J. Austral. Math. Soc. (*Series A*) **30** (1981), 284–296

RAMSEY NUMBERS FOR CERTAIN k-GRAPHS

STEFAN A. BURR and RICHARD A. DUKE

(Received 20 December 1979; revised 1 May 1980)

Communicated by W. D. Wallis

Abstract

We are interested here in the Ramsey number $r(T, C)$, where C is a complete k-uniform hypergraph and T is a "tree-like" k-graph. Upper and lower bounds are found for these numbers which lead, in some cases, to the exact value for $r(T, C)$ and to a generalization of a theorem of Chvátal on Ramsey numbers for graphs. In other cases we show that a determination of the exact values of $r(T, C)$ would be equivalent to obtaining a complete solution to existence question for a certain class of Steiner systems.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 05 C 35; secondary 05 B 35.

1. Introduction

By a k-graph F we mean a k-uniform hypergraph, i.e. a finite set, $V(F)$, of vertices and a collection of edges which are distinct k-subsets of $V(F)$. We shall call a k-graph T *tree-like* if T has exactly k vertices and a single edge or if T can be obtained from any tree-like k-graph T' by adjoining a single k-edge which shares exactly one vertex with T'. (We might have preferred the term "k-tree", but this has been used extensively, as in Beineke and Pippert (1971), to denote a single $(k + 1)$-edge, or a $(k + 1)$-graph formed by adding to a smaller k-tree T' a $(k + 1)$-edge sharing k vertices with T'.) We shall use $T(m, k)$ to denote a tree-like k-graph with $m = q(k - 1) + 1$ vertices (and hence q edges) and $\mathfrak{T}(m, k)$ to denote the class of all tree-like k-graphs of this size. The complete k-graph on n vertices in which each k-subset of the vertex-set is an edge will be denoted by $C(n, k)$.

For k-graphs F and G, the Ramsey number $r(F, G)$ is the smallest integer N such that, for any partition (E_1, E_2) of the edges of $C(N, k)$, either F is a subgraph of the k-graph induced by E_1, or G is a subgraph of the k-graph induced by E_2. As is often done, we will usually think of these partitions as arising from 2-colorings of the edges of $C(N, k)$ with, say, the edges of E_1 colored blue and those of E_2 red. We are interested here in the Ramsey numbers $r(T, C)$, where T is tree-like and C is a complete k-graph, and in the function $f(m, n; k)$ which we define to be the smallest integer N such that for each partition (E_1, E_2) of $C(N, k)$ either the k-graph induced by E_1 contains *some* member of $\mathfrak{T}(m, k)$ or the k-graph induced by E_2 contains $C(n, k)$. Hence $f(m, n; k)$ is the "Ramsey number" of $C(n, k)$ and the class $\mathfrak{T}(m, k)$.

Some of the first results on Ramsey numbers for k-graphs were obtained by Burr, Erdös and Spencer (1975) and by Duke and Harary (1976). The present investigation was suggested in part by the study of 3-graphs in Duke (1975) and by the following result for graphs ($k = 2$) due to Chvátal (1977).

THEOREM (Chvátal). *Let T be a tree with m vertices and K_n the complete graph on n vertices. Then $r(T, K_n) = (m - 1)(n - 1) + 1$.*

Our basic techniques have their origin in Chvátal's original proof of this theorem. The bounds which we obtain for $f(m, n; k)$ and $r(T(m, k), C(n, k))$ are, in many cases, generalizations of the formula in Chvátal's theorem. We find, however, that for $k > 2$ the exact values of these functions depend in many cases on the congruence class of n modulo $k - 1$ and that in some instances determining $f(m, n; k)$ precisely would be equivalent to settling a well-known existence problem in the theory of block designs.

2. Some basic inequalities

We begin by obtaining bounds for $r(T(m, k), C(n, k))$ and $f(m, n; k)$ which are valid for all of the values for which these functions are defined. The proof of the first theorem is a direct generalization of the methods first used by Chvátal to obtain the result mentioned above.

