ON THE CONVERSION OF THE DETERMINANT INTO THE PERMANENT

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1. Introduction. Let $M_m(F)$ be the vector space of m-square matrices over a field F. If X belongs to $M_m(F)$, then x_{ij} will denote the element occuring in row i and column j of X.

Let S_m be the symmetric group of degree m and $\varepsilon: S_m \to F$ the alternating character on S_m (i.e. $\varepsilon(\sigma) = 1$ or -1 according as σ is an even or odd permutation). If X belongs to $M_m(F)$ then the determinant of X and the permanent of X are defined as follows:

$$\det X = \sum_{\sigma \varepsilon S_m} \varepsilon(\sigma) \prod_{i=1}^{m} x_{i\sigma(i)} ;$$

$$\operatorname{per} X = \sum_{\sigma \varepsilon S_m} \prod_{i=1}^{m} x_{i\sigma(i)} .$$

The object of this note is to show that if $m > 2$, then there is no linear transformation $K: M_m(F) \to M_m(F)$ such that $\det K(X) = \operatorname{per} X$ for all X in $M_m(F)$. An early result in this direction is due to Polya [6], who showed that no affixing of $\pm$ signs to the entries of X can (except when $m = 2$) uniformly convert the permanent into the determinant. Recently Marcus and Minc [4] established that if $m > 2$, then there is no linear transformation on matrices to matrices that uniformly converts the permanent into the determinant. In their proof of

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this result Marcus and Minc used r^{th} determinantal and permanental compound matrices, and an induction argument involving some rather complicated computations. The purpose of this note is to give a shorter and somewhat more direct proof of their result.

2. *Results.* We will establish the following:

THEOREM. *If* $m > 2$ *then there does not exist a linear transformation* $K: M_m(F) \to M_m(F)$ *such that* $\det K(X) = \text{per } X$ *for all* X.

Proof. We suppose such a linear transformation K exists. Suppose that K is singular. Then $K(A) = 0$ for some matrix $A \neq 0$. Hence $K(X+A) = K(X)$ for all X. Therefore

$$\text{per } X = \det K(X) = \det K(X+A) = \text{per } (X+A) \quad \text{for all } X.$$

Now note that if P and Q are permutation matrices (i.e. $p_{ij} = \delta_{i\sigma(j)}$, $q_{ij} = \delta_{i\tau(j)}$ for some $\sigma, \tau \in S_m$ where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise) then $\text{per } PXQ = \text{per } X$ for all X. Therefore we may assume that if $A = (a_{ij})$ then $a_{11} \neq 0$. Let B be the following matrix:

$$B = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1m} \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Then

$$B+A = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{12} & 1+a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & 1+a_{mm} \end{bmatrix}.$$

Clearly we have

$$\text{per } B = -a_{11} \neq 0, \quad \text{per } (B+A) = 0.$$

This, however, contradicts the fact that per $(X+A)$ = per X for all X. Therefore we may assume that K is nonsingular.

Let G (resp. H) be the set of all linear transformations T of $M_m(F)$ into itself that satisfy $\det T(X) = \det X$ for all X (resp. per $T(X)$ = per X for all X). It is known [1, 3, 5] that G and H are groups and

1) T belongs to G if and only if there exist fixed nonsingular matrices U and V with $\det UV = 1$ such that either $T(X) = UXV$ or $UX'V$ (X' denotes the transpose of the matrix X).

2) T belongs to H if and only if there exist permutation matrices P and Q and diagonal matrices D and L with per $DL = 1$ such that either $T(X) = DPXQL$ or $DPX'QL$.

Suppose T belongs to H. The map K is nonsingular so K^{-1} exists and it is easy to check that per $K^{-1}(X) = \det X$ for all X. Therefore

$$\det KTK^{-1}(X) = \text{per } TK^{-1}(X) = \text{per } K^{-1}(X) = \det X \text{ for all } X.$$

Hence KTK^{-1} belongs to G. A similar argument shows that if S belongs to G then $K^{-1}SK$ belongs to H. Therefore we may conclude that these two subgroups of the group of nonsingular linear transformations of $M_m(F)$ onto itself are conjugate via K.

We now show that this is not the case and so arrive at the desired contradiction. First note that if T belongs to H we have $T(X) = DPXQL$ or $DPX'QL$, where D, P, Q and L are as above, and $\det DPQL = \pm 1$. Let H_o be the set of T in H with $\det DPQL = 1$. It is clear that H_o is a subgroup of both H and G, and the index of H_o in H is two. Therefore, since H and G are isomorphic, the index of H_o in G is two. Hence we may choose S in G such that G is the disjoint union of the two cosets H_o and SH_o . Further, we know that there exist fixed nonsingular U and V such that $S(X) = UXV$ or $UX'V$. Hence, if T belongs to G it must be of one of the following forms:

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1) $T(X) = DPXQL$ 2) $T(X) = DPX'QL$

3) $T(X) = UDPXQLV$ 4) $T(X) = UDPX'QLV$.

Clearly we may choose A and B with $\det AB = 1$ such that A is different from both DP and UDP and S is different from both QL and QLV, for any diagonal matrices D and L and any permutation matrices P and Q. Define $W: M_m(F) \to M_m(F)$ by $W(X) = AXB$; then W belongs to G and it is immediate that W is neither of form 1) nor of form 3). Suppose $W(X) = AXB = DPX'QL$; then

$$X' = (DP)^{-1} AXB(QL)^{-1} .$$

That is, we can transpose any matrix by pre- and post-multiplication by two fixed matrices. It is well known [2; p.837] that this is not true. Therefore, W cannot be of form 3). A similar argument shows that W is not of form 4). Hence W belongs neither to H_o nor SH_o , so does not belong to G, a contradiction. This completes the proof.

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