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KODAIRA DIMENSION OF UNIVERSAL HOLOMORPHIC SYMPLECTIC VARIETIES

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Abstract We prove that the Kodaira dimension of the n-fold universal family of lattice-polarised holomorphic symplectic varieties with dominant and generically finite period map stabilises to the moduli number when n is sufficiently large. Then we study the transition of Kodaira dimension explicitly, from negative to nonnegative, for known explicit families of polarised symplectic varieties. In particular, we determine the exact transition point in the Beauville–Donagi and Debarre–Voisin cases, where the Borcherds Φ_{12} form plays a crucial role.

Keywords: holomorphic symplectic variety, universal family, Kodaira dimension, modular form, Borcherds product

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1. Introduction

The discovery of Beauville and Donagi [3] that the Fano variety of lines on a smooth cubic fourfold is a holomorphic symplectic variety deformation equivalent to the Hilbert squares of $K3$ surfaces of genus 8 was the first example of explicit geometric construction of polarised holomorphic symplectic varieties. Gradually, further examples – all deformation equivalent to Hilbert schemes of $K3$ surfaces – have been found:

- Iliev and Ranestad [14], with varieties of sums of powers of cubic fourfolds;
- O'Grady [26], with double EPW sextics;
- Debarre and Voisin [8], with the zero loci of sections of a vector bundle on the Grassmannian $G(6,10)$;
- Lehn, Lehn, Sorger and van Straten [18], using the spaces of twisted cubics on cubic fourfolds; and
- Iliev, Kapustka, Kapustka and Ranestad [13], with double EPW cubes.

The moduli spaces $\mathcal{M}$ of these polarised symplectic varieties are unirational by construction. However, if we consider the n-fold fibre product $\mathcal{F}_n \to \mathcal{M}$ of the universal

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family $\mathcal{F} \to \mathcal{M}$ (more or less the moduli space of the varieties with n marked points or its double cover), its Kodaira dimension $\kappa(\mathcal{F}_n)$ is nondecreasing with respect to n [16] and bounded by $\dim \mathcal{M} = 20$ [12]. The main purpose of this paper is to study the transition of $\kappa(\mathcal{F}_n)$ as n grows, especially from $\kappa = -\infty$ to $\kappa \geq 0$, by using modular forms on the period domain. Moreover, we prove that $\kappa(\mathcal{F}_n)$ stabilises to $\dim \mathcal{M}$ at large n for more general families of lattice-polarised holomorphic symplectic varieties.

Our main result is summarised as follows:

Theorem 1.1. *Let $\mathcal{F}_n$ be the n-fold universal family of polarised holomorphic symplectic varieties of Beauville–Donagi (BD), Debarre–Voisin (DV), Lehn–Lehn–Sorger–van Straten (LLSS), Iliev–Ranestad (IR), O'Grady (OG) or Iliev–Kapustka–Kapustka–Ranestad (IKKR) type. Then $\mathcal{F}_n$ is unirational, $\kappa(\mathcal{F}_n) \geq 0$ and $\kappa(\mathcal{F}_n) > 0$ for the bounds of n given in the table.*

	BD	*DV*	*LLSS*	*IR*	*OG*	*IKKR*
unirational	*13*	*5*	*5*	*1*	*0*	*0*
$\kappa \geq 0$	*14*	*6*	*7*	*6*	*11*	*16*
$\kappa > 0$	*23*	*13*	*12*	*12*	*19*	*20*

In all cases, $\kappa(\mathcal{F}_n) = 20$ when n is sufficiently large. The stabilisation $\kappa(\mathcal{F}_n) = \dim \mathcal{M}$ at large n holds more generally for families $\mathcal{F} \to \mathcal{M}$ of lattice-polarised holomorphic symplectic varieties whose period map is dominant and generically finite.

This table means that, in the Beauville–Donagi case, for example, $\mathcal{F}_n$ – the moduli space of Fano varieties of cubic fourfolds with n marked points (or equivalently, cubic fourfolds with n marked lines) – is unirational when $n \leq 13$, has $\kappa(\mathcal{F}_n) \geq 0$ when $n \geq 14$ and has $\kappa(\mathcal{F}_n) > 0$ when $n \geq 23$. In particular, we find the exact transition point from $\kappa = -\infty$ to $\kappa \geq 0$ in the BD and DV cases, and a nearly exact one in the LLSS case. On the other hand, it would not be easy to explicitly calculate a bound for $\kappa = 20$; in fact, we expect that the transition of Kodaira dimension would be sudden, so the actual bound for $\kappa = 20$ would be quite near to the (actual) bound for $\kappa \geq 0$. (In this sense, the bound for $\kappa > 0$ should be temporary.)

Markman [23] gave an analytic construction of general marked universal families over (non-Haussdorff, unpolarised) period domains. Here we take a more ad hoc construction. The space $\mathcal{F}_n$ (birationally) parametrises the isomorphism classes of the n-pointed polarised symplectic varieties except for the two double-EPW cases, where it is a double cover of the moduli space.

Theorem 1.1 in the direction of $\kappa \geq 0$ is proved by using modular forms on the period domain. For a family $\mathcal{F} \to \mathcal{M}$ of lattice-polarised holomorphic symplectic varieties of dimension $2d$ whose period map is dominant and generically finite, we construct an injective map (Theorem 3.1)

$$S_{b+dn}(\Gamma, \det) \hookrightarrow H^0(K_{\bar{\mathcal{F}}_n}), \tag{1.1}$$

where $\bar{\mathcal{F}}_n$ is a smooth projective model of $\mathcal{F}_n$, Γ is an arithmetic group containing the monodromy group, $S_k(\Gamma, \det)$ is the space of Γ-cusp forms of weight k and character det

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and $b = \dim \mathcal{M}$. For the six cases in Theorem 1.1, we construct cusp forms explicitly by using the quasi-pullback of the Borcherds Φ_{12} form [4, 5] and its product with modular forms obtained by the Gritsenko lifting [9]. The same technique of construction should also be applicable to lattice-polarised families, of which more examples would be available.

The proof of unirationality is done by geometric argument, but in the BD and DV cases, we also make use of the "transcendental" results $\kappa(\mathcal{F}_{14}^{BD}) \geq 0$ and $\kappa(\mathcal{F}_{6}^{DV}) \geq 0$ when checking the nondegeneracy of certain maps in the argument (Claim 4.3).

A similar result has been obtained for $K3$ surfaces of low genus g [21], where the quasi-pullback $\Phi_{K3,g}$ of Φ_{12} was crucial too. Moreover, when $3 \leq g \leq 10$, the weight of $\Phi_{K3,g}$ minus 19 coincided with the dimension of a representation space appearing in the projective model of the $K3$ surfaces. In the present paper we see no such a direct identity, but we observe a "switched" identity between $K3$ surfaces of genus 2 and cubic fourfolds (Remark 4.4).

This paper is organised as follows. Section 2 is a recollection of holomorphic symplectic manifolds and modular forms. In Section 3 we construct map (1.1) (Theorem 3.1) and prove the latter half of Theorem 1.1 (Corollary 3.3). The first half of Theorem 1.1 is proved in Sections 4–8.

Throughout this paper, a *lattice* means a free abelian group of finite rank endowed with a nondegenerate integral symmetric bilinear form. A_k, D_l and E_m stand for the *negative-definite* root lattices of respective types. The even unimodular lattice of signature $(1,1)$ is denoted by U. No confusion is likely to occur when U is also used for an open set of a variety. The Grassmannian parametrising r-dimensional linear subspaces of $\mathbb{C}^N$ are denoted by $G(r,N) = \mathbb{G}(r-1,N-1)$. We freely use the fact [24] that if $G = \mathrm{PGL}_N$ acts on a projective variety X and U is a G-invariant Zariski open set of X contained in the stable locus, then a geometric quotient U/G exists. If no point of U has nontrivial stabiliser, then $U \to U/G$ is a principal G-bundle in the étale topology. In that case, every G-linearised vector bundle on U descends to a vector bundle on U/G. Similarly, if V is a representation of SL_N, a geometric quotient $(\mathbb{P}V \times U)/G$ exists as a Brauer–Severi variety over U/G. If Y is a normal G-invariant subvariety of $\mathbb{P}V \times U$, its geometric quotient Y/G exists as the image of Y in $(\mathbb{P}V \times U)/G$.

2. Preliminaries

In this section we recall basic facts about holomorphic symplectic manifolds (§2.1) and orthogonal modular forms (§2.2).

2.1. Holomorphic symplectic manifolds

A compact Kähler manifold X of dimension $2d$ is called a *holomorphic symplectic manifold* if it is simply connected and $H^0(\Omega_X^2) = \mathbb{C}\omega$ for a nowhere degenerate 2-form ω. There exists a nondivisible integral symmetric bilinear form q_X of signature $(3, b_2(X)-3)$ on $H^2(X,\mathbb{Z})$, called the *Beauville form* [2], and a constant c_X called the *Fujiki constant*, such that $\int_X v^{2d} = c_X \cdot q_X(v,v)^d$ for every $v \in H^2(X,\mathbb{Z})$. In particular, for $\omega \in H^0(\Omega_X^2)$,

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we have $q_X(\omega,\omega)=0$ and

$$q_X(\omega,\bar{\omega})^d = C\int_X (\omega\wedge\bar{\omega})^d \tag{2.1}$$

for a suitable constant C.