THEOREM 1. *Let $m = p(k - 1) + 1$ and $n = q(k - 1) + t$, where $k \geq 2$, $p, q \geq 1$, and let t be any integer such that $1 \leq t \leq k - 1$. Then for any $T(m, k)$ in $\mathfrak{T}(m, k)$ we have*

$$(1) \quad \frac{(m-1)(n-t)}{k-1} + t \leqslant f(m, n; k) \leqslant r(T(m, k), C(n, k)) \leqslant \frac{(m-1)(n-1)}{k-1} + 1.$$

https://doi.org/10.1017/S1446788700017183 Published online by Cambridge University Press

PROOF. The lower bound for r follows immediately from the existence of a "canonical" 2-coloring of the edges of $C(N, k)$, where $N = ((m-1)(n-t)/(k-1)) + (t-1)$. In this coloring all of the edges of $(n-t)/(k-1)$ disjoint copies of $C(m-1, k)$ are blue and all of the remaining edges are red. No $T(m, k)$ exists in the blue subgraph and, since each set of n vertices includes at least k vertices in some one of the blue $C(m-1, k)$, there is no red $C(n, k)$.

(This result could also be obtained by using this coloring in each case where $t = 1$, together with the easily obtained fact that $r(T(m, k), C(n+1, k)) \geq r(T(m, k), C(n, k)) + 1$ for any $T(m, k)$.)

The upper bound is established by an inductive argument on the size of $m + n$. The inequality becomes an equality when either $m = k$ or $n = k$, so we may assume that the result holds for all $T(m', k)$ and $C(n', k)$ where $m' + n' < m + n$ with $k < m, n$. It follows that for $N = ((m-1)(n-1)/(k-1)) + 1$, each 2-coloring of $C(N, k)$ must produce either a blue $T(m, k)$ or a red $C(n-1, k)$. In the second case we may remove the vertices of such a $C(n-1, k)$ obtaining a 2-colored $C(N', k)$ with

$$N' = (((m-(k-1)-1)(n-1))/(k-1)) + 1.$$

Let $T'(m-(k-1), k)$ denote a tree-like k-graph obtained by deleting from $T(m, k)$ one "terminal" edge which shares a single vertex x with the remainder of $T(m, k)$. Then $C(N', k)$ must contain a blue copy of T' or a red $C(n, k)$. If there exists a blue T', then either all of the edges of the original $C(N, k)$ having x as a vertex and their remaining vertices in the red $C(n-1, k)$ are red, which would produce a red $C(n, k)$, or one of these edges is blue, producing a blue copy of $T(m, k)$.

COROLLARY 1. *For m and n as in Theorem 1 we have the following whenever $t = 1$:*

$$r(T(m, k), C(n, k)) = \frac{(m-1)(n-1)}{(k-1)} + 1.$$

We shall denote the lower bound in (1) by $g(m, n; k)$ and the upper bound by $h(m, n; k)$. It follows that

$$g(m, n; k) \leq f(m, n; k) \leq h(m, n; k). \tag{2}$$

The following two lemmas provide additional inequalities which will be useful.

LEMMA 1. *For m and n as in Theorem 1 and any tree-like k-graph $T(m, k)$ in $\mathfrak{T}(m, k)$ we have*

$$r(T(m, k), C(n, k)) \geq r(T(m, k), C(n-(k-1), k)) + (m-1). \tag{3}$$

https://doi.org/10.1017/S1446788700017183 Published online by Cambridge University Press

PROOF. Let $R = r(T(m, k), C(n - (k - 1), k))$ and let the edges of a $C(R + (m - 2), k)$ be 2-colored in such a way that for some $R - 1$ vertices the $C(R - 1, k)$ induced by these vertices contains no blue $T(m, k)$ and no red $C(n - (k - 1), k)$, the edges of the complete subgraph induced by the remaining $m - 1$ vertices are all blue, and all other edges are red. This coloring produces no blue $T(m, k)$. Any set of n vertices either contains k or more vertices in the blue $C(m - 1, k)$ or at least $n - (k - 1)$ in the chosen $C(R - 1, k)$ and so has at least one blue edge.

