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If there is given a subgroup S of a (finite) group G, we may ask what information is to be obtained about the structure of G from a knowledge of the location of S in G. Thus, for example, famed theorems of Frobenius and Burnside give criteria for the existence of a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$, and hence in particular for the non-simplicity of G. To aid in locating S in G, and to facilitate exploitation of the transfer, we single out a descending chain of normal subgroups of S. Namely, we introduce the *focal series* of S in G by means of the recursive formulae

${}_0S = S$, ${}_{i+1}S =$ *the subgroup of* G *which is generated by all commutators* $c = [s, g]$, *with* c *and* s *in* ${}_iS$, g *in* G.

For π a set of primes, let us denote by $P(\pi)$ the subgroup of G which is generated by all those elements of G whose orders have no prime divisors in π. A central theorem of our discussion is the following: if π contains all the prime divisors of the index $[S: {}_iS]$ for some i, then every prime which divides $[P(\pi) \cap S: P(\pi) \cap {}_iS]$ also divides $[P(\pi): P(\pi) \cap S]$. From this result we deduce in particular a generalization of Burnside's theorem: if ${}_iS = 1$ for some i, in which case we call S *hyperfocal*, and if S has order prime to its index in G, then there exists a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$. Thus a group is nilpotent if and only if each of its Sylow subgroups is hyperfocal.

The only proof of the theorem of Frobenius in the literature involves group characters. In an attempt to give a purely group theoretic proof, Grün has established a generalization of a special case of this theorem. Our methods lead to a somewhat sharpened form of the theorem of Grün.

Notations. Throughout we shall write *group* for *finite group*. If G is a group,

$Z(G)$ = the centre of G.

${}^iG$ = the ith term of the lower central series of G.

If S is a subgroup of G,

$N(S)$ = the normalizer of S in G.

$C(S)$ = the centralizer of S in G.

When it becomes necessary to emphasize the role of G, we write

$N(S) = N(S \text{ in } G)$, $C(S) = C(S \text{ in } G)$.

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For a subset X and an element g of G,

X^g = the totality of elements $g^{-1}xg$, for x in X.

If A and B are two groups,

$A \otimes B$ = the direct product of A and B.

For two sets X and Y,

$X \subseteq Y$ means that X is part of Y.

$X \subset Y$ means that X is a proper part of Y.

$X \cap Y$ = the totality of elements contained in both X and Y.

1. The subgroups $P(\pi)$ and $G^i(\pi)$. This section is devoted to preliminaries. In particular we define certain subgroups which play an important role in our subsequent discussion. We begin by proving

LEMMA 1.1. *If N is a normal subgroup and S a subgroup of the group G such that $G = NS$, then*

(a) *${}^iS \subseteq N$ if and only if ${}^iG \subseteq N$.*

(b) *${}^iS \subseteq N \cap S \subseteq {}^iG$ if and only if ${}^iG \cap S = N \cap S$.*

(c) *If H is a normal subgroup of S, then NH is a normal subgroup of G.*

Proof. Since $G = NS$, G/N is isomorphic with $S/[N \cap S]$. If ${}^iS \subseteq N$, then ${}^iS \subseteq N \cap S$, and hence it follows that ${}^iG \subseteq N$. On the other hand, since ${}^iS \subseteq {}^iG$, the converse is clear.

By (a), if ${}^iS \subseteq N \cap S$, then ${}^iG \subseteq N$. Hence, if in addition $N \cap S \subseteq {}^iG$, we have ${}^iG \cap S \subseteq N \cap S \subseteq {}^iG \cap S$, so that ${}^iG \cap S = N \cap S$. If conversely we have ${}^iG \cap S = N \cap S$, then ${}^iS \subseteq {}^iG \cap S = N \cap S \subseteq {}^iG$. Hence (b).

To prove (c) we need only show that $H^x \subseteq NH$ for each x in G. But $G = NS$, hence $x = sn$, with s in S and n in N. Thus, since H is a normal subgroup of S, $H^x = H^{sn} = H^n \subseteq NH$. This completes the proof of the lemma.

Now let π be a set of primes. We call a group (element of a group) a π-group (π-element) if all the prime divisors of its order are in π. For G a group we denote by $P(\pi)$ the subgroup of G which is generated by all those elements of G whose orders have no prime divisors in π. Clearly $P(\pi)$ is a fully invariant subgroup of G. Furthermore,

1.2 *$P(\pi)$ is the intersection of all normal subgroups N of G such that G/N is a π-group.*

To verify (1.2), notice first that $G/P(\pi)$ is a π-group, and hence that $P(\pi)$ contains this intersection. If on the other hand N is a normal subgroup of G such that G/N is a π-group, and if p is a prime not in π such that p^a divides the order of G, then p^a divides the order of N. Hence, since N is a normal subgroup of G, N contains all the p-Sylow subgroups of G. Thus we conclude that $P(\pi) \subseteq N$, which completes the verification. We also note the following simple fact:

1.3 *If S is a subgroup of G, and if π is a set of primes which contains no prime divisor of $[G: S]$, then $G = P(\pi)S$.*

For if p is a prime which is not in π, $P(\pi)$ contains (all) p-Sylow subgroups of G, whereas if p is in π, and p^a divides the order of G, then, since p does not divide $[G: S]$, p^a divides the order of S. This proves 1.3.

Now we introduce an additional definition, namely we set

$G^i(\pi)$ = the intersection of all normal subgroups N of G such that G/N is a π-group, and ${}^iG \subseteq N$.
= the intersection of all normal subgroups N of G such that $[G: N]$ is a power of some prime in π, and G/N has a lower central series of length at most i.

That these are indeed two descriptions of the same subgroup is immediate when we remember that if G/N is a nilpotent π-group, then N is the intersection of all normal subgroups M of G which contain N, and have index in G equal to a power of some prime in π.

The subgroup $G^i(\pi)$ is characteristic in G, $G/G^i(\pi)$ is a π-group, and $G^i \subseteq G^i(\pi)$, i.e. $G/G^i(\pi)$ is nilpotent, with a lower central series of length at most i. By 1.2 we have the useful formula

1.31 $\quad G^i(\pi) = {}^iG\,P(\pi)$.

Since furthermore $G^i(\pi)/{}^iG$ is the π-complement of the nilpotent group $G/{}^iG$,

1.4 *no prime divisor of* $[G^i(\pi): {}^iG]$ *is in* π.

The factor group $G/G^1(\pi)$ is an abelian π-group; $G^1(\pi)$ is called the *π-commutator subgroup* of G. There exists a k such that $G^k(\pi) = G^{k+1}(\pi)$ = the intersection of all normal subgroups N of G such that G/N is a nilpotent π-group; $G/G^k(\pi)$ is the so-called *maximal nilpotent π-factor group* of G.

LEMMA 1.5. *For a set π of primes, and a subgroup S of the group G, the following three conditions are equivalent:*

(i) *π contains all the prime divisors of the index* $[G^i(\pi) \cap S: {}^iG \cap S]$.

(ii) $G^i(\pi) \cap S = {}^iG \cap S$.

(iii) *π contains all the prime divisors of the index* $[S: {}^iG \cap S] = [{}^iGS: {}^iG]$.

Proof. $G^i(\pi)/{}^iG$ contains ${}^iG[G^i(\pi) \cap S]/{}^iG$, and this latter group is isomorphic with $[G^i(\pi) \cap S]/[{}^iG \cap S]$. Hence, since the order of $G^i(\pi)/{}^iG$ has no prime divisors in π, it is clear that (i) implies (ii).

$[S: S \cap G^i(\pi)] = [G^i(\pi)S: G^i(\pi)]$ is a divisor of $[G: G^i(\pi)]$. Hence (ii) implies (iii), since $G/G^i(\pi)$ is a π-group.

Finally, it is clear that (i) is a consequence of (iii).

PROPOSITION 1.6. *For π a set of primes, and S a subgroup of the group G, the following three conditions are equivalent:*

(i) $G = G^i(\pi)S$ *and* ${}^iG \cap S = G^i(\pi) \cap S$.

(ii) *π contains all the prime divisors of* $[{}^iGS: {}^iG]$, *but none of* $[G: {}^iGS]$.