A holomorphic symplectic manifold X is said to be of $K3^{[m]}$ *type* if it is deformation-equivalent to the Hilbert scheme of m points on a $K3$ surface. The Beauville lattice of such X is isometric to $L_{2t}=3U\oplus 2E_8\oplus\langle -2t\rangle$, where $t=m-1$ [2]. Let $h\in L_{2t}$ be a primitive vector of norm $2D>0$. The orthogonal complement $h^{\perp}\cap L_{2t}$ is described as follows [10, §3]. For simplicity we assume $(t,D)=1$, which holds in later sections except §8.2. We have either $(h,L_{2t})=\mathbb{Z}$ or $2\mathbb{Z}$. In the former case, h is of *split type*, and $h^{\perp}\cap L_{2t}$ is isometric to $2U\oplus 2E_8\oplus\langle -2t\rangle\oplus\langle -2D\rangle$. In the latter case, h is of *nonsplit type*, and

$$h^{\perp}\cap L_{2t}\simeq 2U\oplus 2E_8\oplus\begin{pmatrix}-2t & t\\ t & -(D+t)/2\end{pmatrix}, \tag{2.2}$$

which has determinant tD. In Sections 4–7, h will be of nonsplit type and the determinant tD will be a prime number of class number 1.

2.2. Modular forms

Let L be a lattice of signature $(2,b)$ with $b\geq 3$. The dual lattice of L is denoted by $L^{\vee}$. We write $A_L=L^{\vee}/L$ for the discriminant group of L. A_L is equipped with a natural $\mathbb{Q}/\mathbb{Z}$-valued bilinear form, which when L is even is induced from a natural $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form. The Hermitian symmetric domain $\mathcal{D}=\mathcal{D}_L$ attached to L is defined as either of the two connected components of the space

$$\{\mathbb{C}\omega\in\mathbb{P}L_{\mathbb{C}}\,|\,(\omega,\omega)=0,(\omega,\bar{\omega})>0\}.$$

Let $\mathrm{O}^+(L)$ be the subgroup of the orthogonal group $\mathrm{O}(L)$ preserving the component $\mathcal{D}$. We write $\tilde{\mathrm{O}}^+(L)$ for the kernel of $\mathrm{O}^+(L)\to\mathrm{O}(A_L)$. When $A_L\simeq\mathbb{Z}/p$ for a prime p, which holds in Sections 4–7, we have $\mathrm{O}(A_L)=\{\pm\mathrm{id}\}$ and so $\mathrm{O}^+(L)=\langle\tilde{\mathrm{O}}^+(L),-\mathrm{id}\rangle$.

Let $\mathcal{L}$ be the restriction of the tautological line bundle $\mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)$ over $\mathcal{D}$. $\mathcal{L}$ is naturally $\mathrm{O}^+(L_{\mathbb{R}})$-linearised. Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$ and χ be a unitary character of Γ. A Γ-invariant holomorphic section of $\mathcal{L}^{\otimes k}\otimes\chi$ over $\mathcal{D}$ is called a *modular form* of weight k and character χ with respect to Γ. When it vanishes at the cusps, it is called a *cusp form* (for the precise definition, see, e.g., [11, 21]). We write $M_k(\Gamma,\chi)$ for the space of Γ-modular forms of weight k and character χ, and $S_k(\Gamma,\chi)$ for the subspace of cusp forms. We denote $M_k(\Gamma)=M_k(\Gamma,1)$. If $\Gamma'\lhd\Gamma$ is a normal subgroup of finite index, the quotient group Γ/Γ' acts on $M_k(\Gamma',\chi)$ by translating Γ'-invariant sections by elements of Γ. We also remark that when $\chi=\det$ and $k\equiv b \bmod 2$, $-\mathrm{id}$ acts trivially on $\mathcal{L}^{\otimes k}\otimes\det$, so that

$$M_k(\langle\Gamma,-\mathrm{id}\rangle,\det)=M_k(\Gamma,\det). \tag{2.3}$$

When $k\not\equiv b \bmod 2$, $M_k(\langle\Gamma,-\mathrm{id}\rangle,\det)$ is zero.

The Hermitian form $(\cdot,\bar{\cdot})$ on $L_{\mathbb{C}}$ defines an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on the line bundle $\mathcal{L}$. This defines a Γ-invariant Hermitian metric on $\mathcal{L}^{\otimes k}\otimes\chi$ which we denote

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by $(\,,\,)_{k,\chi}$. For brevity, we write $(\,,\,)_k$ instead of $(\,,\,)_{k,1}$. Let vol be the $\mathrm{O}^+(L_{\mathbb{R}})$-invariant volume form on $\mathcal{D}$, which exists and is unique up to constant.

Lemma 2.1. *Let $\mathcal{M}'$ be a Zariski open set of $\Gamma\backslash\mathcal{D}$ and $\mathcal{D}'\subset\mathcal{D}$ be its inverse image. Let Φ be a Γ-invariant holomorphic section of $\mathcal{L}^{\otimes k}\otimes\chi$ defined over $\mathcal{D}'$ with $k\geq b$. Then $\Phi\in S_k(\Gamma,\chi)$ if and only if $\int_{\mathcal{M}'}(\Phi,\Phi)_{k,\chi}\mathrm{vol}<\infty$.*

Proof. In [21, Proposition 3.5], this is proved when $\mathcal{D}'=\mathcal{D}$, – that is, $\Phi\in M_k(\Gamma,\chi)$. Hence it suffices here to show that $\int_{\mathcal{M}'}(\Phi,\Phi)_{k,\chi}\mathrm{vol}<\infty$ implies holomorphicity of Φ over $\mathcal{D}$. Let H be an irreducible component of $\mathcal{D}-\mathcal{D}'$. We may assume that H is of codimension 1. If Φ has a pole along H, say of order $a>0$, a local calculation shows that in a neighbourhood of a general point of H, with H locally defined by $z=0$, the integral

$$\begin{aligned}\int_{\varepsilon\leq|z|\leq 1}(\Phi,\Phi)_{k,\chi}\mathrm{vol}&\geq C\int_{\varepsilon\leq|z|\leq 1}|z|^{-2a}dz\wedge d\bar{z}\\&=C\int_0^{2\pi}d\theta\int_{\varepsilon}^1 r^{-2a+1}dr\qquad (z=re^{i\theta})\end{aligned}$$

must diverge as $\varepsilon\to 0$. □

Let $II_{2,26}=2U\oplus 3E_8$ be the even unimodular lattice of signature $(2,26)$. Borcherds [4] discovered a modular form Φ_{12} of weight 12 and character det for $\mathrm{O}^+(II_{2,26})$. The *quasi-pullback* of Φ_{12} is defined as follows [4, 5]. Let L be a sublattice of $II_{2,26}$ of signature $(2,b)$ and $N=L^{\perp}\cap II_{2,26}$. Let $r(N)$ be the number of (-2)-vectors in N. Then

$$\Phi_{12}|_L:=\left.\frac{\Phi_{12}}{\prod_{\delta}(\delta,\cdot)}\right|_{\mathcal{D}_L},$$

where δ runs over all (-2)-vectors in N up to ± 1, is a nonzero modular form on $\mathcal{D}_L$ of weight $12+r(N)/2$ and character det for $\tilde{\mathrm{O}}^+(L)$. Moreover, when $r(N)>0$, $\Phi_{12}|_L$ is a cusp form [11].

In later sections, we will embed $h^{\perp}\cap L_{2t}$ into $II_{2,26}$ by embedding the last rank 2 component of formula (2.2) into E_8. The following model of E_8 will be used:

$$E_8=\{(x_i)\in\mathbb{Q}^8\,|\,(x_i)\in\mathbb{Z}^8\text{ or }(x_i)\in(\mathbb{Z}+1/2)^8,\ x_1+\cdots+x_8\in 2\mathbb{Z}\}.\tag{2.4}$$

Here we take the standard (negative) quadratic form on $\mathbb{Q}^8$. The (-2)-vectors in E_8 are as follows. For $j\neq k$ we define $\delta_{\pm j,\pm k}=(x_i)$ by $x_j=\pm 1$, $x_k=\pm 1$ and $x_i=0$ for $i\neq j,k$. For example, $\delta_{+1,-2}=(1,-1,0,\dots,0)$. For a subset S of $\{1,\dots,8\}$ consisting of even elements, we define $\delta'_S=(x_i)$ by $x_i=1/2$ if $i\in S$ and $x_i=-1/2$ if $i\notin S$. These are the 240 roots of E_8.

We will also use the Gritsenko lifting [9]. Assume that L is even and contains $2U$. We shall specialise to the case $b=20$ for later use. For an odd number k, let $M_k(\rho_L)$ be the space of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ of weight k with values in the Weil representation ρ_L on $\mathbb{C}A_L$. The Gritsenko lifting with $b=20$ is an injective, $\mathrm{O}^+(L)$-equivariant map

$$M_k(\rho_L)\hookrightarrow M_{k+9}(\tilde{\mathrm{O}}^+(L)).$$

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The dimension of $M_k(\rho_L)$ for $k > 2$ can be explicitly computed by using the formula in [6]. A similar formula for the $\mathrm{O}(A_L)$-invariant part $M_k(\rho_L)^{\mathrm{O}(A_L)}$ is given in [20].

3. Cusp forms and canonical forms

In this section we establish, in a general setting, a correspondence between canonical forms on an n-fold universal family of holomorphic symplectic varieties and modular forms on the period domain. This is the basis of this paper. As a consequence, we deduce in Corollary 3.3 the latter half of Theorem 1.1. The first half of Theorem 1.1 will be proved case by case in later sections.