LEMMA 2. *Suppose* $m = p(k - 1) + 1$, $p \geq 2$, *and* $n = q(k - 1) + t$, $q \geq 1$, $1 \leq t \leq k - 1$. *If* $T(m - (k - 1), k)$ *is obtained from* $T(m, k)$ *by deleting one terminal edge, then*

(4) $$r(T(m, k), C(n, k)) \leq r(T(m - (k - 1), k), C(n, k)) + (n - 1).$$

PROOF. Let $N = r(T(m - (k - 1), k), C(n, k)) + (n - 1)$. Then by (1), $N \geq g(m - (k - 1), n; k) + (n - 1)$. By direct computation we obtain

$$g(m - (k - 1), n; k) + n - 1$$
$$= h(m, n - (k - 1); k) + 2(t - 1) + ((m - 1)(k - t)/(k - 1)).$$

Since $1 \leq t \leq k - 1$, it follows that a 2-coloring of $C(N, k)$ which yields no blue $T(m, k)$ must have a red $C(n - (k - 1), k)$. In this case we may remove the vertices of a red $C(n - (k - 1), k)$ from the $C(N, k)$ to obtain a 2-colored $C(N_1, k)$ with $N_1 = N - n + (k - 1) = r(T(m - (k - 1), k), C(n, k)) + (k - 2)$. If this 2-coloring of $C(N_1, k)$ has no red $C(n, k)$, then it has a blue $T(m - (k - 1), k)$. If there is a blue $T(m - (k - 1), k)$ it is disjoint from the red $C(n - (k - 1), k)$ and, as in the proof of the upper bound in Theorem 1, either the one can be extended to a blue $T(m, k)$ or the other to a red $C(n - (k + 1) + 1, k)$ in $C(N, k)$. If a red $C(n - (k - 1) + 1, k)$ is obtained, its vertices can be removed from the $C(N, k)$ to yield a $C(N_2, k)$, where $N_2 = N - n + (k - 2)$, which has either a blue $T(m - (k - 1), k)$ or a red $C(n, k)$. This argument can be repeated until either a red $C(n, k)$ has been obtained or a red $C(n - 1, k)$ is removed from the $C(N, k)$ to obtain a $C(N_{k-1}, k)$, where $N_{k-1} = N - (n - 1) = r(T(m - (k - 1), k), C(n, k))$. Finally, if a $C(N_{k-1}, k)$ is obtained which does not have a red $C(n, k)$, then it does have a blue $T(m - (k - 1), k)$ which is disjoint in $C(N, k)$ from the red $C(n - 1, k)$. Either this $C(n - 1, k)$ is contained in a red $C(n, k)$ or the $T(m - (k - 1), k)$ is contained in a blue $T(m, k)$ in the $C(N, k)$, which completes the proof.

Note that if for some $T(m, k)$ and some n, $r(T(m, k), C(n, k)) = h(m, n; k)$, then by (3) $r(T(m, k), C(n + q(k - 1), k) = h(m, n + q(k - 1); k))$ for each

$q \geqslant 1$. On the other hand, if $r(T(m, k), C(n, k)) < h(m, n; k)$, then by (4) $r(T(m + p(k - 1), k), C(n, k)) < h(m + p(k - 1), n; k)$ for any member of $\mathfrak{T}(m + p(k - 1), k)$ which contains $T(m, k)$ as a subgraph.

3. Results for 3-graphs

For odd $n(t = 1)$ and any $T(m, 3)$, Corollary 1 yields $g(m, n; 3) = r(T(m, 3), C(n, 3)) = h(m, n; 3)$.

For even n, however, (1) gives only $\frac{1}{2}(m - 1)(n - 2) + 2 \leqslant r(T(m, 3), C(n, 3)) \leqslant \frac{1}{2}(m - 1)(n - 1) + 1$. For some tree-like 3-graphs, but not all, we can show that $r(T(m, 3), C(n, 3)) = g(m, n; 3)$ when n is even. The first result in this direction was established by Duke (1975).

