B \backslash G_1 = G_2 \backslash \{0\}$ is an open set in B and $B \backslash G_2 = G_1 \backslash \{0\}$ is an open set in B.

Now $B\setminus\{0\} = (G_1\setminus\{0\}) \cup (G_2\setminus\{0\})$, the union of two disjoint open sets, which is a contradiction (as $B_1\setminus\{0\}$ is a connected set). Therefore, I contains more than one element.

We now show that I is symmetric. Suppose $q \in I$. Then

(ii) $$\|x_1 + qy_1\| \leqslant \|x_1\|, \qquad \|x_2 + qy_2\| \leqslant \|x_2\|.$$

Again,

$$2 \cdot \|x_1\| \leqslant \|x_1 + qy_1\| + \|x_1 - qy_1\|$$
$$\leqslant \|x_1\| + \|x_1 - qy_1\|$$

or

(iii) $$\|x_1\| \leqslant \|x_1 - qy_1\|.$$

Similarly,

(iv) $$\|x_2\| \leqslant \|x_2 - qy_2\|.$$

From (i), (iii), and (iv)

$$\|x_1\| + \|x_2\| \leqslant \|x_1 - qy_1\| + \|x_2 - qy_2\| \leqslant \|x_1\| + \|x_2\|.$$

Therefore from (iii) and (iv)

$$\|x_1 - qy_1\| = \|x_1\|, \qquad \|x_2 - qy_2\| = \|x_2\|$$

which shows that $-q \in I$.

We now show that $I = B$. Let $q \in I$, $q \neq 0$ and $q_1 \in Q$ such that $|q_1 \cdot q| \leqslant 1$. Then as I is symmetric,

$$\|x_1 \pm qy_1\| = \|x_1\|, \qquad \|x_2 \pm qy_2\| = \|x_2\|.$$

By our Lemma 4.20, $[y_1, x_1]_{X_1} = 0 = [y_2, x_2]_{X_2}$. Now

$$\|x_1\|^2 = [x_1 + q_1 \cdot qy, x_1]_{X_1}$$
$$\leqslant \|x_1 + q_1 \cdot qy_1\| \cdot \|x_1\|$$

or

(v) $$\|x_1\| \leqslant \|x_1 + q_1 \cdot qy_1\|.$$

Similarly,

(vi) $$\|x_2\| \leqslant \|x_2 + q_1 \cdot qy_2\|.$$

From (i), (v) and (vi),

$$\|x_1\| + \|x_2\| \leqslant \|x_1 + q_1 \cdot qy_1\| + \|x_2 + q_1 \cdot qy_2\|$$
$$\leqslant \|x_1\| + \|x_2\|.$$

From this we conclude that $\|x_1 + q_1 qy_1\| = \|x_1\|$ and $\|x_2 + q_1 \cdot qy_2\| = \|x_2\|$. Hence $q_1 \cdot q \in I$, and therefore $I = B$.

Hence $\|x_1 + qy_1\| = \|x_1\|$ and $\|x_2 + qy_2\| = \|x_2\|$ for $q \in B$. Since X_1, X_2 are left q.s.c., $y_1 = 0 = y_2$. Hence $v = 0$. Therefore, X is left q.s.c.

For $p = \infty$, let $u = (x_1, 0)$, $v = (0, y_2)$ with $\|x_1\| = 1 = \|y_2\|$. Then $\|u + qv\|_\infty = 1$ for $|q| \leq 1$. Therefore, X is not left q.s.c.

COROLLARY 4.22. *If X is an l_p-sum $(1 \leq p < \infty)$ of a sequence of normed spaces $\{X_i\}$ then X is left q.s.c. if and only if each X_i is left q.s.c.*

PROOF. *Case* I, $1 < p < \infty$. The proof follows from Theorem 4.21 above.

Case II, $p = 1$. Suppose each X_i is left q.s.c. Let $\|x\| = 1$, $\|x + qy\| = 1$ for $q \in B$, $x, y \in X$. Suppose $x = (x_1, x_2, x_3, \ldots)$, $y = (y_1, y_2, y_3, \ldots)$. Then $\sum_{i=1}^{\infty}\|x_i\| = 1$ and $\sum_{i=1}^{\infty}\|x_i + qy_i\| = 1 = \sum_{i=1}^{\infty}\|x_i\|$ for $q \in B$.