Let M be a hyperbolic lattice and L be a lattice of signature $(2, b)$. We say that a smooth algebraic family $\pi : \mathcal{F} \to \mathcal{M}$ of holomorphic symplectic manifolds is *M-polarised with polarised Beauville lattice L* if $R^2\pi_*\mathbb{Z}$ contains a sub local system Λ_{pol} in its $(1,1)$-part whose fibre is isometric to M with the orthogonal complement isometric to L. Let $\Lambda_{per} = (\Lambda_{pol})^{\perp} \cap R^2\pi_*\mathbb{Z}$ and choose an isometry $(\Lambda_{per})_{x_0} \simeq L$ at some base point $x_0 \in \mathcal{M}$. If a finite-index subgroup Γ of $\mathrm{O}^+(L)$ contains the monodromy group of Λ_{per}, we can define the period map

$$\mathcal{P} : \mathcal{M} \to \Gamma\backslash\mathcal{D}_L, \qquad x \mapsto [H^{2,0}(\mathcal{F}_x) \subset (\Lambda_{per})_x \otimes \mathbb{C}].$$

By Borel's extension theorem, $\mathcal{P}$ is a morphism of algebraic varieties. Our interest will be in the case $\mathrm{rk}(M) = 1$, but the proof of the following theorem works in the general lattice-polarised setting as well:

Theorem 3.1. *Let L be a lattice of signature $(2, b)$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Let $\mathcal{F} \to \mathcal{M}$ be a smooth algebraic family of lattice-polarised holomorphic symplectic manifolds of dimension $2d$ with polarised Beauville lattice L whose monodromy group is contained in Γ. Assume that the period map $\mathcal{P} : \mathcal{M} \to \Gamma\backslash\mathcal{D}$ is dominant and generically finite. If $\mathcal{F}_n = \mathcal{F} \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \mathcal{F}$ (n times) and $\bar{\mathcal{F}}_n$ is a smooth projective model of $\mathcal{F}_n$, we have a natural injective map*

$$S_{b+dn}(\Gamma, \det) \hookrightarrow H^0(K_{\bar{\mathcal{F}}_n}) \tag{3.1}$$

which makes the following diagram commutative:

$$\begin{tikzcd} \bar{\mathcal{F}}_n \arrow[r, dashed, "\phi_K"] \arrow[d, dashed] & {|K_{\bar{\mathcal{F}}_n}|^\vee} \arrow[d, dashed, "(3.1)^\vee"] \\ \Gamma\backslash\mathcal{D} \arrow[r, dashed, "\phi"'] & \mathbb{P}S_{b+dn}(\Gamma, \det)^\vee. \end{tikzcd} \tag{3.2}$$

Here ϕ_K is the canonical map of $\bar{\mathcal{F}}_n$ and ϕ is the rational map defined by the sections in $S_{b+dn}(\Gamma, \det)$. Furthermore, if the period map $\mathcal{P}$ is birational and Γ does not contain $-\mathrm{id}$, map (3.1) is an isomorphism.

Proof. Let $\mathcal{M}' = \mathcal{P}(\mathcal{M}) \subset \Gamma\backslash\mathcal{D}$ and $\mathcal{D}' \subset \mathcal{D}$ be the inverse image of $\mathcal{M}'$. Shrinking $\mathcal{M}$ as necessary, we may assume that both $\mathcal{M} \to \mathcal{M}'$ and $\mathcal{D}' \to \mathcal{M}'$ are unramified. We take the universal cover $\tilde{\mathcal{M}} \to \mathcal{M}$ of $\mathcal{M}$ and pull back the family: write $\tilde{\mathcal{F}} = \mathcal{F} \times_{\mathcal{M}} \tilde{\mathcal{M}}$ with

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the projection $\pi : \tilde{\mathcal{F}} \to \tilde{\mathcal{M}}$. We obtain a lift $\tilde{\mathcal{P}} : \tilde{\mathcal{M}} \to \mathcal{D}' \subset \mathcal{D}$ of the period map $\mathcal{P}$ which is equivariant with respect to the monodromy representation $\pi_1(\mathcal{M}) \to \Gamma$. Since $\mathcal{P}$ is unramified, $\tilde{\mathcal{P}}$ is unramified too. We first construct an injective map

$$H^0(\mathcal{D}', \mathcal{L}^{\otimes b+dn} \otimes \det)^{\Gamma} \hookrightarrow H^0(\mathcal{F}_n, K_{\mathcal{F}_n}), \tag{3.3}$$

where $H^0(\mathcal{F}_n, K_{\mathcal{F}_n})$ means the space of holomorphic (rather than regular) canonical forms on $\mathcal{F}_n$.

We have a natural $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant isomorphism $K_{\mathcal{D}} \simeq \mathcal{L}^{\otimes b} \otimes \det$ of line bundles over $\mathcal{D}$ (see, e.g., [11, 21]), and hence a $\pi_1(\mathcal{M})$-equivariant isomorphism

$$K_{\tilde{\mathcal{M}}} \simeq \tilde{\mathcal{P}}^* K_{\mathcal{D}} \simeq \tilde{\mathcal{P}}^*(\mathcal{L}^{\otimes b} \otimes \det) \tag{3.4}$$

over $\tilde{\mathcal{M}}$. Here $\pi_1(\mathcal{M})$ acts on $\tilde{\mathcal{P}}^*\mathcal{L}$, $\tilde{\mathcal{P}}^*\det$ through the Γ-action on $\mathcal{L}$, det and the monodromy representation $\pi_1(\mathcal{M}) \to \Gamma$.

On the other hand, by the definition of the period map, we have a canonical isomorphism $\pi_*\Omega_\pi^2 \simeq \tilde{\mathcal{P}}^*\mathcal{L}$ sending a symplectic form to its cohomology class. Since $\pi : \tilde{\mathcal{F}} \to \tilde{\mathcal{M}}$ is a family of holomorphic symplectic manifolds, both $\pi_*\Omega_\pi^2$ and π_*K_π are invertible sheaves, and the homomorphism $(\pi_*\Omega_\pi^2)^{\otimes d} \to \pi_*K_\pi$ defined by the wedge product is isomorphic. Therefore we have a natural isomorphism $\pi_*K_\pi \simeq \tilde{\mathcal{P}}^*\mathcal{L}^{\otimes d}$. Since the natural homomorphism $\pi^*\pi_*K_\pi \to K_\pi$ is isomorphic, we find that $K_\pi \simeq \pi^*\tilde{\mathcal{P}}^*\mathcal{L}^{\otimes d}$. By construction this is $\pi_1(\mathcal{M})$-equivariant. If we write $\tilde{\mathcal{F}}_n = \mathcal{F}_n \times_{\mathcal{M}} \tilde{\mathcal{M}}$ with the projection $\pi_n : \tilde{\mathcal{F}}_n \to \tilde{\mathcal{M}}$, this shows that

$$K_{\pi_n} \simeq \pi_n^* \tilde{\mathcal{P}}^* \mathcal{L}^{\otimes dn} \tag{3.5}$$

as $\pi_1(\mathcal{M})$-linearised line bundles on $\tilde{\mathcal{F}}_n$. Combining formulas (3.4) and (3.5), we obtain a $\pi_1(\mathcal{M})$-equivariant isomorphism

$$K_{\tilde{\mathcal{F}}_n} \simeq \pi_n^* \tilde{\mathcal{P}}^*(\mathcal{L}^{\otimes b+dn} \otimes \det)$$

over $\tilde{\mathcal{F}}_n$. Hence pullback of sections of $\mathcal{L}^{\otimes b+dn} \otimes \det$ over $\mathcal{D}'$ by $\tilde{\mathcal{P}} \circ \pi_n$ defines a $\pi_1(\mathcal{M})$-equivariant injective map

$$H^0(\mathcal{D}', \mathcal{L}^{\otimes b+dn} \otimes \det) \hookrightarrow H^0(\tilde{\mathcal{F}}_n, K_{\tilde{\mathcal{F}}_n}). \tag{3.6}$$

Taking the invariant parts by Γ and $\pi_1(\mathcal{M})$, respectively, we obtain map (3.3).

Next we prove that restriction of map (3.3) gives the desired map (3.1). Let Φ be a Γ-invariant section of $\mathcal{L}^{\otimes b+dn} \otimes \det$ over $\mathcal{D}'$ and $\omega \in H^0(K_{\mathcal{F}_n})$ be the image of Φ by map (3.3). We shall show that

$$\int_{\mathcal{F}_n} \omega \wedge \bar{\omega} = C \int_{\mathcal{M}'} (\Phi, \Phi)_{b+dn,\,\det} \mathrm{vol}$$

for some constant C. Our assertion then follows from Lemma 2.1 and the standard fact that ω extends over a smooth projective model of $\mathcal{F}_n$ if and only if $\int_{\mathcal{F}_n} \omega \wedge \bar{\omega} < \infty$.