LEMMA 3 (Duke (1975)). *We note that* $\mathfrak{T}(5, 3)$ *has only one member* $T(5, 3)$. *For* n *even and this unique member of* $\mathfrak{T}(5, 3)$ *we have*

$$r(T(5, 3), C(n, 3)) = g(5, n; 3) = 2n - 2.$$

By a *k-path* on m vertices, denoted by $P(m, k)$, we mean that unique member of $\mathfrak{T}(m, k)$ in which no vertex is contained in more than two edges and in which no edge is incident to more than two other edges. The only member of $\mathfrak{T}(2k - 1, k)$ is the k-path $P(2k - 1, k)$. A vertex contained in one of the two terminal edges of a k-path P and contained in no other edge of P will be called a terminal vertex of P.

LEMMA 4. *For each 3-path* $P(m, 3)$ *we have*

$$r(P(m, 3), C(4, 3)) = g(m, 4; 3) = m + 1.$$

PROOF. Clearly $r(P(3, 3), C(4, 3)) = g(3, 4; 3) = 4$ and by Lemma 3, $r(P(5, 3), C(4, 3)) = g(5, 4; 3) = 6$. The result for $m = 2p + 1$, $p \geqslant 3$, is established by induction on p.

Let Q be an integer, $Q \geqslant 3$, and assume that $r(P(m, 3), C(4, 3)) = m + 1$ for each $m = 2p + 1$ with $p < Q$. Consider a 2-coloring of the edges of $C(m + 1, 3)$, where $m = 2Q + 1$, which has no red $C(4, 3)$. There must be a blue $P(m - 2, 3)$. Let x be a terminal vertex of such a blue $P(m - 2, 3)$ and y a terminal vertex in the other terminal edge. Let the vertices of $C(m + 1, 3)$ not in this $P(m - 2, 3)$ be a, b and c. If there is a blue edge with one vertex in $\{x, y\}$ and two in $\{a, b, c\}$, we have a blue $P(m, 3)$. Assume, then, that each such edge is red. Since there is no red $C(4, 3)$, $\{a, b, c\}$ is blue. Similarly, one of the edges $\{x, y, b\}$ and $\{x, y, c\}$ must be blue. Hence there is a blue $P(m, 3)$ consisting of

https://doi.org/10.1017/S1446788700017183 Published online by Cambridge University Press

the blue $P(m-2, 3)$ with one terminal edge deleted, and $\{a, b, c\}$ and one of the edges $\{x, y, b\}$ and $\{x, y, c\}$ added.

THEOREM 2. *For each 3-path $P(m, 3)$ and each $n \geqslant 3$ we have*

(5) $$r(P(m, 3), C(n, 3)) = g(m, n; 3).$$

PROOF. For odd n the result follows from Corollary 1. For even n we proceed by induction on $n + m$. The result is immediate for $m = 3$ and was established above for $n = 4$. Let $M \geqslant 5$ and $N \geqslant 6$ be integers with M odd and N even. Suppose $r(P(m, 3), C(n, 3)) = g(m, n; 3)$ for each odd m and even n with $m + n < M + N$. Consider a 2-coloring of $C(g(M, N; 3), 3)$ with no red $C(N, 3)$. Since $g(M, N; 3) \geqslant g(M-2, N; 3)$, there is a blue $P(M-2, 3)$. If the vertices of such a 3-path are removed, we obtain a 2-colored complete 3-graph with $g(M, N; 3) - M + 2 = g(M, N-2; 3) + 1$ vertices. If this 3-graph has no blue $P(M, 3)$, it has a red $C(N-2, 3)$, say C_1. Let the vertices of one terminal edge of the $P(M-2, 3)$ be x, y and z, with x and y the terminal vertices, and let w be a terminal vertex of the other terminal edge. There is a vertex, say a, of C_1 such that $\{a, x, y\}$ is blue. Consider the original 2-colored $C(g(M, N; 3), 3)$ with a and the vertices of the blue $P(M-2, 3)$ removed. The resulting $C(g(M, N-2; 3), 3)$ has a red $C(N-2, 3)$, C_2. As before there is a vertex b of C_2 for which the edge $\{b, y, w\}$ is blue. Note that $a \neq b$. Then the blue $P(M-2, 3)$ with $\{x, y, z\}$ deleted and the edges $\{a, x, y\}$ and $\{b, y, w\}$ added yields a blue $P(M, 3)$.