(iii) $G = G^j(\pi)S$ *and* ${}^jG \cap S = G^j(\pi) \cap S$, *for all* $j \leqslant i$.

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Hence there exist a set π of primes which satisfies the equivalent conditions (i), (ii), *and* (iii) *if and only if the indices* $[G\colon {}^iGS]$ *and* $[{}^iGS\colon {}^iG]$ *are relatively prime.*

Proof. Assume (i), then by Lemma 1.5, all the prime divisors of $[{}^iGS\colon {}^iG]$ are in π. And

$$[G\colon {}^iGS] = [G^i(\pi)\colon G^i(\pi) \cap {}^iGS] = [G^i(\pi)\colon {}^iG(G^i(\pi) \cap S)]$$
$$= [G^i(\pi)\colon {}^iG({}^iG \cap S)] = [G^i(\pi)\colon {}^iG],$$

hence by 1.4, $[G\colon {}^iGS]$ has no prime divisors in π. Hence (i) implies (ii).

Now asume (ii). By Lemma 1.5, ${}^iG \cap S = G^i(\pi) \cap S$. Furthermore, if p is a prime in π, p does not divide $[G\colon {}^iGS]$, and hence if p^a divides the order of G, p^a divides the order of ${}^iGS$. If on the other hand q is a prime which is not in π, $P(\pi)$ contains (all) q-Sylow subgroups of G. Hence $G = P(\pi){}^iGS = G^i(\pi)S$, by formula 1.31. Hence (ii) implies (i), and we have proved the equivalence of (i) and (ii).

If $j \leqslant i$, $[G\colon {}^jGS]$ is a divisor of $[G\colon {}^iGS]$, and $[{}^jGS\colon {}^jG]$ is a divisor of $[{}^iGS\colon {}^iG]$. Hence if π satisfies (ii), π contains all the prime divisors of $[{}^jGS\colon {}^jG]$, but none of $[G\colon {}^jGS]$. Hence, since (ii) implies (i), (ii) implies (iii). And it is clear that (iii) implies (i). Thus the three conditions are equivalent as stated.

It is clear that the existence of a set π of primes satisfying (ii) implies that $[G\colon {}^iGS]$ and $[{}^iGS\colon {}^iG]$ are relatively prime. If on the other hand these indices are relatively prime, then the totality of prime divisors of $[{}^iGS\colon {}^iG]$ satisfies (ii).

Proposition 1.7. *Let S be a subgroup of the group G, and assume that there exists a set π of primes such that $G = G^i(\pi)S$ and ${}^iG \cap S = G^i(\pi) \cap S$. Then if N is a normal subgroup of G such that $G = NS$ and ${}^jG \cap S = N \cap S$, for some $j \leqslant i$, $N = G^j(\pi)$.*

Proof. In view of Proposition 1.6 it will suffice to consider the case $i = j$. Assume then that N is a normal subgroup of G such that $G = NS$ and ${}^iG \cap S = N \cap S$. Since ${}^iS \subseteq {}^iG \cap S = N \cap S$, we have ${}^iG \subseteq N$ by Lemma 1.1(a). By Lemma 1.5, π contains all the prime divisors of $[{}^iGS\colon {}^iG]$, hence since $[G\colon N] = [S\colon N \cap S] = [S\colon {}^iG \cap S]$, G/N is a π-group. Hence N contains $G^i(\pi)$, and hence $N = G^i(\pi)$, since

$$G/N \simeq S/[{}^iG \cap S] \simeq G/G^i(\pi).$$

In the next section we shall introduce the focal series of S in G, the ith term ${}_iS$ of which will be a normal subgroup of S, such that ${}^iS \subseteq {}_iS \subseteq {}^iG$. In §3 we shall obtain a useful sufficient condition for the validity of the relation $SG = P(\pi)S$ and $P(\pi) \cap S \subseteq {}_iS$. Here we prove

Proposition 1.8. *Let H be a normal subgroup of the subgroup S of G, such that ${}^iS \subseteq H \subseteq {}^iG$. Then for π a set of primes, the following conditions are equivalent:*
(A) *$G = P(\pi)S$ and $P(\pi) \cap S \subseteq H$.*
(B) *$N = P(\pi)H$ is a normal subgroup of G such that $G = NS$ and $H = N \cap S$.*

(C) $H = {}^iG \cap S = G^i(\pi) \cap S$, ${}^iG \subseteq P(\pi)H$, *and* $G = G^i(\pi)S$.
(D) $H = {}^iG \cap S$, ${}^iG \subseteq P(\pi)H$, *and* π *contains all the prime divisors of* $[{}^iGS: {}^iG]$, *but none of* $[G: {}^iGS]$.
(E) $H = {}^iG \cap S$, $G = P(\pi)S$, *and* π *contains all the prime divisors of* $[{}^iGS: {}^iG]$.
(F) $P(\pi) \cap S \subseteq H$, $G = P(\pi)S$, *and no prime divisor of* $[G: {}^iGS]$ *is in* π.

Proof. Assume (A). Then $N = P(\pi)H$ is a normal subgroup of G by Lemma 1.1 (c). Furthermore, $NS = P(\pi)HS = P(\pi)S = G$, and $N \cap S = P(\pi)H \cap S = [P(\pi) \cap S]H = H$. Hence (B) is a consequence of (A).

Now assume (B). Since ${}^iS \subseteq H = N \cap S \subseteq {}^iG$, and $G = NS$, we have by Lemma 1.1 (a) and (b) that ${}^iG \subseteq N$, and ${}^iG \cap S = N \cap S = H$. Since ${}^iG \subseteq N = P(\pi)H$, $G^i(\pi) = P(\pi){}^iG \subseteq N \subseteq G^i(\pi)$, so that $N = G^i(\pi)$. Now we see that (B) implies (C). The equivalence of (C) and (D) is a consequence of Proposition 1.6.

Now it is easy to prove that (C) implies (A), from which the equivalence of the first four conditions will follow. Assume (C). Then $G^i(\pi) = P(\pi)H$ since ${}^iG \subseteq P(\pi)H$, hence

$$P(\pi)S = P(\pi)HS = G^i(\pi)S = G,$$

and

$$P(\pi) \cap S \subseteq P(\pi)H \cap S = G^i(\pi) \cap S = H.$$

Hence (C) implies (A).

If the equivalent conditions (A), (C), and (D) are satisfied, then it is clear that (E) and (F) are also satisfied. Assume (E), then it follows, using Lemma 1.5 that

$$H = {}^iG \cap S = G^i(\pi) \cap S = P(\pi)\,{}^iG \cap S \supseteq P(\pi) \cap S.$$

Hence (E) implies (A). Finally assume (F). Then $H \subseteq {}^iG \cap S \subseteq P(\pi)H \cap S = H$, hence $H = {}^iG \cap S$. Moreover,

$$[{}^iGS: {}^iG] = [S: S \cap {}^iG] = [S: H] = [P(\pi)S: P(\pi)H],$$

from which it follows that π contains all the prime divisors of $[{}^iGS: {}^iG]$. Hence (F) implies (D). We have now proved the equivalence of the six conditions as stated.

COROLLARY 1.9. *Assuming the hypotheses of proposition* 1.8, *we have*
(a) $[G: {}^iGS]$ *and* $[S: H]$ *are relatively prime.*
(b) *For a normal subgroup* N *of* G, *the following conditions are equivalent:*
(i) $G = NS$ *and* $H = N \cap S$.
(ii) $N = G^i(\pi) = P(\pi)H$.
(iii) $N = G^i(\sigma)$, σ *a set of primes which contains all the prime divisors of* $[S: H]$, *but none of* $[G: {}^iGS]$.

Proof. By Proposition 1.8, π contains all the prime divisors of $[S: H]$, but none of $[G: {}^iGS]$. Hence (a).

If we set $N = P(\pi)H$, then we have by proposition 1.8, that $G = NS$, $H = N \cap S$, and $N = G^i(\pi)$. Now (b) is an easy consequence of Propositions 1.6 and 1.7.