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Since the problem is local, it suffices to take an arbitrary small open set $U \subset \tilde{\mathcal{M}}$ and prove

$$\int_{\pi_n^{-1}(U)} \omega \wedge \bar{\omega} = C \int_{\tilde{\mathcal{P}}(U)} (\Phi, \Phi)_{b+dn,\,\det} \mathrm{vol} \tag{3.7}$$

for some constant C independent of U. In what follows, C stands for any such unspecified constant. Since U is small, we may decompose Φ as $\Phi = \Phi_1 \otimes \Phi_2^{\otimes dn}$, with Φ_1 a local section of $\mathcal{L}^{\otimes b} \otimes \det$ and Φ_2 a local section of $\mathcal{L}$. Let ω_1 be the canonical form on $U \simeq \tilde{\mathcal{P}}(U)$ corresponding to Φ_1, and ω_2 be the relative symplectic form on $\tilde{\mathcal{F}}|_U \to U$ corresponding to $\tilde{\mathcal{P}}^* \Phi_2$. On the one hand, we have

$$\omega_1 \wedge \bar{\omega}_1 = C(\Phi_1, \Phi_1)_{b,\,\det} \mathrm{vol} \tag{3.8}$$

(see, e.g., [21, §3.1]). On the other hand, at each fibre X of $\tilde{\mathcal{F}}|_U$ the pointwise Petersson norm $(\Phi_2, \Phi_2)_1 = (\Phi_2, \bar{\Phi}_2)$ is nothing but the pairing $q_X(\omega_2, \bar{\omega}_2)$ in the Beauville form of X. Since

$$q_X(\omega_2, \bar{\omega}_2)^d = C \int_X (\omega_2 \wedge \bar{\omega}_2)^d$$

by equation (2.1), we find that

$$(\Phi_2^{\otimes dn}, \Phi_2^{\otimes dn})_{dn} = C \int_{X^n} (p_1^* \omega_2 \wedge \cdots \wedge p_n^* \omega_2)^d \wedge (p_1^* \bar{\omega}_2 \wedge \cdots \wedge p_n^* \bar{\omega}_2)^d, \tag{3.9}$$

where $p_i \colon X^n \to X$ is the ith projection. Since $(p_1^* \omega_2 \wedge \cdots \wedge p_n^* \omega_2)^d$ is the canonical form on X^n corresponding to the value of $\Phi_2^{\otimes dn}$ at $[X] \in U$, equations (3.8) and (3.9) imply equation (3.7). Thus we obtain map (3.1). Since this map is defined by pullback of sections of line bundles, diagram (3.2) is commutative.

Finally, when $\mathcal{P}$ is birational, we may assume as before that it is an open immersion. If Γ does not contain $-\mathrm{id}$, Γ acts on $\mathcal{D}$ effectively, and the monodromy group coincides with Γ. We can kill the monodromy by pulling back the family $\mathcal{F} \to \mathcal{M}$ to $\mathcal{D}'$ instead of to $\tilde{\mathcal{M}}$. If we rewrite $\tilde{\mathcal{F}}_n = \mathcal{F}_n \times_{\mathcal{M}} \mathcal{D}'$, then map (3.6) is isomorphic. Taking the Γ-invariant part, we see that map (3.3) is isomorphic. Finally, taking the subspace of finite norm, we see that map (3.1) is isomorphic. This completes the proof of Theorem 3.1. □

Remark 3.2. The last statement of Theorem 3.1 can also be proved more directly by using the line bundles on $\mathcal{M} \subset \Gamma \backslash \mathcal{D}$ coming from the Γ-linearised line bundles $\mathcal{L}$, det on $\mathcal{D}$.

Corollary 3.3. *If n is sufficiently large, then $\kappa(\mathcal{F}_n) = b$.*

Proof. Since $\mathcal{F}_n \to \mathcal{F}_{n-1}$ is a smooth family of holomorphic symplectic varieties, $\kappa(\mathcal{F}_n)$ is nondecreasing with respect to n by Iitaka's subadditivity conjecture known in this case [16]. We also have the bound $\kappa(\mathcal{F}_n) \le \dim \mathcal{M} = b$ by Iitaka's addition formula [12].

We take a weight k_0 such that $S_{k_0}(\Gamma, \det) \neq \{0\}$. Then we take a weight k_1 such that $k_1 \equiv b - k_0 \bmod d$ and that $\Gamma \backslash \mathcal{D} \dashrightarrow \mathbb{P} M_{k_1}(\Gamma)^\vee$ is generically finite onto its image. (When Γ contains $-\mathrm{id}$, we must have $k_0 \equiv b \bmod 2$, so $b - k_0 + d\mathbb{Z}$ contains sufficiently large even k_1.) Since

$$S_{k_0}(\Gamma, \det) \cdot M_{k_1}(\Gamma) \subset S_{k_0+k_1}(\Gamma, \det),$$

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Theorem 3.1 implies that for $n_0 = (k_0 + k_1 - b)/d$, the image of the canonical map of $\bar{\mathcal{F}}_{n_0}$ has dimension $\geq b$. Hence $\kappa(\mathcal{F}_{n_0}) \geq b$, and so $\kappa(\mathcal{F}_n) = b$ for all $n \geq n_0$. □

This proves the latter half of Theorem 1.1. In the following sections, we apply Theorem 3.1 to the six explicit families in Theorem 1.1. In practice, one needs to identify the group Γ. For example, according to [22, Remark 8.5] and [11, Remark 3.15], the monodromy group of a family of polarised symplectic manifolds of $K3^{[2]}$ type with polarisation vector h is contained in $\tilde{\mathrm{O}}^+(h^\perp \cap L_2)$.

4. Fano varieties of cubic fourfolds

In this section we prove Theorem 1.1 for the case of Fano varieties of cubic fourfolds [3]. Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold. The Fano variety $F(Y) \subset \mathbb{G}(1,5)$ of Y is the variety parametrising lines on Y, which is smooth of dimension 4. Beauville and Donagi [3] proved that $F(Y)$ is a holomorphic symplectic manifold of $K3^{[2]}$ type polarised by the Plücker, and its polarised Beauville lattice is isometric to $L_{cub} = 2U \oplus 2E_8 \oplus A_2$. In fact, the polarised Beauville lattice of $F(Y)$ is isomorphic to the primitive part of $H^4(Y,\mathbb{Z})$ as polarised Hodge structures, where the intersection form on $H^4(Y,\mathbb{Z})$ is (-1)-scaled.

Let $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the parameter space of smooth cubic fourfolds. By GIT [24], the geometric quotient U/PGL_6 exists as an affine variety of dimension 20. Let $\Gamma = \tilde{\mathrm{O}}^+(L_{cub})$. The period map $U/\mathrm{PGL}_6 \to \Gamma\backslash\mathcal{D}$ is an open immersion by Voisin [28], and the complement of its image was determined by Looijenga [19] and Laza [17].

Lemma 4.1 (cf. [17])**.** *The cusp form* $\Phi_{12}|_{L_{cub}}$ *has weight* 48*. Moreover,* $S_{66}(\Gamma, \det)$ *and* $S_{68}(\Gamma, \det)$ *have dimension* ≥ 2*.*

Proof. Write $L = L_{cub}$. The weight of $\Phi_{12}|_L$ is computed in [17]. ($A_2^\perp \simeq E_6$ has 72 roots.) We have $\dim M_k(\rho_L) = [(k+3)/6]$ by computing the formula in [6]. The product of $\Phi_{12}|_L$ with the Gritsenko lift of $M_9(\rho_L)$ and $M_{11}(\rho_L)$ proves the second assertion. □

We consider the parameter space of smooth cubic fourfolds with n marked lines,

$$F_n = \{\,(Y, l_1, \cdots, l_n) \mid Y \in U,\, l_1, \cdots, l_n \in F(Y)\,\} \subset U \times \mathbb{G}(1,5)^n,$$

and let $\mathcal{F}_n = F_n/\mathrm{PGL}_6$. Then $\mathcal{F}_n$ is smooth over the open locus of U/PGL_6 where cubic fourfolds have no nontrivial stabiliser. By Lemma 4.1, with $48 = 20 + 2\cdot 14$ and $66 = 20 + 2\cdot 23$, we see that $\mathcal{F}_{14}$ has positive geometric genus and $\kappa(\mathcal{F}_{23}) > 0$. (Cusp forms of weight 68 will be used in Section 6.) It remains to prove that $\mathcal{F}_{13}$ is unirational.

Proposition 4.2. F_{13} *is rational.*

Proof. Consider the second projection $\pi\colon F_{13} \to \mathbb{G}(1,5)^{13}$. If $(l_1, \ldots, l_{13}) \in \pi(F_{13})$, the fibre $\pi^{-1}(l_1, \ldots, l_{13})$ is a nonempty open set of the linear system of cubics containing $l_1, \ldots, l_{13}$, which we denote by

$$\mathbb{P}V(l_1, \ldots, l_{13}) = \mathbb{P}\mathrm{Ker}(H^0(\mathcal{O}_{\mathbb{P}^5}(3)) \to \oplus_{i=1}^{13} H^0(\mathcal{O}_{l_i}(3))).$$

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This shows that F_{13} is birationally a $\mathbb{P}^N$-bundle over $\pi(F_{13})$ with

$$N = \dim F_{13} - \dim \pi(F_{13}) \geq \dim F_{13} - \dim \mathbb{G}(1,5)^{13} = 3.$$

Hence we are reduced to the following assertion:

Claim 4.3. $\pi\colon F_{13} \to \mathbb{G}(1,5)^{13}$ is dominant.

Assume to the contrary that π is not dominant. Then we have $\dim V(l_1,\dots,l_{13}) \geq 5$ for a general point $(l_1,\dots,l_{13})$ of $\pi(F_{13})$. Consider the similar projection $\pi'\colon F_{14} \to \mathbb{G}(1,5)^{14}$ in $n = 14$. Since $\mathcal{F}_{14} = F_{14}/\mathrm{PGL}_6$ cannot be uniruled as just proved, we must have $\dim V(l_1,\dots,l_{14}) = 1$ for general $(l_1,\dots,l_{14}) \in \pi'(F_{14})$. On the other hand, $V(l_1,\dots,l_{14})$ can be written as

$$V(l_1,\dots,l_{14}) = \mathrm{Ker}(V(l_1,\dots,l_{13}) \overset{\rho}{\to} H^0(\mathcal{O}_{l_{14}}(3))),$$

where ρ is the restriction map. Hence for general $(l_1,\dots,l_{13}) \in \pi(F_{13})$, we have $\dim V(l_1,\dots,l_{13}) = 5$, ρ is surjective and $\pi(F_{13})$ is of codimension 1 in $\mathbb{G}(1,5)^{13}$.