COROLLARY 2. *For each m and n in the domain of $f(m, n; 3)$ we have*

(6) $$f(m, n; 3) = g(m, n; 3).$$

The class $\mathfrak{T}(7, 3)$ contains only two members, the 3-path $P(7, 3)$ and a 3-graph having three edges with a single vertex as their common intersection. We shall call the one k-graph in $\mathfrak{T}(m, k)$ which has a vertex which is contained in every one of its edge the *k-star* on m vertices and denote this k-star by $S(m, k)$. Hence $P(2k-1, k) = S(2k-1, k)$ and $\mathfrak{T}(3k-2, k) = \{P(3k-2, k), S(3k-2, k)\}$.

LEMMA 5. *For each integer n, $n \geqslant 3$, we have*

$$r(S(7, 3), C(n, 3)) = g(7, n; 3).$$

PROOF. Again the result follows for odd n from Corollary 1.

For even n we wish to show that $r(S(7, 3), C(n, 3)) = 3n - 4$. Consider a 2-coloring of $C(3N-4, 3)$ where N is even and $N \geqslant 4$, and assume this 2-coloring produces no blue $S(7, 3)$. Since $3N - 4 = g(7, N-1; 3) + 1$ and

$N - 1$ is odd, there is a red $C(N - 1, 3)$. Call this red 3-graph A and consider the 2-colored $C(2N - 3, 3)$ obtained by removing the vertices of A from the $C(3N - 4, 3)$. Since $g(5, N - 1; 3) = 2N - 3$, it follows from Theorem 2 that there is a blue $S(5, 3)$ or a red $C(N - 1, 3)$ disjoint from A in the $C(3N - 4, 3)$. If there is a blue $S(5, 3)$, then the vertex in both of its edges is either on a blue edge whose remaining vertices are in A producing a blue $S(7, 3)$, or, with A, it forms a red $C(N, 3)$.

Thus we may assume that there is a red $C(N - 1, 3)$ disjoint from A. Call it B. Let C denote the $C(N - 2, 3)$ obtained when the vertices of both A and B are removed. For any vertex not in A we may assume that there is a blue edge containing this vertex and having its other vertices in A. The same is true for B. Let the vertices of A be $\{a_1, a_2, \ldots, a_{N-1}\}$, the vertices of B be $\{b_1, b_2, \ldots, b_{N-1}\}$, and of C be $\{c_1, c_2, \ldots, c_{N-2}\}$. It follows that for each c_t there are blue edges $\{c_t, a_i, a_j\}$ and $\{c_t, b_k, b_l\}$. If $\{c_t, a_i, a_j\}$ and $\{c_s, a_k, a_l\}$, $t \neq s$, are both blue, then either $\{a_i, a_j\} = \{a_k, a_l\}$ or these pairs are disjoint, for otherwise there would be a blue $S(7, 3)$ with edges of the form $\{c_t, a_i, a_j\}$, $\{c_s, a_k, a_j\}$ and $\{a_j, b_i, b_j\}$. Similarly, if $\{c_t, b_i, b_j\}$ and $\{c_s, b_k, b_l\}$, $t \neq s$, are both blue, then the intersection of $\{b_i, b_j\}$ and $\{b_k, b_l\}$ is empty or has two vertices. Thus if we choose one pair $\{a_i, a_j\}$ for each c_t such that $\{c_t, a_i, a_j\}$ is blue, we obtain a list of pairs of vertices of A no two of which intersect in a single point. Since $N - 1$ is odd, there is a vertex a^* of A which is in none of the chosen pairs. Let b^* in B be selected in the same way.