2. The focal series. If X, Y, and Z are subgroups of a group G, we shall denote by $[X, Y; Z]$ the subgroup of G which is generated by all the commutators $[x, y] = x^{-1} y^{-1} xy$, with x in X and y in Y, which are in Z. In case $Z = G$, $[X, Y; G]$ is the usual commutator subgroup of X by Y, which we shall denote simply by $[X, Y]$. We recall that the *lower central series* of G is defined by the recursive formulae: ${}^0G = G$, ${}^{i+1}G = [{}^iG, G]$. In the next paragraph we shall introduce a generalization of the notion of lower central series.

Let S be a subgroup of the group G, and consider the subgroups ${}_iS = {}_i(S, G)$ defined inductively by the formulae: ${}_0S = S$, ${}_{i+1}S = [{}_iS, G; {}_iS]$. It is clear from this definition that

2.1 $\quad {}_{i+1}S \subseteq {}_iS$.

Furthermore, obvious induction arguments give

2.2 $\quad {}^iS \subseteq {}_iS \subseteq {}^iG \cap S$;

2.3 *if α is any endomorphism of G such that $S^\alpha \subseteq S$, then $({}_iS)^\alpha \subseteq {}_iS$.*

It follows in particular that the subgroups ${}_iS$ constitute a descending chain of subgroups of S, each normal in $N(S)$, in which all possible factors are nilpotent, and adjacent factors are abelian. We call this chain the *focal series of S in G.* In case $S = G$, the focal series coincides with the lower central series of G. In case $S = N$ is a normal subgroup of G, the focal series of N in G is the N-central series of G in the sense of Baer [**1**]. Our main concern is with the case in which S is a proper, non-normal subgroup of G.

In this section we establish a number of lemmas concerning focal series, which we shall use in the sequel.

If M and N are normal subgroups of G, then ${}_i(MN) = {}_iM\, {}_iN$; indeed, this is a standard formula of the commutator calculus. We prove

LEMMA 2.4. *If the subgroup S of the group G is the direct product $S = A \otimes B$ of two of its subgroups A and B which have relatively prime orders, then ${}_iS = {}_iA \otimes {}_iB$.*

Proof. If the transform s^g of an element s of S by an element g of G is in S, i.e. if the commutator $[s, g]$ is in S, we have $s^g = ab$, with a in A and b in B; and $s = xy$ with x in A and y in B. If we let n denote the order of B, we can find an integer k such that $x = x^{kn} = (xy)^{kn} = s^{kn}$. Hence

$$x^g = g^{-1}(s)^{kn}g = (g^{-1}sg)^{kn} = (s^g)^{kn} = (ab)^{kn} = a^{kn},$$

so that x^g is in A, i.e. $[g, x]$ is an element of A. Similarly $[g, y]$ is in B, hence $[g, s] = [g, xy] = [g, y][g, x]^y$ is an element of ${}_1A \otimes {}_1B$. But ${}_1S$ is generated by such commutators $[g, s]$ so that ${}_1S \subseteq {}_1A \otimes {}_1B$. The reverse inequality is obvious, hence ${}_1S = {}_1A \otimes {}_1B$. Now the lemma follows by an obvious induction.

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Many examples show that the formula ${}_i(ST) = {}_iS\ {}_iT$ is not valid for arbitrary pairs of subgroups. For instance

Example 1. Suppose that N is the direct product of two cyclic subgroups $A = \{a\}$ and $B = \{b\}$, each of order p, p a prime. Consider the group G formed by adjoining to N the automorphism c of N defined by $a^c = a$, and $b^c = ab^{-1}$. One verifies easily that (i) c has order 2, (ii) $A = Z(G)$, so that ${}_1A = [A, A] = 1$, and (iii) ${}_1B = 1$. The commutator $[ab, c] = ab^{-2}$ is an element $\neq 1$ of ${}_1N$, hence $1 = {}_1A\ {}_1B \subset {}_1N$. This example shows that the hypothesis that the orders of A and B are relatively prime is needed for Lemma 2.4.

Let us consider furthermore the commutators c_i defined recursively by: $c_1 = [ab, c]$ $c_{i+1} = [c_i, c]$. Set $n_i = (-1)^{i+1}\ 2^i$, then by a straightforward induction argument one verifies that $c_i = (ab^{-2})^{n_i}$. Hence c_i is an element of ${}_iN$, and if p is an odd prime, $c_i \neq 1$. Hence in this case, N is not part of the hypercentre of G, for otherwise, since N is normal in G, we would have ${}_iN = 1$ for some i.

LEMMA 2.5. *If N is a normal subgroup of G such that $G = NS$, then ${}_iS \subseteq {}^iS[N \cap S]$.*

Proof. We prove by induction on i that ${}_iS \subseteq {}^iSN$. Since ${}_0S = {}^0S = S$, this statement is true for $i = 0$. Let us assume that it is true for some i, $0 \leqslant i$. ${}_{i+1}S$ is generated by commutators $c = [s, g]$, with s in ${}_iS$, g in G. Since $G = NS$, $g = nt$, with n in N, t in S. Hence, since N is a normal subgroup of G, $c = [s, g] = [s, nt] = [s, t][s, n]^t \equiv [s, t]$ modulo N. By the induction hypothesis, $s = mr$, with m in N and r in ${}^iS$. Hence $[s, t] = [mr, t] = [m, t]^r\,[r, t] \equiv [r, t]$ modulo N, again using the fact that N is a normal subgroup of G. But r is in ${}^iS$ and t is in S, hence $[r, t]$ is in ${}^{i+1}S$. Now we have proved that c is in ${}^{i+1}SN$, from which it follows that ${}_{i+1}S \subseteq {}^{i+1}SN$. This completes the induction. Thus ${}_iS \subseteq {}^iSN \cap S = {}^iS[N \cap S]$, by Dedekind's law.

PROPOSITION 2.6. *If N is a normal subgroup and S a subgroup of G such that $G = NS$ and ${}_iS = N \cap S$, then ${}_iS = {}^iG \cap S$, and more generally, for $j \leqslant i$, ${}_jS = {}^jS[{}^iG \cap S]$.*

Proof. Since ${}_iS$ is a normal subgroup of G and ${}^iS \subseteq {}_iS \subseteq {}^iG$, we have by Lemma 1.1 (b) that ${}^iG \cap S = N \cap S$, and hence that ${}_iS = {}^iG \cap S$. Using this fact together with Lemma 2.5, we have, for $j \leqslant i$,

$${}_jS \subseteq {}^jS[N \cap S] = {}^jS[{}^iG \cap S] = {}^jS\ {}_iS \subseteq {}_jS.$$

Hence ${}_jS = {}^jS[{}^iG \cap S]$, which proves the proposition.

That the formula ${}_iS = {}^iG \cap S$ is not universally true can be easily seen. For instance

Example 2. Let S be a subgroup of order p in a p-group G; $S = \{s\}$. If for an element g in G the commutator $[g, s]$ is in S, g induces an automorphism of S of order a divisor of $p - 1$, since S has order p. But, since g has order a power of

p, the order of this automorphism must divide p. Thus g must induce the identity automorphism in S, hence $[g, s] = 1$, so that ${}_1S = 1$. If now S is part of ${}^1G$, $1 = {}_1S \subset {}^1G \cap S$.

We wish to apply certain theorems of Burnside and Grün concerning conjugacy of elements in subgroups. First:

(Burnside.) *Two subsets of a group G which are normalized by a Sylow subgroup S of G (in particular, two elements of the centralizer of S in G) are conjugate in G only if they are conjugate in the normalizer of S in G.*

When we recall the generalized Sylow theorems of P. Hall for solvable groups, we have by an identical argument to that used in proving this result of Burnside (see for example [**9**, p. 139]) that:

If G is a solvable group, and if S is a subgroup of G with order prime to its index in G, then two subsets of G which are normalized by S (in particular, two elements of the centralizer of S in G) are conjugate in G only if they are conjugate in the normalizer of S in G.