The last property implies that the similar projection $\pi''\colon F_{12} \to \mathbb{G}(1,5)^{12}$ in $n = 12$ must be dominant, because otherwise $\pi(F_{13})$ would be dense in the inverse image of $\pi''(F_{12}) \subset \mathbb{G}(1,5)^{12}$ by the projection $\mathbb{G}(1,5)^{13} \to \mathbb{G}(1,5)^{12}$, which contradicts the $\mathfrak{S}_{13}$-invariance of $\pi(F_{13})$. This in turn shows that

$$\dim V(l_1,\dots,l_{12}) = \dim F_{12} - \dim \mathbb{G}(1,5)^{12} + 1 = 8$$

for a general point $(l_1,\dots,l_{12})$ of $\mathbb{G}(1,5)^{12}$. However, since $V(l_1,\dots,l_{13}) \to H^0(\mathcal{O}_{l_{14}}(3))$ is surjective, $V(l_1,\dots,l_{12}) \to H^0(\mathcal{O}_{l_{14}}(3))$ is surjective too. Hence $\dim V(l_1,\dots,l_{12},l_{14}) = 4$. But since $(l_1,\dots,l_{12},l_{14})$ is a general point of $\pi(F_{13})$, this is absurd. This proves Claim 4.3 and so finishes the proof of Proposition 4.2. □

Remark 4.4. In the analogous case of $K3$ surfaces of genus g [21], when $3 \leq g \leq 10$, the weight of the quasi-pullback $\Phi_{K3,g}$ of Φ_{12} coincided with

$$\mathrm{weight}(\Phi_{K3,g}) = \dim V_g + 19 = \dim V_g + \dim(\mathrm{moduli})$$

for a representation space V_g related to the projective model of the $K3$ surfaces. Here, for $\Phi_{K3,2}$ and $\Phi_{cubic} = \Phi_{12}|_{L_{cub}}$, the "switched" equalities

$$\mathrm{weight}(\Phi_{K3,2}) - 19 = 56 = h^0(\mathcal{O}_{\mathbb{P}^5}(3))$$
$$\mathrm{weight}(\Phi_{cubic}) - 20 = 28 = h^0(\mathcal{O}_{\mathbb{P}^2}(6))$$

hold. Is this accidental?

5. Debarre–Voisin fourfolds

In this section we prove Theorem 1.1 for the case of Debarre–Voisin fourfolds [8]. Let $\mathcal{E}$ be the dual of the rank 6 universal sub vector bundle over the Grassmannian $G(6,10)$. The space $H^0(\bigwedge^3 \mathcal{E})$ is naturally isomorphic to $\bigwedge^3(\mathbb{C}^{10})^\vee$. Debarre and Voisin [8] proved that the zero locus $X_\sigma \subset G(6,10)$ of a general section σ of $\bigwedge^3 \mathcal{E}$ is a holomorphic symplectic manifold of $K3^{[2]}$ type, and the polarisation given by the Plücker has Beauville norm 22

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and is of nonsplit type. The polarised Beauville lattice is hence isometric to

$$L_{DV} = 2U \oplus 2E_8 \oplus K, \qquad K = \begin{pmatrix} -2 & 1 \\ 1 & -6 \end{pmatrix}.$$

Let $\Gamma = \tilde{\mathrm{O}}^+(L_{DV})$.

Lemma 5.1. *There exists an embedding $K \hookrightarrow E_8$ with $r(K^\perp) = 40$. The resulting cusp form $\Phi_{12}|_{L_{DV}}$ has weight* 32*. Moreover, $S_{46}(\Gamma, \det)$ has dimension* ≥ 2.

Proof. Let v_1, v_2 be the basis of K in the matrix expression. We embed K into E_8, in model (2.4) of E_8, by

$$v_1 \mapsto (1, -1, 0, \dots, 0), \quad v_2 \mapsto (0, 1, 1, 2, 0, \dots, 0).$$

The roots of E_8 orthogonal to these two vectors are $\delta_{\pm i, \pm j}$ with $i, j \geq 5$ and $\pm\delta'_S$ with $1, 2, 3 \in S$ and $4 \notin S$. The total number is $24 + 16 = 40$. Hence the weight of $\Phi_{12}|_{L_{DV}}$ is $12 + 20 = 32$. Working out the formula in [6], we also see that $\dim M_k(\rho_{L_{DV}}) = (k-1)/2$. Taking the product of $\Phi_{12}|_{L_{DV}}$ with the Gritsenko lift of $M_5(\rho_{L_{DV}})$, we obtain the last assertion. □

Let U be the open locus of $\mathbb{P}(\bigwedge^3 \mathbb{C}^{10})^\vee$ where X_σ is smooth of dimension 4 and $[\sigma]$ is PGL_{10}-stable with no nontrivial stabiliser. The period map $U/\mathrm{PGL}_{10} \to \Gamma \backslash \mathcal{D}$ is generically finite and dominant [8]. Consider the incidence

$$F_n = \{ ([\sigma], p_1, \cdots, p_n) \in U \times G(6,10)^n \mid p_i \in X_\sigma \} \subset U \times G(6,10)^n$$

and let $\mathcal{F}_n = F_n/\mathrm{PGL}_{10}$. By Lemma 5.1, with $32 = 20 + 2 \cdot 6$ and $46 = 20 + 2 \cdot 13$, we see that $\mathcal{F}_6$ has positive geometric genus and $\kappa(\mathcal{F}_{13}) > 0$. It remains to show that $\mathcal{F}_5$ is unirational.

Proposition 5.2. *F_5 is rational.*

Proof. Consider the second projection $\pi \colon F_n \to G(6,10)^n$. The fibre $\pi^{-1}(p_1, \dots, p_n)$ over $(p_1, \dots, p_n) \in \pi(F_n)$ is a nonempty open set of the linear system $\mathbb{P}V(p_1, \dots, p_n) \subset \mathbb{P}H^0(\bigwedge^3 \mathcal{E})$ of sections vanishing at $p_1, \dots, p_n$. When $n = 5$, we have

$$\dim V(p_1, \dots, p_5) \geq h^0(\wedge^3 \mathcal{E}) - 5 \cdot \mathrm{rk}(\wedge^3 \mathcal{E}) = 20,$$

so $F_5 \to \pi(F_5)$ is birationally a $\mathbb{P}^N$-bundle with $N \geq 19$. Furthermore, by the same argument as for Claim 4.3, the result $\kappa(\mathcal{F}_6) \geq 0$ enables us to conclude that $F_5 \to G(6,10)^5$ is dominant. Therefore F_5 is rational. □

6. Lehn–Lehn–Sorger–van Straten eightfolds

In this section we prove Theorem 1.1 for the case of Lehn–Lehn–Sorger–van Straten eightfolds [18]. They have the same parameter space and period space as the Beauville–Donagi case.

Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold which does not contain a plane. The space $M^{gtc}(Y)$ of *generalised twisted cubics* on Y is defined as the closure of the locus of twisted cubics on Y in the Hilbert scheme $\mathrm{Hilb}_{3m+1}(Y)$. By [18], $M^{gtc}(Y)$ is smooth and

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irreducible of dimension 10, and there exists a natural contraction $M^{gtc}(Y) \to X(Y)$ to a holomorphic symplectic manifold $X(Y)$ with general fibres $\mathbb{P}^2$. The variety $X(Y)$ is of $K3^{[4]}$ type [1] and has a polarisation of Beauville norm 2 and nonsplit type (see [7, footnote 22]). Hence its polarised Beauville lattice is isometric to the lattice $L_{cub} = 2U \oplus 2E_8 \oplus A_2$ considered in Section 4, and the monodromy group is evidently contained in $\mathrm{O}^+(L_{cub})$. We can reuse Lemma 4.1: since $\mathrm{O}^+(L_{cub}) = \langle \tilde{\mathrm{O}}^+(L_{cub}), -\mathrm{id}\rangle$ and the weights in Lemma 4.1 are even, the cusp forms there are not just $\tilde{\mathrm{O}}^+(L_{cub})$-invariant but also $\mathrm{O}^+(L_{cub})$-invariant, as remarked in equation (2.3).