Now for each c_t in C there is a blue edge $\{c_t, a_i, a_j\}$ which does not contain a^* and a blue edge $\{c_t, b_k, b_l\}$ not containing b^*. Since there is no blue $S(7, 3)$, any other blue edge containing c_t must meet one of these two blue edges in more than one point. It now follows that every edge is red in the $C(N, 3)$ induced by a^*, b^*, and the vertices of C, which completes the proof.

The next result follows immediately from this lemma and Theorem 2.

THEOREM 3. *For each member of* $\mathfrak{T}(7, 3)$ *we have*

$$r(T(7, 3), C(n, 3)) = g(7, n; 3).$$

This Theorem, together with (4), yields

THEOREM 4. *For n even and each $T(m, 3)$, $m \geq 7$, we have*

$$r(T(m, 3), C(n, 3)) \leq \frac{1}{2}(m - 1)(n - 1) - 1 < h(m, n; 3).$$

For each member of $\mathfrak{T}(9, 3)$, other than $S(9, 3)$, it can be shown that $r(T(9, 3), C(4, 3)) = g(9, 4; 3) = 10$. We have not been able to show, however,

https://doi.org/10.1017/S1446788700017183 Published online by Cambridge University Press

that $r(S(9, 3), C(4, 3)) = 10$, and do not know whether $r(S(m, 3), C(4, 3)) = g(m, 4; 3) = m + 1$ in general.

4. Results for k-graphs, $k \geqslant 4$

All of our results for $k = 3$ suggest the possibility that $r(T(m, 3), C(m, 3)) = g(m, n; 3)$ in every case. From Corollary 1 we have that $r(T(m, 4), C(n, 4)) = g(m, n; 4)$ whenever n is of the form $n = 3p + t$, $t \neq 1$.

LEMMA 6. $r(T(7, 4), C(5, 4)) = h(7, 5; 4) = 9$.

PROOF. From (1) we have $r(T(7, 4), C(5, 4)) \leqslant 9$. It suffices, therefore, to show that there is a 2-coloring of $C(8, 4)$ in which there is no blue $T(7, 4)$ and no red $C(5, 4)$. Such a coloring is obtained as follows: Let the vertices of $C(8, 4)$ be the integers i, $1 \leqslant i \leqslant 8$. Color the edges in the following table blue and all other edges red.

$\{1, 2, 3, 4\}$	$\{1, 2, 5, 6\}$	$\{1, 3, 5, 7\}$	$\{1, 4, 5, 8\}$
$\{5, 6, 7, 8\}$	$\{1, 2, 7, 8\}$	$\{1, 3, 6, 8\}$	$\{1, 4, 6, 7\}$
	$\{2, 3, 6, 7\}$	$\{2, 4, 6, 8\}$	$\{3, 4, 7, 8\}$
	$\{2, 3, 5, 8\}$	$\{2, 4, 5, 7\}$	$\{3, 4, 5, 6\}$

Each $C(5, 4)$ in this $C(8, 4)$ contains one of $\{1, 2, 3, 4\}$ and $\{5, 6, 7, 8\}$ or has two vertices in one of these and three in the other. In the latter case it may be checked that the $C(5, 4)$ must contain one of the remaining blue edges. Since each pair of these 4-sets meet in either 0 or 2 vertices, there is no blue $T(7, 4)$.

It can be shown that this 2-coloring is essentially the only one producing no blue $T(7, 4)$ and no red $C(5, 4)$.

Lemma 6 and (3) yield the following:

THEOREM 5. *For* $n = 3p + 2, p \geqslant 1$, *we have*

$$r(T(7, 4), C(n, 4)) = h(m, n; 4) = 2n - 1.$$

The next result provides our first example in which the Ramsey number is strictly between the bounds of (1).

THEOREM 6. $g(7, 6; 4) < r(T(7, 4), C(6, 4)) = 10 < h(7, 6; 4)$.