Now let S be a p-Sylow subgroup of a group G, and denote by $B(S)$ the weak closure of $Z(S)$ in S, i.e., the subgroup generated by all the conjugates to $Z(S)$ which are contained in S. Following Grün, we call G *p-regular* if each p-Sylow subgroup of G which contains $Z(S)$ also contains $B(S)$. Notice that this definition is independent of the particular p-Sylow subgroup S. For p-regular groups, a result of the type we are interested in is the following theorem of Grün [**6**, §4, Satz 2]:

If S is a p-Sylow subgroup of a p-regular group G, then two subsets of G normalized by $Z(S)$ (in particular, two elements of S) which are conjugate in G are conjugate in the normalizer of $B(S)$ in G.

p-regularity is a generalization of the concept, also due to Grün, of p-normality. A group is called *p-normal* if the centre of a p-Sylow subgroup is the centre of each p-Sylow subgroup in which it is contained. Thus G is p-normal if and only if $Z(S) = B(S)$ for S a p-Sylow subgroup of G.

The applications of the theorems of Burnside and Grün which we have in mind are the following:

LEMMA 2.7. *If S is a Sylow subgroup of the group G, or if S is a subgroup of a solvable group G, with order prime to its index in G, and if H is any subgroup of the centralizer of S in G, then the focal series of H in G coincides with the focal series of H in the normalizer of S in G.*

LEMMA 2.8. *If S is a p-Sylow subgroup of a p-regular group G, then the focal series of S in G coincides with the focal series of S in the normalizer of $B(S)$ in G.*

The proofs of these lemmas are fairly obvious. Suppose for example that S is a Sylow subgroup of G or that the order and index of S in the solvable group G

are relatively prime, and suppose that H is a subgroup of the centralizer $C(S)$ of S in G. We shall prove by induction that ${}_i(H, G) = {}_i(H, N(S))$, where $N(S)$ is the normalizer of S in G. Since ${}_0(H, G) = {}_0(H, N(S)) = H$, the proposition is true for $i = 0$. Assume its validity for some $i \geqslant 0$. ${}_{i+1}(H, G)$ is generated by commutators $c = [h, g]$ with c and h in ${}_i(H, G)$, g in G. But $c = h^{-1} h^g$, so that h^g is also in ${}_i(H, G)$. Hence it follows from the above mentioned theorem of Burnside that there exists an element k in $N(S)$ such that $h^g = h^k$, so that $c = [h, g] = [h, k]$. By the induction assumption c and h are in ${}_i(H, N(S))$, hence c is in ${}_{i+1}(H, N(S))$. Thus ${}_{i+1}(H, G) \subseteq {}_{i+1}(H, N(S))$, which completes the induction, since the opposite inequality is clear. This proves Lemma 2.7. The proof of Lemma 2.8 follows exactly similar lines, so we omit it.

3. Focal chains: a divisibility theorem. We define a *focal chain* in a group G to be a set of subgroups S_i of G subject to the conditions

(1) $S_{i+1} \subseteq S_i$,

(2) ${}_1(S_i, G) \subseteq S_{i+1}$.

Notice that since $[S_i, S_i] \subseteq {}_1(S_i, G)$, S_{i+1} is a normal subgroup of S_i, and S_i/S_{i+1} is abelian.

If S is a subgroup of G, then the focal series of S is the minimal focal chain in G beginning with S, that is, the focal series of S is a focal chain beginning with S, and if the subgroups S_i form another such focal chain, then ${}_iS \subseteq S_i$. The intersection of S with the lower central series of G, i.e. the set of subgroups $S \cap {}^iG$, is a second example of a focal chain beginning with S.

If the subgroups S_i form a focal chain in G, and if H is a subgroup of G, then the subgroups $S_i \cap H$ also form a focal chain in G.

In case there exists a focal chain in G which begins with the subgroup S of G and terminates with the subgroup H of S, we shall say that H is *chained* to S in G. If H is chained to S, then ${}_iS \subseteq H$ for some i. That the converse is not true is easily seen; for instance:

Example 3. Let the group N be the direct product of two cyclic groups $\{a\}$ and $\{b\}$ each of order p, p a prime, and form the group G by adjoining to N the automorphism c of N defined by: $a^c = a$, $b^c = ab$. Then ${}^1G = {}_1N = \{a\}$, but $\{a\} = Z(G)$, hence ${}^2G = {}_2N = 1$. Thus ${}_2N \subseteq \{b\}$, but there exists no focal chain from N to $\{b\}$ since ${}_1N$ not $\subseteq \{b\}$.

THEOREM 3.1. *Let S be a subgroup of the group G, and assume that the subgroup H of S is chained to S in G. Assume moreover that the set π of primes contains all the prime divisors of the index $[S\colon H]$. Then every prime divisor of $[P(\pi) \cap S\colon P(\pi) \cap H]$ divides $[P(\pi)\colon P(\pi) \cap S]$.*

To prove the theorem we need two lemmas.

LEMMA 3.2. *If S is a subgroup of the group G, then there exists a homomorphism σ of G into $S/{}_1S$ such that $x^\sigma = {}_1S\, x^{[G:S]}$ for x in S.*

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Proof. The transfer of Schur [**8**; **9**, chap. V] is a homomorphism τ of G into $S/[S, S]$ which may be computed according to the formula

$$x^\tau = [S, S]\prod_{i=1}^{r} t_i^{-1}x^{f_i}\, t_i, \qquad x \in G,$$

where $t_i^{-1}\, x^{f_i}\, t_i$ is an element of S, and $f_1 + \ldots + f_r = [G\colon S]$. Since $[S, S] \subseteq {}_1S \subseteq S$, there exists a natural homomorphism ν of $S/[S, S]$ onto $S/{}_1S$. Since the factors $t_i^{-1}\, x^{f_i}\, t_i$ are in S, it follows that for x in S,

$$t_i^{-1}\, x^{f_i}\, t_i = x^{f_i}[x^{f_i}, t_i] \equiv x^{f_i} \qquad \mod {}_1S.$$

Hence if we set $\sigma = \tau\nu$ we have for each element x in S

$$x^\sigma = {}_1S\prod_{i=1}^{r} x^{f_i} = {}_1S\, x^{f_1+f_2+\ldots+f_r} = {}_1S\, x^{[G:S]}.$$

Thus σ has the required property, which proves the lemma.

LEMMA 3.3. *Let H be a normal subgroup of the subgroup S of G, and let π be a set of primes which contains all the prime divisors of $[S\colon H]$. Then if we set $A = P(\pi) \cap S$ and $B = P(\pi) \cap H$ we have*

$$A^{[P(\pi):A]} \subseteq {}_1(A, P(\pi))B.$$

Proof. By Lemma 3.2 there exists a homomorphism λ of $P(\pi)$ into $A/{}_1(A, P(\pi))B$ such that $x^\lambda = {}_1(A, P(\pi))B\, x^{[P(\pi):A]}$ for x in A. Since π contains all the prime divisors of $[S\colon H]$, π contains all the prime divisors of the divisor

$$[A : B] = [P(\pi) \cap S\colon P(\pi) \cap H]$$

of $[S\colon H]$. Hence, since $P(\pi)$ is generated by elements whose orders have no prime divisors in π, it follows that $\lambda = 0$. But this implies that $A^{[P(\pi):A]} \subseteq {}_1(A, P(\pi))B$, which is the desired result.

On the basis of Lemma 3.3 it is easy to give a

Proof of Theorem 3.1. Assume that there exists a focal chain $S_i\, (i = 0, 1, \ldots, k)$ in G, such that $S_0 = S$ and $S_k = H$. Assume also that the set π of primes contains all the prime divisors of $[S\colon H]$. Then the subgroups $T_i = P(\pi) \cap S_i$ form a focal chain with $T_0 = P(\pi) \cap S$, $T_k = P(\pi) \cap H$. Thus T_{i+1} is a normal subgroup of T_i, and π contains all the prime divisors of the divisor

$$[T_i\colon T_{i+1}] = [P(\pi) \cap S_i\colon P(\pi) \cap S_{i+1}]$$

of $[S\colon H]$. Hence by Lemma 3.3, we have

3.4 $$T_i^{[P(\pi):T_i]} \subseteq {}_1(T_i, P(\pi))\, T_{i+1} = T_{i+1}.$$

Now we shall prove by induction on i that every prime divisor of $[P(\pi)\colon T_i]$ divides $[P(\pi)\colon T_0]$. This statement is clearly true for $i = 0$; assume then that it is true for some i, $0 \leqslant i < k$. Since $T_{i+1} \subseteq T_i \subseteq P(\pi)$, we have

$$[P(\pi)\colon T_{i+1}] = [P(\pi)\colon T_i][T_i\colon T_{i+1}],$$

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and by 3.4, every prime divisor of $[T_i: T_{i+1}]$ divides $[P(\pi): T_i]$. Now it follows from the induction hypothesis that every prime divisor of $[P(\pi): T_{i+1}]$ divides $[P(\pi): T_0]$, completing the induction. The theorem follows.