Let $H = \mathrm{Hilb}^{gtc}(\mathbb{P}^5)$ be the irreducible component of the Hilbert scheme $\mathrm{Hilb}_{3m+1}(\mathbb{P}^5)$ that contains the locus of twisted cubics in $\mathbb{P}^5$. Then H is smooth of dimension 20, and we have $M^{gtc}(Y) = H \cap \mathrm{Hilb}_{3m+1}(Y)$ for Y above [18]. Let $U \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ be the parameter space of smooth cubic fourfolds which does not contain a plane and has no nontrivial stabiliser in PGL_6. The period map $U/\mathrm{PGL}_6 \to \Gamma\backslash\mathcal{D}$, where $\Gamma = \mathrm{O}^+(L_{cub})$, is generically finite and dominant [18, 1]. We consider the incidence

$$M_n^{gtc} = \{\, (Y, C_1, \dots, C_n) \in U \times H^n \mid C_i \in M^{gtc}(Y)\,\} \subset U \times H^n.$$

As noticed in [18], the construction of $X(Y)$ can be done in families. This produces a smooth family $X \to U$ of symplectic eightfolds and a contraction $M_1^{gtc} \to X$ over U with general fibres $\mathbb{P}^2$. Taking the n-fold fibre product $X_n = X \times_U \cdots \times_U X$, we obtain a morphism $M_n^{gtc} \to X_n$ over U with general fibres $(\mathbb{P}^2)^n$. Let $\mathcal{F}_n = X_n/\mathrm{PGL}_6$. By Lemma 4.1, now with $48 = 20 + 4\cdot 7$ and $68 = 20 + 4 \cdot 12$ ($d = 4$ in place of $d = 2$) and with $\Gamma = \mathrm{O}^+(L_{cub})$ in place of $\tilde{\mathrm{O}}^+(L_{cub})$, we see that $\mathcal{F}_7$ has positive geometric genus and $\kappa(\mathcal{F}_{12}) > 0$. It remains to show that $\mathcal{F}_5$ is unirational.

Proposition 6.1. *M_5^{gtc} is unirational.*

Proof. We enlarge M_n^{gtc} to the complete incidence over $|\mathcal{O}_{\mathbb{P}^5}(3)|$:

$$(M_n^{gtc})^* = \{\, (Y, C_1, \dots, C_n) \in |\mathcal{O}_{\mathbb{P}^5}(3)| \times H^n \mid C_i \subset Y\,\}.$$

The fibre of the projection $\pi\colon (M_n^{gtc})^* \to H^n$ over $(C_1, \dots, C_n) \in H^n$ is the linear system $\mathbb{P}V(C_1, \dots, C_n) \subset |\mathcal{O}_{\mathbb{P}^5}(3)|$ of cubics containing $C_1, \dots, C_n$. When $n = 5$, we have $\dim V(C_1, \dots, C_5) \geq 6$ for any $(C_1, \dots, C_5) \in H^5$, so π is surjective, and there is a unique irreducible component of $(M_5^{gtc})^*$ of dimension ≥ 105 that is birationally a $\mathbb{P}^N$-bundle over H^5 with $N \geq 5$. On the other hand, M_5^{gtc} is an open set of the unique irreducible component of $(M_5^{gtc})^*$ of dimension 105 that dominates $|\mathcal{O}_{\mathbb{P}^5}(3)|$. We want to show that these two irreducible components coincide: then $M_5^{gtc} \to H^5$ is dominant, and M_5^{gtc} is birationally a $\mathbb{P}^5$-bundle over H^5 and hence unirational.

Let $(C_1, \dots, C_5)$ be a general point of H^5. By genericity we may assume that each C_i is smooth and spans a 3-plane $P_i \subset \mathbb{P}^5$, $P_i \cap P_j$ is a line and $C_i \cap P_j = \emptyset$. Let $(Y, C_1, \dots, C_5)$ be a general point of $\pi^{-1}(C_1, \dots, C_5) = \mathbb{P}V(C_1, \dots, C_5)$. It suffices to show that generalisation of $(Y, C_1, \dots, C_5)$ – that is, small perturbation inside $(M_5^{gtc})^*$ – contains $(Y', C_1', \dots, C_5')$ with $Y' \in U$.

We may assume that Y is irreducible and contains no 3-plane, because the locus of $(Y, C_1, \dots, C_5)$ with Y reducible or containing a 3-plane has dimension < 105. Since each C_i is smooth, the results of [18, §2] tell us that the cubic surface $S_i = Y \cap P_i$ either

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has at most ADE singularities, or is integral but nonnormal (singular along a line) or is reducible. By comparison of dimension again, we may assume that at least one, say S_1, is of the first type.

Now $(C_2,\dots,C_5)$ is a general point of H^4. The projection $M_4^{gtc}\to H^4$ is dominant, as can be checked similarly in an inductive way. Therefore there exists a cubic fourfold $Y''\in U$ containing $C_2,\dots,C_5$. Let Y' be a general member of the pencil $\langle Y, Y''\rangle$. Since $Y''\in U$, we have $Y'\in U$. Since both Y and Y'' contain $C_2,\dots,C_5$, Y' contains $C_2,\dots,C_5$ too. In the fixed 3-plane P_1, the cubic surface $S'=Y'\cap P_1$ degenerates to the cubic surface $S_1=Y\cap P_1$ with at most ADE singularities, so S' has at most ADE singularities too. By [18, Theorem 2.1], the nets of twisted cubics on cubic surfaces degenerate flatly in such a family. Therefore we have a twisted cubic $C'\subset S'$ which specialises to $C_1\subset S_1$ as Y' specialises to Y. Therefore $(Y',C',C_2,\dots,C_5)\in M_5^{gtc}$ specialises to $(Y,C_1,C_2,\dots,C_5)$. This proves our assertion. □

7. Varieties of sums of powers of cubic fourfolds

In this section we prove Theorem 1.1 for the case of Iliev–Ranestad fourfolds [14]. Let H be the irreducible component of the Hilbert scheme $\mathrm{Hilb}_{10}|\mathcal{O}_{\mathbb{P}^5}(1)|$ of length 10 subschemes of $|\mathcal{O}_{\mathbb{P}^5}(1)|$ that contains the locus of 10 distinct points. For a cubic fourfold $Y\subset\mathbb{P}^5$ with defining equation $f\in H^0(\mathcal{O}_{\mathbb{P}^5}(3))$, its variety of sums of 10 powers $\mathrm{VSP}(Y)=\mathrm{VSP}(Y,10)$ is defined as the closure in H of the locus of distinct $([l_1],\dots,[l_{10}])$ such that $f=\sum_i\lambda_i l_i^3$ for some $\lambda_i\in\mathbb{C}$. Iliev and Ranestad [14, 15] proved that when Y is general, $\mathrm{VSP}(Y)$ is a holomorphic symplectic manifold of $K3^{[2]}$ type, with polarisation of Beauville norm 38 and nonsplit type. (See also [25] for the computation of polarisation.) Hence its polarised Beauville lattice is isometric to

$$L_{IR}=2U\oplus 2E_8\oplus K,\qquad K=\begin{pmatrix}-2 & 1\\ 1 & -10\end{pmatrix}.$$

Let $\Gamma=\tilde{\mathrm{O}}^+(L_{IR})$.

Lemma 7.1. *There exists an embedding $K\hookrightarrow E_8$ with $r(K^\perp)=40$. The resulting cusp form $\Phi_{12}|_{L_{IR}}$ has weight 32. Moreover, $S_{44}(\Gamma,\det)$ has dimension ≥ 2.*

Proof. Let v_1,v_2 be the basis of K in the matrix expression. We embed $K\hookrightarrow E_8$ by sending, in model (2.4) of E_8,

$$v_1\mapsto(1,-1,0,\cdots,0),\quad v_2\mapsto(0,1,3,0,\cdots,0).$$

The roots of E_8 orthogonal to these two vectors are $\delta_{\pm i,\pm j}$ with $i,j\geq 4$, whose number is $2\cdot5\cdot4=40$. Hence $\Phi_{12}|_{L_{IR}}$ has weight $12+20=32$. Furthermore, computing the formula in [6], we see that $\dim M_k(\rho_{L_{IR}})=[(5k-3)/6]$. The product of $\Phi_{12}|_{L_{IR}}$ with the Gritsenko lift of $M_3(\rho_{L_{IR}})$ implies the last assertion. □

Let U be the open locus of $|\mathcal{O}_{\mathbb{P}^5}(3)|$ where $\mathrm{VSP}(Y)$ is smooth of dimension 4 and Y is smooth with no nontrivial stabiliser. The period map $U/\mathrm{PGL}_6\to\Gamma\backslash\mathcal{D}$ is generically

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finite and dominant [14, 15]. Consider the incidence

$$\mathrm{VSP}_n = \{ (Y, \Gamma_1, \dots, \Gamma_n) \in U \times H^n \mid \Gamma_i \in \mathrm{VSP}(Y) \} \subset U \times H^n$$

and let $\mathcal{F}_n = \mathrm{VSP}_n/\mathrm{PGL}_6$. By Lemma 7.1, with $32 = 20+2\cdot 6$ and $44 = 20+2\cdot 12$, we see that $\mathcal{F}_6$ has positive geometric genus and $\kappa(\mathcal{F}_{12}) > 0$. On the other hand, as observed in [14], VSP_1 is birationally a $\mathbb{P}^9$-bundle over H and hence rational. Therefore $\mathcal{F}_1$ is unirational. This proves Theorem 1.1 in the present case.

Remark 7.2. There also exist embeddings $K \hookrightarrow E_8$ with $r(K^\perp) = 30$ (send v_2 to $(0,1,1,2,2,0,0,0)$ or to $(0,1,1,1,1,1,1,2)$), but the resulting cusp form has weight 27, which is not of the form $20+2n$. This, however, suggests that $\kappa \geq 0$ would actually start from at least $n = 4$.

8. Double EPW series

In this section we prove Theorem 1.1 for double EPW sextics [26] and double EPW cubes [13]. They share some common features: both are parametrised by the Lagrangian Grassmannian $LG = LG(\bigwedge^3 \mathbb{C}^6)$, where $\bigwedge^3 \mathbb{C}^6$ is equipped with the canonical symplectic form $\bigwedge^3 \mathbb{C}^6 \times \bigwedge^3 \mathbb{C}^6 \to \bigwedge^6 \mathbb{C}^6$. Both are constructed as double covers of degeneracy loci related to $\bigwedge^3 \mathbb{C}^6$. And both have $L_{EPW} = 2U \oplus 2E_8 \oplus 2A_1$ as the polarised Beauville lattices. Thus they share the same parameter space and essentially the same period space.