PROOF. Since, in general, $r(T(m, k), C(n + 1, k) \geqslant r(T(m, k), C(n, k))) + 1$, we need only show that $r(T(7, 4), C(6, 4)) \leqslant 10$. Consider a 2-coloring of the $C(10, 4)$ which has no blue $T(7, 4)$. By the previous result there is a red $C(5, 4)$. Let the vertices of $C(10, 4)$ be the integers i, $1 \leqslant i \leqslant 10$. Let A denotes a red $C(5, 4)$ and suppose that its vertices are 1, 2, 3, 4 and 5. Let B denote the $C(5, 4)$ with vertices 6, 7, 8, 9 and 10. Either there is a red $C(6, 4)$ or for each vertex b in B there is a blue edge having b as a vertex and its other vertices in A. In the latter case each edge of B is red. Then each vertex of A is also contained in a blue edge having no other vertices in A. We may assume that $\{1, 6, 7, 8\}$ is blue. There is a blue edge containing 6 and 1 and the other vertices of A, say 2 and 3. The blue edge containing 4 and three vertices of B would not contain 6 as this would form a $T(7, 4)$ with $\{1, 2, 3, 6\}$. Hence it would include both 7 and 8, and we may assume that its fourth vertex would be 9. The blue edge containing 7 and three vertices of A would include 1, 4, and one of 2 and 3. Hence the blue edge containing 5 and three vertices of B could not include 7. If it contained 6, a blue $T(7, 4)$ would be formed with $\{1, 2, 3, 6\}$, while if it contained 8, a blue $T(7, 4)$ would be formed with $\{1, 6, 7, 8\}$. But this leaves only vertices 9 and 10 in B, which is impossible.

While $r(T(7, 4), C(6, 4)) < h(7, 6; 4)$, the next result, together with Corollary 1 and Theorem 5, show that this is the only case in which $r(T(7, 4), C(n, 4)) < h(7, n; 4)$.

THEOREM 7. *For $n = 3p + 3, p \geqslant 2$, we have*

$$r(T(7, 4), C(n, 4)) = h(m, n; 4) = 2n - 1.$$

PROOF. In light of Lemma 1, we need only show that there is a 2-coloring of a $C(16, 4)$ which has no blue $T(7, 4)$ and no red $C(9, 4)$. Such a coloring results when the edges of each of two disjoint copies of $C(8, 4)$ are given the 2-coloring described in the proof of Lemma 6 and each of the edges which has vertices in both of these two complementary sets is colored red.

The 14 blue edges in the coloring of $C(8, 4)$ used in Lemma 6 and used twice for Theorem 7 are such that each set of three vertices of the $C(8, 4)$ occurs in precisely one of these 14 4-sets. This collection of 14 sets therefore constitutes a block design known as a Steiner system. In general, a Steiner system $S(t, k, v)$ is a system of subsets of size k, called blocks, from a v-set X such that each t-subset of X is contained in exactly one block. The 14 edges colored blue in the $C(8, 4)$ form a Steiner system $S(3, 4, 8)$. Our next result shows that this connection between the Ramsey numbers and Steiner systems is not limited to 4-graphs.

THEOREM 8. *For each integer* $k \geqslant 2$ *we have* $r(T(2k-1, k), C(k+1, k)) = h(2k-1, k+1; k) = 2k+1$ *when a Steiner system* $S(k-1, k, 2k)$ *exists. If no such Steiner system exists, then*

$$r(T(2k-1, k), C(k+1, k)) = g(2k-1, k+1; k) = 2k.$$

PROOF. Suppose that for some $k \geqslant 2$ there exists a Steiner system $S(k-1, k, 2k)$. By using the formulas given by de Vries (1977) for the reduced intersection numbers of a Steiner system, we find that in any such system with these parameters the complement of each block is again a block and that no two blocks intersect in a single point. It follows that if the blocks of a Steiner system $S(k-1, k, 2k)$ are used as the blue edges in a 2-coloring of the $C(2k, k)$, then no blue $T(2k-1, k)$ will result. Furthermore if any $(k+1)$-set is chosen from among the vertices of this $C(2k, k)$, the remaining $k-1$ vertices lie in exactly one blue edge. The complementary k-edge is also blue and is contained in the subgraph induced by the $(k+1)$-set chosen. It follows that in this case $r(T(2k-1, k), C(k+1, k)) = h(2k-1, k+1; k)$.