Remark 1. If H is a subgroup of a subgroup S of G, then for π a set of primes, the statements

(i) π contains all the prime divisors of $[S: H]$,

(ii) π contains all the prime divisors of $[P(\pi) \cap S: P(\pi) \cap H]$,

are equivalent.

$$\text{For } [S: H] = [S: (P(\pi) \cap S) H][(P(\pi) \cap S) H: H]$$
$$= [P(\pi) S: P(\pi) H][P(\pi) \cap S: P(\pi) \cap H],$$

and $[P(\pi)S: P(\pi) H]$ divides the order of the π-group $G/P(\pi)$.

Remark 2. In case ${}_iS = 1$ for some i, Theorem 3.1 implies that for $\pi =$ the totality of prime divisors of the order of S, every prime divisor of $P(\pi) \cap S$ divides $[P(\pi): P(\pi) \cap S]$. That this does not imply $P(\pi) \cap S = 1$ is shown by the following example of a group which is generated by elements of order prime to the prime number p, but which contains a centre element of order p.

Example 4. Let A be the direct product of two cyclic groups $\{a\}$ and $\{b\}$ each of order p, p an odd prime. Let i be an integer $1 < i < p$, then the equations: $a^c = a$, $b^c = b^i$, and $a^d = a$, $(ab)^d = (ab)^i$ define automorphisms c and d of A, which have the same order n, equal to the multiplicative order of i modulo p. In particular, n is prime to p. In the finite group G, obtained by adjoining to A the group of automorphisms U of A generated by c and d, a is a centre element of order p, and c and d have order n prime to p. Furthermore the element bc has order prime to p, for $(bc)^n = b^m c^n = b^m$, where

$$m = 1 + i + i^2 + \ldots + i^{n-1} = (i^n - 1)(i - 1)^{-1},$$

since $i - 1$ is prime to p. Hence p divides m, so that $(bc)^n = b^m = 1$. Similarly the order of $(ab)d$ is prime to p. But the elements c, d, bc, abd generate G. Thus G is generated by elements of order prime to p, and contains a centre element of order p.

COROLLARY 3.5. *If the subgroup H of S is chained to S, and if π is a set of primes which contains all the prime divisors of $[S: H]$, but none of $[G: S]$, then $G = P(\pi)S$ and $P(\pi) \cap S \subseteq H$.*

Proof. Since π contains no prime divisors of $[G: S]$, $G = P(\pi)S$ by 1.3. Since furthermore π contains all the prime divisors of $[S: H]$, this index is relatively prime to

$$[G: S] = [P(\pi)S: S] = [P(\pi): P(\pi) \cap S].$$

Hence $P(\pi) \cap S = P(\pi) \cap H$ by Theorem 3.1. This proves the Corollary.

Now we obtain, as an application of Corollary 3.5, Proposition 1.8, Corollary 1.9, and Proposition 2.6.

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THEOREM 3.6. *Assume that the normal subgroup H of the subgroup S of G satisfies the condition*

3.61 *H is chained to S in G, ${}^iS \subseteq H \subseteq {}^iG$, and $[G\colon S]$ and $[S\colon H]$ are relatively prime.*

Then if π is any set of primes which contains all the prime divisors of $[S\colon H]$, but none of $[G\colon S]$ (for example, if π is the totality of prime divisors of $[S\colon H]$) we have

(i) *$N = P(\pi)H$ is a normal subgroup of G such that $G = NS$ and $H = N \cap S$.*

(ii) *For N a normal subgroup of G, the following are equivalent statements:*

(a) *$G = NS$ and $H = N \cap S$;*

(b) *$N = P(\pi)H = G^i(\pi)$;*

(c) *$N = G^i(\sigma)$, σ a set of primes which contains all the prime divisors of $[S\colon H]$, but none of $[G\colon {}^iGS]$.*

(iii) *$H = {}^iG \cap S$, and in case $H = {}_iS$, we have ${}_jS = {}^jS[{}^iG \cap S]$ for $j \leqslant i$.*

Remark 3. If the indices $[G\colon S]$ and $[S\colon {}_kS]$ are relatively prime for some k, and if $i \leqslant k$, then the hypotheses of Theorem 3.6 are satisfied for $H = {}_iS$ and $\pi =$ the totality of prime divisors of $[S\colon {}_kS]$. Thus, for example, if S is a p-Sylow subgroup of G, then for each i, all the conclusions of Theorem 3.6 are valid with $H = {}_iS$ and $\pi = p$.

Remark 4. Simple examples show that the condition that the indices $[G\colon S]$ and $[S\colon H]$ be relatively prime is not necessary for the conclusion (i) of Theorem 3.6. Indeed, suppose that A is a p-group, and that B is a group which is generated by elements of order prime to the prime number p, but that the order of B is divisible by p. Set $G = A \otimes B$, then $B = P(p)$, and it is clear that for $S = A$, $H = {}_iA$, and $\pi = p$, (i) is satisfied for each i.

For a second example, suppose that U is a group which is generated by elements whose orders are not divisible by p, but such that p divides the order of U, and U admits an automorphism a of order a power of p—for instance, if p divides the order of $U/Z(U)$, U has an inner automorphism of order p. Form the group G by adjoining a to U. Then $U = P(p)$, and for $S =$ the cyclic subgroup of G generated by a, $H = {}_iS$, and $\pi = p$; (i) is fulfilled for each i.

COROLLARY 3.7. *Let H and K be respectively normal subgroups of the subgroups S and T of G, subject to the condition* 3.61 *of Theorem* 3.6. *Assume furthermore that the totalities of prime divisors of $[S\colon H]$ and $[T\colon K]$ coincide. Then if H is conjugate to K, S is conjugate to T.*

Proof. If there exists x such that $H = K^x$, set $R = T^x$. Then H is a normal subgroup of R, and satisfies the condition 3.61 with respect to R. Moreover, since $[R\colon H] = [T\colon K]$, the totalities of prime divisors of $[S\colon H]$ and $[R\colon H]$ coincide. If π denotes this totality, we have by Theorem 3.6 that $N = P(\pi)H$ is a normal subgroup of G such that $G = NS = NR$, and $H = N \cap S = N \cap R$.

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Both S and R are part of the normalizer $N(H)$. Set $M = N \cap N(H)$, then M is a normal subgroup of $N(H)$, and $H \subseteq M$. For X any subgroup of G which contains H and is part of $N(H)$, set $X^* = X/H$. Then S^* and R^* are nilpotent representative subgroups of the normal subgroup M^* of $N(H)^*$. Since $[N(H): M] = [G: N] = [S: H]$, and $[M: H]$ divides $[N: H] = [G: S]$, M^* has order prime to its index in $N(H)^*$. Hence it follows [**9**, p. 132] that S^* is conjugate to R^* in $N(H)^*$, and hence that S is conjugate to R in $N(H)$. Now the corollary is proved, since the relation of conjugacy is transitive.

As a further application of Theorem 3.6 we shall prove the following generalization of a theorem of Grün [**4**; **6**].

Corollary 3.8. *If S is a p-Sylow subgroup of a p-regular group G, and if N_B is the normalizer of the weak closure $B = B(S)$ of $Z(S)$ in S* (see §2 for the necessary definitions) *then $S \cap {}^iG = S \cap {}^iN_B$, and if we denote this subgroup by $S(i)$ we have the isomorphisms*

$$S/S(i) \simeq N_B/N_B{}^i(p) \simeq G/G^i(p).$$

Proof. By Theorem 3.6 we have the equations

$$G = G^i(p)S, \; {}_i(S, G) = G^i(p) \cap S = {}^iG \cap S,$$

and

$$N_B = N_B{}^i(p)S, \; {}_i(S, N_B) = N_B{}^i(p) \cap S = {}^iN_B \cap S.$$

By Lemma 2.8, ${}_i(S, G) = {}_i(S, N_B)$. Now the Corollary follows.