The presence of covering involution requires extra care in the construction of the universal (or perhaps we should say “tautological”) family over a Zariski open set of the moduli space.

8.1. Double EPW sextics

We recall the construction of double EPW sextics following [26, 27]. Let F be the vector bundle over $\mathbb{P}^5$ whose fibre over $[v] \in \mathbb{P}^5$ is the image of $\mathbb{C}v \wedge (\bigwedge^2 \mathbb{C}^6) \to \bigwedge^3 \mathbb{C}^6$. For $[A] \in LG$ we write $Y_A[k] \subset \mathbb{P}^5$ for the locus of those $[v] \in \mathbb{P}^5$ such that $\dim(A \cap F_v) \geq k$. We say that A is generic if $Y_A[3] = \emptyset$ and $\mathbb{P}A \cap G(3,6) = \emptyset$ in $\mathbb{P}(\bigwedge^3 \mathbb{C}^6)$. In that case, $Y_A = Y_A[1]$ is a sextic hypersurface in $\mathbb{P}^5$ singular along $Y_A[2]$, $Y_A[2]$ is a smooth surface and Y_A has a transversal family of A_1-singularities along $Y_A[2]$. Let $\lambda_A \colon F \to (\bigwedge^3 \mathbb{C}^6/A) \otimes \mathcal{O}_{\mathbb{P}^5}$ be the composition of the inclusion $F \hookrightarrow \bigwedge^3 \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}^5}$ and the projection $\bigwedge^3 \mathbb{C}^6 \otimes \mathcal{O}_{\mathbb{P}^5} \to (\bigwedge^3 \mathbb{C}^6/A) \otimes \mathcal{O}_{\mathbb{P}^5}$. Then $\mathrm{coker}(\lambda_A) = i_* \zeta_A$ for a coherent sheaf ζ_A on Y_A, where $i : Y_A \hookrightarrow \mathbb{P}^5$ is the inclusion. Let $\xi_A = \zeta_A \otimes \mathcal{O}_{Y_A}(-3)$. If we choose a Lagrangian subspace B of $\bigwedge^3 \mathbb{C}^6$ transverse to A, we can define a multiplication $\xi_A \times \xi_A \to \mathcal{O}_{Y_A}$. Although B is necessary for the construction, the resulting multiplication does not depend on the choice of B [27, p. 152]. Then let $X_A = \mathrm{Spec}(\mathcal{O}_{Y_A} \oplus \xi_A)$. This is a double cover of Y_A. If A is generic, X_A is a holomorphic symplectic manifold of $K3^{[2]}$ type. The polarisation (pullback of $\mathcal{O}_{\mathbb{P}^5}(1)$) has Beauville norm 2 and is of split type, and the polarised Beauville lattice is isometric to L_{EPW}. If $LG^\circ \subset LG$ is the open locus of generic A, the period map $LG^\circ/\mathrm{PGL}_6 \to \Gamma \backslash \mathcal{D}$, where $\Gamma = \tilde{\mathrm{O}}^+(L_{EPW})$, is birational according to [26, §6] and [22, §8].

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Lemma 8.1. *The cusp form* $\Phi_{12}|_{L_{EPW}}$ *has weight* 42. *Moreover,* $S_{58}(\Gamma,\det)$ *has dimension* ≥ 2.

Proof. We embed $2A_1$ in E_8 in any natural way. Then $(2A_1)^\perp \simeq D_6$ has 60 roots, so $\Phi_{12}|_{L_{EPW}}$ has weight 42. Working out the formula in [6], we see that $\dim M_k(\rho_{L_{EPW}}) = [k/3]$. The product of $\Phi_{12}|_{L_{EPW}}$ with the Gritsenko lift of $M_7(\rho_{L_{EPW}})$ implies the second assertion. $\square$

The construction of double EPW sextics can be done over a Zariski open set of the moduli space as follows (cf. [27]). Let $LG' \subset LG^\circ$ be the open locus where A has no nontrivial stabiliser, and $\pi_1\colon LG'\times\mathbb{P}^5 \to LG'$, $\pi_2\colon LG'\times\mathbb{P}^5\to\mathbb{P}^5$ be the projections. Let $Y=\cup_A Y_A \subset LG'\times\mathbb{P}^5$ be the universal family of EPW sextics over LG'.

Lemma 8.2. *There exists a* PGL_6*-invariant Zariski open set* LG'' *of* LG' *such that* $\mathcal{O}_{LG''\times\mathbb{P}^5}(Y)\simeq \pi_2^*\mathcal{O}_{\mathbb{P}^5}(6)$ *as* PGL_6*-linearised line bundles over* $LG''\times\mathbb{P}^5$.

Proof. Consider the quotient $\mathcal{Y}=Y/\mathrm{PGL}_6$, which is a divisor of the Brauer–Severi variety $\mathcal{P}=(LG'\times\mathbb{P}^5)/\mathrm{PGL}_6$ over $\mathcal{M}=LG'/\mathrm{PGL}_6$. Each fibre of $\mathcal{Y}\to\mathcal{M}$ is a canonical divisor of the fibre of $\pi:\mathcal{P}\to\mathcal{M}$. This implies that $\mathcal{O}_{\mathcal{P}}(\mathcal{Y})\simeq K_\pi\otimes\pi^*\mathcal{O}_{\mathcal{M}}(D)$ for some divisor D of $\mathcal{M}$. Removing the support of D from $\mathcal{M}$, we obtain $\mathcal{O}_{\mathcal{P}}(\mathcal{Y})\simeq K_\pi$ over its complement. Pulling back this isomorphism to $LG'\times\mathbb{P}^5\to LG'$, we obtain the desired PGL_6-equivariant isomorphism. $\square$

We rewrite $LG''=LG'$ and $Y|_{LG''}=Y$. Let E be the universal quotient vector bundle of rank 10 over LG'. We have a natural homomorphism $\lambda\colon \pi_2^*F\to\pi_1^*E$ over $LG'\times\mathbb{P}^5$ whose restriction to $\{A\}\times\mathbb{P}^5$ is λ_A, and $\mathrm{coker}(\lambda)=i_*\zeta$ for a coherent sheaf ζ on Y where $i\colon Y\to LG'\times\mathbb{P}^5$ is the inclusion. As was done in [27], if we choose $B\in LG$ and let $U_B\subset LG'$ be the open locus of those A transverse to B, we have a multiplication $\zeta\times\zeta\to\mathcal{O}_Y(Y)$ over $Y|_{U_B}$. Since the multiplication does not depend on the choice of B at each fibre, we obtain an SL_6-equivariant multiplication $\zeta\times\zeta\to\mathcal{O}_Y(Y)$ over the whole Y. If we set $\xi=\zeta\otimes\mathcal{O}_Y(-3)$, Lemma 8.2 enables us to pass to an SL_6-equivariant multiplication $\xi\times\xi\to\mathcal{O}_Y$. Since the scalar matrices in SL_6 act trivially on ξ, ξ is actually PGL_6-linearised and this multiplication is PGL_6-equivariant.

Now taking $X=\mathrm{Spec}(\mathcal{O}_Y\oplus\xi)$, we obtain a universal family of double EPW sextics over LG' acted on by PGL_6. Let $\mathcal{M}=LG'/\mathrm{PGL}_6$, $\mathcal{F}=X/\mathrm{PGL}_6$ and $\mathcal{F}_n=\mathcal{F}\times_{\mathcal{M}}\cdots\times_{\mathcal{M}}\mathcal{F}$ (n times). Note that this is not a moduli space even birationally, as we have not divided out by the covering involution. By Lemma 8.1, with $42=20+2\cdot 11$ and $58=20+2\cdot 19$, we see that $\mathcal{F}_{11}$ has positive geometric genus and $\kappa(\mathcal{F}_{19})>0$. This proves Theorem 1.1 in the case of double EPW sextics.

8.2. Double EPW cubes

We recall the construction of double EPW cubes following [13]. For $[U]\in G(3,6)$, we write $T_U=(\bigwedge^2 U)\wedge\mathbb{C}^6\subset\bigwedge^3\mathbb{C}^6$. For $[A]\in LG$, let $D_k^A\subset G(3,6)$ be the locus of those $[U]$ with $\dim(A\cap T_U)\geq k$. We say that A is generic if $D_4^A=\emptyset$ and $\mathbb{P}A\cap G(3,6)=\emptyset$ in $\mathbb{P}(\bigwedge^3\mathbb{C}^6)$. In that case, D_2^A is a sixfold singular along D_3^A, D_3^A is a smooth threefold and the singularities of D_2^A are a transversal family of $\frac{1}{2}(1,1,1)$ quotient singularity along D_3^A.

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Let $\tilde{D}_2^A \to D_2^A$ be the blowup at D_3^A and $E \subset \tilde{D}_2^A$ be the exceptional divisor. Then $\tilde{D}_2^A$ is smooth and E is a smooth bicanonical divisor of $\tilde{D}_2^A$ [13, p. 254]. Take the double cover $\tilde{Y}_A \to \tilde{D}_2^A$ branched over E and contract the $\mathbb{P}^2$-ruling of the ramification divisor by using pullback of some multiple of $\mathcal{O}_{D_2^A}(1)$. This produces a holomorphic symplectic manifold Y_A of $K3^{[3]}$ type [13, Theorem 1.1].