Let $G_i = \{q: q \in B, \|x_i + qy_i\| \leq x_i\|\}$ and $\bar{G}_2 = \{q: q \in B, \|\bar{x} + q\bar{y}\| \leq \|\bar{x}\|\}$, where $\bar{x} = (0, x_2, x_3, \ldots)$, $\bar{y} = (0, y_2, y_3 \ldots)$. Then $G \cup \bar{G}_2 = B$ and each G_i and $\bar{G}_2$ are closed sets.

Arguing as in the case of two spaces, we have $I = G_1 \cap \bar{G}_2$ contains more than one element. Also arguing as in the case of two spaces we can show that

$$\|x_1 + qy_1\| = \|x_1\| \quad \text{and} \quad \|\bar{x} + q\bar{y}\| = \|\bar{x}\| \quad \text{for } q \in B.$$

Therefore $G_1 = B$ and hence $y_1 = 0$ as X_1 is left q.s.c. Arguing in the same manner we get $G_i = B$ for all i and hence $y_i = 0$ for all i as the X_i's are left q.s.c. Therefore $y = 0$ and hence X is left q.s.c.

COROLLARY 4.23. *Theorem 4.21 and Corollary 4.22 also hold for left Q-c.s.c. and c.s.c. spaces over Q and c.s.c. spaces over C.*

NOTE 4.24. For $p = 2$, Theorem 4.21 above is proved for c.s.c. spaces over C in Istratescu (1979a).

COROLLARY 4.25. *Let X be the product space $X_1 \times X_2$ with l_p norm, $1 \leq p \leq \infty$. For $1 < p < \infty$, X is s.c. if and only if X_i $(i = 1, 2)$ is s.c. For $p = 1$ or $p = \infty$, X is never s.c.*

COROLLARY 4.26. *If X is an l_p-sum $(1 < p < \infty)$ of a sequence of normed spaces $\{X_i\}$ then X is s.c. if and only if each X_i is s.c.*

We recall the following definition from Jamison (1982).

DEFINITION 4.27. Let X be a complex Banach space and f an analytic function defined on a domain D with values in X. The space X is said to have the strong maximum modulus property (SMMP) if and only if either $\| f(z) \|$ has no maximum on D or $f(z)$ is constant on D.

THEOREM 4.28. *If X is an l_p-sum $(1 \leqslant p < \infty)$ of a sequence of complex Banach spaces $\{X_i\}$ then X has SMMP if and only if each X_i has SMMP.*

PROOF. The result follows from Corollary 4.23 above and Theorem 3.1 of Throp and Whitley (1967).

NOTE 4.29. For $p = 1$, Theorem 4.28 above is stated without proof in Jamison (1982).

Acknowledgement

References

P. R. Beesack, E. Hughes and M. Ortel (1979), 'Rotund complex normed linear spaces', *Proc. Amer. Math. Soc.* **75**, 42–44.

J. R. Giles (1967), 'Classes of semi inner product spaces', *Trans. Amer. Math. Soc.* **129**, 436–446.

S. Gudder and D. Strawther (1976), 'Strictly convex normed linear spaces', *Proc. Amer. Math. Soc.* **59**, 263–267.

J. R. Holub (1975), 'Rotundity, orthogonality and characterizations of inner product spaces', *Bull. Amer. Math. Soc.* **81** (6), 1087–1089.

V. I. Istratescu (1979a), 'On complex strictly convex spaces-I', *J. Math. Anal. Appl.* **70**, 423–429.

V. I. Istratescu (1979b), 'On complex strictly convex spaces-II', *J. Math. Anal. Appl.* **71**, 580–589.

J. E. Jamison (1982), 'Banach spaces with the strong maximum modulus property', *Abstracts Amer. Math. Soc.* **3**, 355.

G. Köethe (1969), *Topological vector spaces*, Volume I (Springer-Verlag, Berlin).

G. Lumer (1961), 'Semi inner product spaces', *Trans. Amer. Math. Soc.* **100**, 29–43.

D. K. Sen (1981), 'Characterization of strict convexity', *Bull. Calcutta Math. Soc.* **73**, 93–97.

A. Torgashev (1974), 'A generalisation of inner and semi inner product spaces', *Math. Vesnik*, **11(26)**, 301–311.

A. Torgashev (1976), 'On the numerical range of an operator in a normed space', *Math. Vesnik.* **13(28)**, 118–124.

F. Torrance (1970), 'Strictly convex spaces via semi inner product space orthogonality', *Proc. Amer. Math. Soc.* **26**, 108–110.

E. Throp and R. Whitley (1967), 'The strong maximum modulus theorem for analytic functions into a Banach space', *Proc. Amer. Math. Soc.* **18**, 640–646.

Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Calcutta - 700019
India