4. Hyperfocality. In analogy with the terminology of centre, hypercentre, etc., we shall refer to a subgroup S of the group G as a *focal* subgroup of G if ${}_1S = 1$, and more generally, as a *hyperfocal* subgroup of G if ${}_kS = 1$ for some k. Thus a focal subgroup is abelian, and a hyperfocal subgroup is nilpotent.

The hypercentre $H(G)$ of G may be defined in several equivalent ways. We recall only, that using our terminology,

$H(G)$ = *the product of all normal, hyperfocal subgroups of G,*

from which it is clear that the hypercentre, and hence any hypercentral subgroup, is hyperfocal. We have already discussed other examples of hyperfocal subgroups. Example 1 of §2 shows that a (hyper) focal subgroup need not be part of the hypercentre. It furthermore shows that the product of two (hyper) focal subgroups need not be hyperfocal, and indeed that a (hyper) focal subgroup need not be hyperfocal modulo the centre. In Example 2 of §2 we proved that every subgroup of order p in a p-group is hyperfocal, which shows in particular that a focal subgroup which is part of the hypercentre need not be part of the centre.

A further indication of the relation between the concepts of hypercentrality and hyperfocality is given by the following strengthening of a theorem of Baer [**2**].

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PROPOSITION 4.1. *If p is a prime, and if the subgroup S of the group G is a p-group, then the following conditions are equivalent:*

(i) $S \subseteq H(G)$.

(ii) $P(p) \subseteq C(S)$.

(iii) $P(p) \subseteq N(S)$, *and* $S \cap P(p)$ *hyperfocal in* $P(p)$.

Proof. The equivalence of the conditions (i) and (ii) is the theorem of Baer. If $P(p) \subseteq C(S)$, then $S \cap P(p) \subseteq Z(P(p))$, hence, since $C(S) \subseteq N(S)$, (ii) implies (iii).

To complete the proof, we shall show that (iii) implies (ii). Assume (iii). Then $S \cap P(p)$ is a normal hyperfocal subgroup of $P(p)$, and hence belongs to the hypercentre of $P(p)$. It follows that since (i) implies (ii), $S \cap P(p)$ is part of the centre of $P(p)$. Now let x be an element of G whose order is prime to p (by definition $P(p)$ is generated by such elements) and let s be an element of S. Since x is in $P(p)$, the commutator $c = [x, s]$ is in $P(p)$, and since $P(p) \subseteq N(S)$, c is in S. Thus x is in $S \cap P(p)$, and hence is an element of the centre of $P(p)$. Hence $cx = xc$, from which we prove by a standard method that the order of c divides the order of x. But c is in S, and hence has order a power of p. It follows therefore that $c = 1$. We have proved that (ii) is a consequence of (iii), which completes the proof of the proposition.

Now we come to the problem of characterizing in a group G those subgroups S which are hyperfocal. In order to be able to apply results of §3, we proceed under the assumption that the order of S is prime to its index in G. For the sake of brevity, we shall refer to a subgroup which satisfies this condition as a *P-subgroup*. We begin by proving

PROPOSITION 4.2 (a) *If S is a P-subgroup of a group G and if $Z(S)$ is part of the hypercentre of $N(S)$, then $Z(S)$ is part of the centre of $N(S)$.*

(b) *If S is a Sylow subgroup of G, or if S is a P-subgroup of a solvable group G, then the following conditions are equivalent:*

(i) $Z(S)$ *is focal in* G.

(ii) $Z(S)$ *is hyperfocal in* G.

(iii) $Z(S)$ *is part of the hypercentre of* $N(S)$.

(iv) $Z(S)$ *is part of the centre of* $N(S)$.

Proof. If S is a P-subgroup of G, S is a P-subgroup of $N(S)$. Hence if U is the subgroup which is generated by all the elements of $N(S)$ which have orders prime to the order of S, and hence to the order of $Z(S)$, $N(S) = SU$. If $Z(S)$ is part of the hypercentre of $N(S)$, $Z(S)$ commutes elementwise with U, by the theorem of Baer (Proposition 4.1), and hence in this case $Z(S)$ is part of the centre of $N(S)$. This proves (a).

Assume now that S is a Sylow subgroup of G, or that S is a P-subgroup of the solvable group G. That (i) implies (ii) is trivial. Since $Z(S)$ is normal in $N(S)$, $Z(S)$ is part of the hypercentre of $N(S)$ whenever it is hyperfocal in $N(S)$. But (ii) implies that $Z(S)$ is hyperfocal in $N(S)$, hence (ii) implies (iii). That

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(iii) implies (iv) is a consequence of part (a) of the present proposition. Finally we conclude the implication of (i) by (iv) from Lemma 2.7.

THEOREM 4.3. *If S is a P-subgroup of the group G, and if π denotes the totality of prime divisors of the order of S, then the following conditions are equivalent:*

(i) *S is hyperfocal in G.*
(ii) *S is nilpotent, and $1 = P(\pi) \cap S$.*
(iii) *S is nilpotent, and $P(\pi)$ = the totality of elements in G of order prime to the order of S.*
(iv) *S is nilpotent, and there exists a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$.*
(v) *S is nilpotent, and ${}_iS = {}^iS$ for $0 \leqslant i$.*
(vi) *S has only the identity element in common with the hypercommutator subgroup of G* (the hypercommutator subgroup of G is defined to be the smallest term of the lower central series of G).
(vii) *S is nilpotent, there exists a subgroup T of G such that ${}_iN(S) = {}_iS \otimes {}_iT$, and ${}_i(S, G) = {}_i(S, N(S))$ for $0 \leqslant i$.*
(viii) *S is nilpotent, $N(S) = SC(S)$, and ${}_i(S, G) = {}_i(S, N(S))$ for $0 \leqslant i$.*

THEOREM 4.3′. *If S is a Sylow subgroup of G, or if S is a P-subgroup of the solvable group G, then the following are equivalent conditions:*

(i) *S is hyperfocal in G.*
(ix) *S is nilpotent, and ${}^iN(S) = {}_iS_i(C(S), N(S))$ for $0 \leqslant i$.*
(x) *S is nilpotent, and ${}^iN(S) = {}_iS_iC(S)$ for $0 \leqslant i$.*
(xi) *$Z(S) \subseteq Z(N(S))$, and ${}_kS \subseteq Z(S)$ for some k.*
(xii) *S is part of the hypercentre of $N(S)$, and ${}^iN(S) = {}_iN(S)$ for $0 \leqslant i$.*

Remark 5. In case S is a Sylow subgroup of G, we strike out the condition "S is nilpotent" whenever it occurs, and the resulting twelve conditions remain equivalent.

Proof of Theorems 4.3 *and* 4.3′. For convenience, let us first record the following fact:

4.31 *If S is a subgroup, N a normal subgroup of the group G such that $G = NS$ and $1 = N \cap S$, then ${}_iS = {}^iS$.*

For then, using Lemma 2.5, ${}_iS \subseteq {}^iS[N \cap S] = {}^iS$. But ${}^iS \subseteq {}_iS$, hence ${}_iS = {}^iS$.

Let us now collect a number of conclusions resulting from the hypothesis that S is a P-subgroup of G. By Corollary 3.5,

(a) *$G = P(\pi)S$, and $P(\pi) \cap S \leqslant {}_iS$ for $0 \leqslant i$.*

Furthermore, by Theorem 3.6,

(b) *${}_iS = {}^iG \cap S$, for $0 \leqslant i$.*

By a theorem of Schur [**9**, p. 132] there exists a subgroup U of G such that

(c) *$C(S) = [S \cap C(S)] \otimes U = Z(S) \otimes U$*

and also a subgroup W of G such that

(d) *$N(S) = SW$, $1 = S \cap W$.*

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Now assume (i), that is, assume that there is a k such that ${}_kS = 1$. Then by (a), $P(\pi) \cap S = 1$, so that (i) implies (ii), since hyperfocal subgroups are nilpotent.