The polarisation has Beauville norm 4 and divisibility 2, so the polarised Beauville lattice is isometric to L_{EPW} by [10]. The monodromy group is evidently contained in $\mathrm{O}^+(L_{EPW})$ (but whether it is smaller seems unclear to me). The quotient $\mathrm{O}^+(L_{EPW})/\tilde{\mathrm{O}}^+(L_{EPW})$ is $\mathfrak{S}_2$ generated by the switch of the two copies of A_1, say $\iota \in \mathrm{O}^+(L_{EPW})$. Construction of cusp forms becomes more delicate than the previous cases, as $\Phi_{12}|_{L_{EPW}}$ is *anti*-invariant under ι.

Lemma 8.3. *Let $\Gamma = \mathrm{O}^+(L_{EPW})$. Then $S_{68}(\Gamma, \det) \neq \{0\}$ and $S_{80}(\Gamma, \det)$ has dimension ≥ 2.*

Proof. We abbreviate $L = L_{EPW}$. We first verify that $\Phi_{12}|_L$ is ι-anti-invariant. Let ι' be the involution of the D_6 lattice induced by the involution of its Dynkin diagram. Then $\iota \oplus \iota'$ extends to an involution $\tilde{\iota}$ of $II_{2,26}$. The modular form Φ_{12} is $\tilde{\iota}$-invariant. If we run δ over the positive roots of D_6, the product $\prod_\delta (\delta, \cdot)$ is also $\tilde{\iota}$-invariant, because ι' permutes the positive roots of D_6. Therefore $\Phi_{12}/\prod_\delta(\delta, \cdot)$ as a section of $\mathcal{L}^{\otimes 42} \otimes \det$ over $\mathcal{D}_{II_{2,26}}$ is $\tilde{\iota}$-invariant. Since $\det(\tilde{\iota}) = 1$ while $\det(\iota) = -1$, $\Phi_{12}|_L$ as a section of $\mathcal{L}^{\otimes 42} \otimes \det$ over $\mathcal{D}_L$ is anti-invariant under ι.

In order to construct ι-invariant cusp forms of character det, we take the product of $\Phi_{12}|_L$ with the Gritsenko lift of the ι-anti-invariant part of $M_k(\rho_L)$. By the formulae in [6] and [20], we see that $\dim M_k(\rho_L) = [k/3]$ and $\dim M_k(\rho_L)^\iota = [(k+2)/4]$ for $k > 2$ odd. We also require the congruence condition $42 + k + 9 \equiv 20 \mod 3$, namely $k \equiv 2 \mod 3$. Now, when $k = 17$ (resp., $k = 29$), the ι-anti-invariant part has dimension 1 (resp., 2). This proves our claim. □

We can do the double-cover construction over a Zariski open set of the moduli space. Let $LG^\circ \subset LG$ be the open set of generic $[A]$ which is PGL_6-stable and has no nontrivial stabiliser. Let $D_2 = \cup_A D_2^A \subset LG^\circ \times G(3,6)$ be the universal family of D_2^As. We have the geometric quotients $\mathcal{M} = LG^\circ/\mathrm{PGL}_6$, $\mathcal{Z} = D_2/\mathrm{PGL}_6$ with projection $\mathcal{Z} \to \mathcal{M}$. The relative $\mathcal{O}(2)$ descends. Let $\tilde{\mathcal{Z}} \to \mathcal{Z}$ be the blowup at $\mathrm{Sing}(\mathcal{Z})$, $\mathcal{B} \subset \tilde{\mathcal{Z}}$ be the exceptional divisor and $\pi : \tilde{\mathcal{Z}} \to \mathcal{M}$ be the projection. As in the proof of Lemma 8.2, we may shrink $\mathcal{M}$ to a Zariski open set $\mathcal{M}' \subset \mathcal{M}$ so that $\mathcal{B}|_{\mathcal{M}'} \sim 2K_\pi$. Then we can take the double cover of $\tilde{\mathcal{Z}}|_{\mathcal{M}'}$ branched over $\mathcal{B}|_{\mathcal{M}'}$. Contracting the ramification divisor relatively by using pullback of a multiple of the relative $\mathcal{O}(2)$, we obtain a universal family $\mathcal{F} \to \mathcal{M}'$ of double EPW cubes over $\mathcal{M}'$. Then let $\mathcal{F}_n = \mathcal{F} \times_{\mathcal{M}'} \cdots \times_{\mathcal{M}'} \mathcal{F}$ (n times).

The period map $\mathcal{M} \to \Gamma \backslash \mathcal{D}$ is generically finite and dominant [13, Proposition 5.1]. By Lemma 8.3, with $68 = 20 + 3 \cdot 16$ and $80 = 20 + 3 \cdot 20$, we see that $\mathcal{F}_{16}$ has positive geometric genus and $\kappa(\mathcal{F}_{20}) > 0$. This proves Theorem 1.1 in the case of double EPW cubes.

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References

[1] N. Addington and M. Lehn, On the symplectic eightfold associated to a Pfaffian cubic fourfold, *J. Reine Angew. Math.* **731** (2017), 129–137.

[2] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, *J. Differential Geom.* **18** (1983), 755–782.

[3] A. Beauville and R. Donagi, La variétés des droites d'une hypersurface cubique de dimension 4, *C. R. Math. Acad. Sci. Paris* **301** (1985), 703–706.

[4] R. Borcherds, Automorphic forms on $O_{s+2,2}(\mathbb{R})$ and infinite products, *Invent. Math.* **120**(1) (1995), 161–213.

[5] R. Borcherds, L. Katzarkov, T. Pantev and N. I. Shepherd-Barron, Families of $K3$ surfaces, *J. Algebraic Geom.* **7**(1) (1998), 183–193.

[6] J. H. Bruinier, On the rank of Picard groups of modular varieties attached to orthogonal groups, *Compos. Math.* **133**(1) (2002), 49–63.

[7] O. Debarre, 'Hyperkähler manifolds', Preprint, 2018, https://arxiv.org/abs/1810.02087.

[8] O. Debarre and C. Voisin, Hyper-Kähler fourfolds and Grassmann geometry, *J. Reine Angew. Math.* **649** (2010), 63–87.

[9] V. Gritsenko, Modular forms and moduli spaces of abelian and $K3$ surfaces, *St. Petersburg Math. J.* **6**(6) (1995), 1179–1208.

[10] V. Gritsenko, K. Hulek and G. Sankaran, Moduli spaces of irreducible symplectic manifolds, *Compos. Math.* **146**(2) (2010), 404–434.

[11] V. A. Gritsenko, K. Hulek and G. K. Sankaran, Moduli of $K3$ surfaces and irreducible symplectic manifolds, in *Handbook of Moduli, I*, pp. 459–525 (International Press, Sommerville, 2013).

[12] S. Iitaka, *Algebraic Geometry*, Graduate Texts in Mathematics, **76** (Springer, New York, 1982.

[13] A. Iliev, G. Kapustka, M. Kapustka and K. Ranestad, EPW cubes, *J. Reine Angew. Math.* **748** (2019), 241–268.

[14] A. Iliev and K. Ranestad, $K3$ surfaces of genus 8 and varieties of sums of powers of cubic fourfolds, *Trans. Amer. Math. Soc.* **353**(4) (2001), 1455–1468.

[15] A. Iliev and K. Ranestad, Addendum to "K3 surfaces of genus 8 and varieties of sums of powers of cubic fourfolds", *C. R. Acad. Bulgare Sci.* **60** (2007), 1265–1270.

[16] Y. Kawamata, Minimal models and the Kodaira dimension of algebraic fiber spaces, *J. Reine. Angew. Math.* **363** (1985), 1–46.

[17] R. Laza, The moduli space of cubic fourfolds via the period map, *Ann. of Math. (2)* **172** (2010), 673–711.

[18] C. Lehn, M. Lehn, C. Sorger and D. van Straten, Twisted cubics on cubic fourfolds, *J. Reine Angew. Math.* **731** (2017), 87–128.

[19] E. Looijenga, The period map for cubic fourfolds, *Invent. Math.* **177** (2009), 213–233.

[20] S. Ma, Equivariant Gauss sum of finite quadratic forms, *Forum Math.* **30**(4) (2018), 1029–1047.

[21] S. Ma, 'Mukai models and Borcherds products', Preprint, 2019, https://arxiv.org/abs/1909.03946.

[22] E. Markman, A survey of Torelli and monodromy results for holomorphic-symplectic varieties, in *Complex and Differential Geometry*, pp. 257–322, (Springer, Berlin, 2011).

[23] E. Markman, 'On the existence of universal families of marked irreducible holomorphic symplectic manifolds', Preprint, 2017, https://arxiv.org/abs/1701.08690, to appear in Kyoto J. Math.

https://doi.org/10.1017/S1474748021000013 Published online by Cambridge University Press

[24] D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory* (Springer, Berlin, 1994).
[25] G. Mongardi, 'Automorphisms of Hyperkähler manifolds', Ph.D. thesis, Università Roma Tre, 2013, available at https://arxiv.org/abs/1303.4670.
[26] K. G. O'Grady, Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics, *Duke Math. J.* **134**(1) (2006), 99–137.
[27] K. G. O'Grady, Double covers of EPW-sextics, *Michigan Math. J.* **62** (2013), 143–184.
[28] C. Voisin, Théorème de Torelli pour les cubiques de $\mathbb{P}^5$, *Invent. Math.* **86** (1986), 577–601.

https://doi.org/10.1017/S1474748021000013 Published online by Cambridge University Press