Instability, Chaos and Predictability
in Celestial Mechanics and
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VECTOR LAYPUNOV FUNCTIONALS AND STABILITIES AND CHAOTICITIES OF FUNCTIONAL DIFFERENTIAL EQUATIONS OCCURRING IN CELESTIAL MECHANICS AND STELLAR DYNAMICS

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ABSTRACT

Vector Laypunov functions and stabilities and chaoticities of Functional Differential Equations occurring in Celestial Mechanics and Stellar Dynamics have been discussed. Fuzzy dynamical systems are more realistic while considering problems occurring in Celestial Mechanics and Stellary Dynamics.

DISCUSSION

La Salle used Vector Laypunov Functions and the Principle of invariance to have geometric pictures of the regions of Stabilities in [4] which in turn are extended to Difference Equations in [14] and difference equations are exploited to obtain qualitative results in Chaos, and stabilities with rich applications [5 (i),5(ii),13]. All the researchers in Celestial Mechanics and Stellary Dynamics, while investigating several complicated problems, could consider differential equations, ignoring the time delay equations which are equally important because the origin and evolution of the solar system along with the other planets and the Stars depend on their past history also. We feel the Vector Laypunov Functionals could be conveniently used even to deal with the Chaoticities besides Stabilities, instabilities and conditional stabilities. Computational aspects has not been considered although it is equally important. These results can easily be extended to functional differential system $z'(t) = F(z_t, \lambda_t)$ where t is the time, $z_t(\theta) = z(t+\theta)$, $-r \leq \theta \leq 0$, $\lambda_t(\theta) = \lambda(t+\theta)$, $-r \leq \theta \leq 0$

λ_t is a fast oscillating functional parameter which is more general than the fast oscillating parameter considered by

Richard Bellman and his collaborators in [15]. Analogeous to the system (3.01) in [1]. We consider (m+n) dimensional multi-frequency oscillatory functional differential

$$dx/dt = \lambda f(x_t, y_t)$$

$$dy/dt = \omega(x_t) + \lambda g(x_t, y_t) \qquad (1)$$

with n fast and m slow variables, $x_t = x(t+\theta)$, $-r \leq \theta \leq 0$, x_t, $f \in C_1([-r,0], R^m)$, y_t, ω, $g \in C_2([-r,0], R^n)$. $\bar{C}_1, \bar{C}_2$ and $C([-r,0], R^{n+m}) = C$ are spaces of continuous functions f and g are defined and 2π-periodic with respect to the second variable y_t. System (1) can be reduced to the multifrequency oscillatory system of ordinary differential equations of the type (3.01) in [1]. System (1) also deals with the three-body problem dealing with the study of the motion of each of the three objects P_o, P_1 and P_2 having arbitrary masses m_o, m_1 and m_2 respectively and attracting one another in accordance with the Newton's Law of gravitation and the three-body problem can be interpreted as a two-planet problem where in the bodies with the masses m_1 and m_2 are the two planets and the body with the mass m_o is the SUN. And the differential equations of motion of the planets will be of the form

$$\frac{d^2x_1}{dt^2} + \frac{G(m_o + m_1)x_1}{r_1^3} = Gm_2\left(\frac{x_2 - x_1}{\Delta_{12}^3} - \frac{x_1}{r_1^3}\right)$$

$$\frac{d^2y_1}{dt^2} + \frac{G(m_o + m_1)y_1}{r_1^3} = Gm_2\left(\frac{y_2 - y_1}{\Delta_{12}^3} - \frac{y_1}{r_1^3}\right)$$

$$\frac{d^2z_1}{dt^2} + \frac{G(m_o + m_1)z_1}{r_1^3} = Gm_2\left(\frac{z_2 - z_1}{\Delta_{12}^3} - \frac{z_1}{r_1^3}\right)$$

$$\frac{d^2x_2}{dt^2} + \frac{G(m_o + m_2)x_2}{r_2^3} = Gm_1\left(\frac{x_1 - x_2}{\Delta_{12}^3} - x\frac{x_2}{r_2^3}\right)$$

$$\frac{d^2y_2}{dt^2} + \frac{G(m_o + m_2)y_2}{r_2^3} = Gm_1\left(\frac{y_1 - y_2}{\Delta_{12}^3} - \frac{y_2}{r_2^3}\right)$$

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$$\frac{d^2z_2}{dt^2} + \frac{G(m_o + m_2)z_2}{r_2^3} = Gm_1 \left(\frac{z_1 - z_2}{\Delta_{12}^3} - \frac{z_2}{r_2^3} \right) \quad (2)$$

where (x_1, y_1, z_1) are the rectangular coordinates of the planet P_1 and (x_2, y_2, z_2) are the rectangular coordinates of the planet P_2 with the SUN as the origin, G is the gravitational constant and

$$\Delta_{12}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$r_i^2 = x_i^2 + y_i^2 + z_i^2, \quad i = 1,2.$$

Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$, $F(z_t, \lambda) = F(x_t, y_t, \lambda) =$

$$= \begin{pmatrix} \lambda f(x_t, y_t) \\ \omega(x_t) + \lambda g(x_t, y_t) \end{pmatrix}$$

Functional differential equation (1) can now be written as

$$z'(t) = F(z_t, \lambda) \quad (3)$$

where $F: C([-r,0], R^{m+n}) \times (0,\infty) \to C([-r,0], R^{m+n})$ with the initial function $z(t) = \phi(t)$, $-r \le t \le 0$. Consider Vector Laypunov functional $V(z_t)$: $C_3(C, R^p)$ and as usual its derivative is taken along the solutions of (3)

$$V'(z_t) = \lim_{t \to 0^+} \inf \left[\frac{V(z_t(\phi) - V(\phi)}{t} \right]$$

Define $E = \{z_t : V'(z_t) = 0,\ z_t \in C([-r,0], R^{m+n})\}$ and let M be the largest invariant set in E, that is M is the union of all solutions of (3) defined on $R = (-\infty, \infty)$ that remains for all t in E. Consider a set

$$V^{-1}(c) = \{z_t : V(z_t) = c,\ z_t \in C\}$$

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Theorem: Let $V(z_t)$ be a Laypunov functional of (3) defined on C satisfying the above conditions. If $z_t(\phi)$ is a solution of (3) that is compact for all $t \geq 0$, then there is a constant vector $c \in R^{m+n}$ such that $z_t(\phi) \to M \cap V^{-1}(c)$ as $t \to \infty$.

Proof of this Theorem is analogeous to that of Theorem 3.1 in [4].

By using this Theorem we can have a picture of the regions of stability and also instability for system (3).

Even for Difference Equations Laypunov Functions have been exploited to obtain similar results in [14]. These difference equations are used to get results of qualitative nature in Chaos, besides stabilities in [5(i), 5(ii),13].

Remarks: (1) Instead of (3), we may consider Functional Differential Equation

$$z'(t) = F(z_t, \lambda_t) \tag{4}$$

where $\lambda_t = \lambda(t+\theta), \ -r \leq \theta \leq 0$

$$\lambda_t \in C_4([-r,0], R^q), \quad F: C \times C_4 \to C$$

and λ_t is a functional parameter.

The above results and the Theorem can be extended to (4).

Our Functional Differential Equation (4) is much more general than equation (1) in [15] considered by Bellman and his collaborators.

(2) Fuzzy Dynamical Systems are more realistic while considering problems occurring in Celestial Mechanics and Stellary Dynamics. The above results can also be obtained by considering Generalized Dynamical Systems and Fuzzy Dynamical Systems occurring in Celestial Mechanics and Stellar Dynamics, besides Stochastic Dynamical Systems.

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