If we assume next that $P(\pi) \cap S = 1$, then the order of $P(\pi)$ divides $[G:S]$, which is by hypothesis prime of the order of S. By definition, $P(\pi)$ contains the totality of elements of G of order prime to the order of S, hence $P(\pi)$is equal to this totality. Conversely, if $P(\pi)$ contains only elements of order prime to the order of S, $1 = P(\pi) \cap S$. Thus (ii) and (iii) are equivalent.

Since S is a P-group, $G = P(\pi)S$, hence if $1 = P(\pi) \cap S$, $N = P(\pi)$ satisfies the condition (iv), that is (ii) implies (iv). Furthermore, if there exists a normal subgroup N of G satisfying (iv), ${}_iS = {}^iS$ by 4.31. Hence (iv) implies (v). That (v) implies (i) is an immediate consequence of the definitions of hyperfocality and nilpotency. We have now proved the equivalence of the conditions (i) through (v).

The equivalence of (i) and (vi) is an immediate consequence of the formula (b), when we recall that the hypercommutator subgroup of G is the smallest term of the lower central series of G.

Next assume the existence of a normal subgroup N of G such that $G = NS$, and $1 = N \cap S$. Then by 4.31, ${}_iS = {}^iS$, and by Dedekind's law, $N(S) = [N(S) \cap N] \otimes S$. Hence by 4.31, ${}_i(S, N(S)) = {}^iS$, so that ${}_iS = {}_i(S, N(S))$. Moreover, $T = N(S) \cap N$ has order prime to the order of S, and hence by Lemma 2.4, ${}_iN(S) = {}_iS \otimes {}_iT$. This proves that (vii) is a consequence of (iv).

In case $i = 0$, condition (vii) implies that $N(S) = S \otimes T$, but this clearly implies that $N(S) = SC(S)$. Hence (vii) implies (viii). Conversely, if we assume that $N(S) = SC(S)$, then by (c),

$$N(S) = S([S \cap C(S)] \otimes U) = S \otimes U.$$

Hence by Lemma 2.4, (viii) implies (vii). Thus (vii) and (viii) are equivalent.

Assuming (vii), we have by 4.31 that ${}_i(S, N(S)) = {}^iS$, and hence that ${}_iS = {}^iS$. This means that (v) is a consequence of (vii). The equivalence of the conditions (i) through (viii) is now established, proving Theorem 4.3.

In proving Theorem 4.3′, we may use the equivalence (i) through (viii). Now assume once more that $N(S) = SC(S)$, then, since S and $C(S)$ are normal subgroups of $N(S)$, ${}^iN(S) = {}_i(S, N(S))\ {}_i(C(S)N(S))$. Hence (viii) implies that ${}^iN(S) = {}_iS\ {}_i(C(S), N(S))$, so that (viii) implies (ix). If we assume that G is solvable, or that S is a Sylow subgroup of the group G, then the equivalence of (ix) and (x) is immediate by Lemma 2.7.

Assume (ix), then in particular $N(S) = SC(S)$, and this condition implies by (c) that S is a direct factor of $N(S)$. Hence by 4.31, ${}_i(S, N(S)) = {}^iS$. Since by (ix), S is nilpotent, there is a k such that $1 = {}^kS = {}_k(S, N(S))$, i.e., S is part of the hypercentre of $N(S)$. Now we conclude by (ix) that ${}_kS \subseteq {}_k(C(S), N(S)) \cap S \subseteq C(S) \cap S = Z(S)$. Furthermore, since S is part of the hypercentre of $N(S)$, so is $Z(S)$. Hence by Proposition 4.2, $Z(S)$ is part of the centre of $N(S)$. Hence (ix) implies (xi).

Now assume that G is solvable, or that S is a Sylow subgroup of the group G, and assume (xi). Then ${}_{k+1}S \subseteq {}_1Z(S)$, and by Lemma 2.7, and condition (xi), ${}_1Z(S) = (Z(S), N(S)) = 1$, hence ${}_{k+1}S = 1$. Thus in case G is solvable, or S is a Sylow subgroup of G, (i) is a consequence of (xi), and then the conditions (ix), (x), and (xi) are equivalent to each other, and to the preceding conditions.

Finally, we must consider the condition (xii). Assume that G is solvable, or that S is a Sylow subgroup. First assume the preceding equivalent conditions. By (i), S is part of the hypercentre of $N(S)$. By (vii), ${}_iN(S) = {}_iS \otimes {}_iT$, and ${}^iN(S) = {}_i(S, N(S)) \otimes {}_i(T, N(S))$, and ${}_i(S, N(S)) = {}_iS$. Since furthermore $T \subseteq C(S)$, we have by Lemma 2.7 that ${}_i(T, N(S)) = {}_iT$. Hence ${}_iN(S) = {}^iN(S)$. Thus in this case (xii) is a consequence of the preceding equivalent conditions.

Now assume (xii). By (d), there exists a subgroup W of G such that $N(S) = SW$ and $1 = S \cap W$. Since S is part of the hypercentre of $N(S)$, it follows from Proposition 4.1 that $W \subseteq C(S)$, and hence $N(S) = S \otimes W$. Hence ${}^iN(S) = {}^iS \otimes {}^iW$, and by Lemma 2.4, ${}_iN(S) = {}_iS \otimes {}_iW$. Since ${}^iN(S) = {}_iN(S)$, and since ${}^iS \subseteq {}_iS$, ${}^iW \subseteq {}_iW$, we conclude that ${}_iS = {}^iS$. Hence, since a subgroup of the hypercentre is nilpotent, (xii) implies (v). This completes the proof of Theorem 4.3′.

It should be remarked that the subgroup N of condition (iv) is uniquely determined,

$N = P(\pi) =$ *the totality of elements of G with orders prime to the order of S.*

COROLLARY 4.4. *For a P-subgroup S of the group G, the following four conditions are equivalent:*

(i) *S is a focal subgroup of G.*

(ii) *S is an abelian hyperfocal subgroup of G.*

(iii) *S is abelian, and there exists a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$.*

(iv) *$S \cap [G, G] = 1$.*

If S is a Sylow subgroup of G, or if G is solvable, the following condition is equivalent to each of the preceding conditions.

(v) *$N(S) = C(S)$.*

Proof. The implication of (ii) by (i) is clear, and (iii) follows from (ii) by Theorem 4.3. If (iii) is true, then G/N is isomorphic with the abelian group S, and hence is abelian. Thus $[G, G] \subseteq N$, so that $1 = S \cap [G, G]$. But then $1 = S \cap [G, G] \supseteq {}_1S$, so that ${}_1S = 1$. Thus (iii) implies (iv) and (iv) implies (i), which proves the equivalence of (i) through (iv).

In case S is abelian, (v) is equivalent with condition (xi) of Theorem 4.3′ and (v) implies that S is abelian. Hence, if we assume that S is a Sylow subgroup of G or that G is solvable, we have the equivalence of (v) with the preceding conditions by Theorem 4.3′, which completes the proof.

The implication of (iii) by (v) is the classical theorem of Burnside.

We also mention the following

COROLLARY 4.5. *If S is a P-subgroup of the group G, the following are equivalent conditions:*

(i) *S is a normal hyperfocal subgroup of G.*
(ii) *S is part of the hypercentre of G.*
(iii) *S is a nilpotent direct factor of G.*
(iv) *S is nilpotent, and $G = SC(S)$.*

This corollary is easily verified by reference to Theorem 4.3 when we remember that a normal hyperfocal subgroup is part of the hypercentre, and notice that each of the conditions (ii) and (iv) imply that S is a normal subgroup of G.

Now it is easy to construct further examples of hyperfocal subgroups which are not part of the hypercentre. For instance, in the group G formed by adjoining to the group A an automorphism a of order prime to the order of A, the cyclic group Z generated by a is a representative subgroup for the normal subgroup A, but is not a direct factor. Hence by Corollary 4.4, Z is focal in G, but by Corollary 4.5, Z is not part of the hypercentre.

THEOREM 4.6. *Suppose that the order of the group G is of the form, mn, with m and n relatively prime. Then*

(a) *The following two conditions are equivalent:*
(i) *G contains a hyperfocal subgroup of order m;*
(ii) *a p-Sylow subgroup of G is hyperfocal in G for each prime divisor p of m.*

(b) *If G contains a hyperfocal subgroup S of order m, any subgroup of order m is conjugate to S, and any subgroup of order a divisor of m is contained in a subgroup of order m, and hence is hyperfocal in G.*

Proof. If the prime p divides m, and if S is a subgroup of G of order m, then S contains a p-Sylow subgroup P of G. If S is hyperfocal in G then so is P. Hence (i) implies (ii). Let us assume (ii), then if p^a is the highest power of the prime p which divides m, there exists by Theorem 4.3 a normal subgroup N_p of G such that $[G: N_p] = p^a$. The intersection N for all primes p of the subgroups N_p is a normal subgroup of G, $[G: N] = m$, and G/N is nilpotent. Now it follows by the theorem of Schur that there exists a subgroup S of G of order m, i.e. such that $G = NS$ and $1 = N \cap S$. Since S is isomorphic with G/N, S is nilpotent, and it therefore follows from Theorem 4.3 that S is a hyperfocal subgroup of G. This proves (a).

If now S is any hyperfocal subgroup of G of order m, there exists by Theorem 4.3 a normal subgroup N of G such that $G = NS$ and $1 = N \cap S$. S is nilpotent, and hence G/N is nilpotent. That any subgroup of order m is conjugate to S now follows from a well-known theorem [**9**, p. 132]. Suppose that the order of the subgroup T of G divides m, and consider the subgroup $H = NT$ of G. Since $G = NS$, we have by Dedekind's law that $H = N[H \cap S]$; and H/N is nilpotent, as a subgroup of the nilpotent group G/N. Thus, by the theorem just quoted, there exists an element x in H such that $T = [H \cap S]^x = H \cap S^x$, so that $T \subseteq S^x$. Since S is hyperfocal, so is S^x, and hence, so is T. This completes the proof.

Taking m = the order of G in the preceding proposition, we have

COROLLARY 4.7. *A group G is nilpotent if and only if for each prime p there exists a p-Sylow subgroup of G which is hyperfocal.*

This corollary also follows from the fact that for S a p-Sylow subgroup of G, ${}_iS = {}^iG \cap S$.

COROLLARY 4.8. *A subgroup S of a group G is hyperfocal in G if and only if for each prime p there exists a p-Sylow subgroup of S which is hyperfocal in G.*

Proof. That a hyperfocal subgroup of G has the above mentioned property is clear. Assume that S is a subgroup of G which, for each prime, has a Sylow subgroup hyperfocal in G. Then the same is true in S, hence S is nilpotent by Corollary 4.7. Thus S has only one p-Sylow subgroup for each prime p, and is the direct product of these. Now the result follows from Lemma 2.4.

The study of hyperfocal subgroups is now reduced to the study of primary hyperfocal subgroups.

5. A theorem of Grün. The methods of §3 admit of further application, leading in particular to a theorem of Grün. Let S be a subgroup of the group G, and let H be a subgroup of S. Let π denote the totality of prime divisors of $[S\colon H]$, and set $A = P(\pi) \cap S$, $B = P(\pi) \cap H$. Following Grün, we introduce the subgroup

S_* = the (normal) subgroup of S which is generated by all the intersections $S \cap S^g \neq S$, for g in G.

THEOREM 5.1. *If H is a normal subgroup of S, if the indices $[G\colon S]$ and $[S\colon H]$ are relatively prime, and if $S_* \subseteq H$, then*

(i) $G = P(\pi)S$.

(ii) $N = P(\pi)H$ *is a normal subgroup of* G.

(iii) $A = [A, N(A \text{ in } P(\pi))]B$.

Proof. Conditions (i) follows from 1.3, and hence (ii) is a consequence of Lemma 1.1. By Lemma 3.3, we have $A^{[P(\pi):A]} \subseteq {}_1(A, P(\pi))B$. But $[P(\pi)\colon A] = [G\colon S]$ is relatively prime to the divisor $[A\colon B]$ of $[S\colon H]$, hence $A = A^{[P(\pi):A]}B$. We conclude that $A = {}_1(A, P(\pi))B$. The subgroup ${}_1(A, P(\pi))B$ is generated by commutators $c = [a, x]$, with c and a in A, x in $P(\pi)$. Since a and c are in A, a^x is in A, hence since $A \subseteq S$, a and a^x are in $S \cap S^x$. If both a and a^x are in S_*, c is in $S_* \cap P(\pi)$. If either a or a^x is not in S_*, it follows that $S = S^x$, that is, x is in

$$N(S \text{ in } G) \cap P(\pi) \subseteq N(A \text{ in } P(\pi)),$$

and hence c is in $[A, N(A \text{ in } P(\pi))]$. This proves that

$${}_1(A, P(\pi)) = [A, N(A \text{ in } P(\pi))][S_1 \cap P(\pi)].$$

Thus

$$A = {}_1(A, P(\pi))B = [A, N(A \text{ in } P(\pi))][S_* \cap P(\pi)]B = [A, N(A \text{ in } P(\pi))]B,$$

since $S_* \cap P(\pi) \subseteq H \cap P(\pi) = B$. Thus (iii) is proved.

COROLLARY 5.2. *If* $S = N(S \text{ in } G)$, *and* $S_* \subseteq H$, *then* (i) *and* (ii) *of Theorem* 5.1 *hold, together with*
(iii′) $A = [A, A]B$, *that is*, A/B *is perfect.*

Proof. First we have

5.21 (Grün) $S = N(S \text{ in } G)$ *implies that* $[G: S]$ *is prime to* $[S: S_*]$.

For suppose that P is a p-Sylow subgroup of S which is not part of S_*. Then there exists a p-Sylow subgroup P^* of G which contains P. If $P \subset P^*$, then $P \subset N(P \text{ in } P^*)$. But $S \cap P^* = P$, hence there is an element x in $N(P \text{ in } P^*)$ which is not in S. Thus $P = P^x$ is part of $S \cap S^x$, which implies that x is in $N(S \text{ in } G) = S$. Since this is impossible, $P = P^*$, which proves 5.21.

Hence, since we have assumed that $S_* \subseteq H$, the indices $[G: S]$ and $[S: H]$ are relatively prime. Thus we conclude (i) and (ii) together with $A = [A, N(A \text{ in } P(\pi))]B$ from Theorem 5.1. Since $S_1 \subseteq H$,

$$P(\pi) \cap S_* \subseteq P(\pi) \cap H = B.$$

In case $P(\pi) \cap S_* = A$, we have $A = B$, so that (iii′) follows in this case. If on the other hand $P(\pi) \cap S_* \subset A$, and if x is an element of $N(A \text{ in } P(\pi))$, $A = A^x \subseteq S \cap S^x$, and hence x is in $N(S \text{ in } G)$. Hence

$$N(A \text{ in } P(\pi)) \subseteq N(S \text{ in } G) \cap P(\pi) = S \cap P(\pi) = A,$$

and (iii′) follows in this case.

Corollary 5.2 has as an immediate consequence the following result of Grün [**5**]:

COROLLARY 5.3. *If S is a subgroup of the group G such that* $S = N(S \text{ in } G)$, *if the normal subgroup H of S contains every intersection $S \cap S^g$ for g not in S, and if S/H is solvable, then there exists a normal subgroup N of G such that $G = NS$ and $H = N \cap S$.*

Since $S/{}_iS$ is nilpotent, Corollary 5.3 implies that the condition

5.31 $S = N(S \text{ in } G)$, *and* $S_* \subseteq {}_iS$

is sufficient for the existence of a normal subgroup N of G such that $G = NS$ and ${}_iS = N \cap S$. This criterion is subsumed by Theorem 3.6, since 5.31 implies that $[G: S]$ and $[S: {}_iS]$ are relatively prime.

Added in proof. Since this paper was submitted for publication, Professor R. Brauer has communicated to the author theorems concerning the subgroup denoted here by ${}_1S = {}_1(S, G)$ which he obtained as applications of his profound characterization of group characters. These theorems are contained in our results or are easily obtained by our methods. They appear in Professor Brauer's paper, *A characterization of the characters of groups of finite order*, Annals of Mathematics, *57* (1953), 357-377.

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