In the other direction, suppose $r(T(2k-1, k), C(k+1, k)) = 2k+1$. Then there must be a 2-coloring of a $C(2k, k)$ which has no blue $T(2k-1, k)$ and no red $C(k+1, k)$. Consider the blue edges of this coloring. If A is any blue k-edge and x is a vertex of A, then the complete subgraph induced by x and the vertices not in A contains a blue edge B. The edge B must be disjoint from A since it can not meet A in the single vertex x. It follows that the complement of each blue edge is again a blue edge. Now let D be any set of $k-1$ vertices of the $C(2k, k)$. The $C(k+1, k)$ induced by the vertices not in D contains a blue edge whose complement is a blue edge containing D. If D were contained in two blue edges, then one of them would meet the complement of the other in just one point. The blue edges thus form a Steiner system $S(k-1, k, 2k)$.

Thus far we have seen from Chvátal's theorem and Lemmas 3 and 6 that the value $r(T(2k-1, k), C(k+1, k))$ is $2k+1$ for $k = 2$ and $k = 4$, but that it is $2k$ for $k = 3$. This reflects the fact that a pair of disjoint edges in the complete graph on 4 vertices constitute a Steiner system $S(1, 2, 4)$ and blue edges in our coloring of $C(8, 4)$ are the blocks of an $S(3, 4, 8)$, while no Steiner triple system $S(2, 3, 6)$ exists. It is known that there is no $S(4, 5, 10)$, but that a Steiner system $S(5, 6, 12)$ associated with the Mathieu group M_{12} does exist. Therefore $r(T(9, 5), C(6, 5)) = 10$ and $r(T(11, 6), C(7, 6)) = 13$. No Steiner system $S(k-1, k, 2k)$ with $k > 6$ has been discovered, and it has been shown by Krammer and Mesner (1975) that in order for such a system to exist, $k+1$ must be a prime. This leads to the next theorem.

THEOREM 9. *If* $k+1$ *is not a prime, then*

$$r(T(2k-1, k), C(k+1, k)) = g(2k-1, k+1; k) = 2k.$$

https://doi.org/10.1017/S1446788700017183 Published online by Cambridge University Press

Mendlesohn and Hung (1972) have shown that there is no $S(9, 10, 20)$, so the smallest value of $k > 6$ for which a Steiner system $S(k-1, k, 2k)$ might exist is $k = 12$. Such a system would have over 200,000 blocks. When an $S(k-1, k, 2k)$ does exist, the reasoning used in the proofs of Theorems 5 and 7 can be used to yield the following result.

COROLLARY 3. *If there exists a Steiner system* $S(k-1, k, 2k)$, *then for all* $n \geqslant k^2 - 2k + 1$ *we have*

$$r(T(2k-1, k), C(n, k)) = h(2k-1, n; k) = 2n - 1.$$

This result shows that for certain k and m we may have $f(m, n; k) = h(m, n; k)$ for all sufficiently large n. Nonetheless, our final theorem shows that $f(m, k+1; k)$ is approximately equal to $g(m, k+1; k)$ for large m.

THEOREM 10. *For each* $k \geqslant 3$ *and every* k*-path* $P(m, k)$ *we have*

$$r(P(m, k), C(k+1, k)) \leqslant m + (k-2) = g(m, k+1; k) + k - 3.$$

PROOF. For $k = 3$ the result follows from Lemma 4. For fixed $k \geqslant 4$ the proof is a straightforward generalization of the inductive argument given for that lemma. The induction is grounded in the fact that for $k \geqslant 4$ we have

$$m + k - 2 \geqslant h(2k-1, k+1; k).$$

The inductive step consists of showing that a 2-coloring of a $C(m+k-2, k)$ with no red $C(k+1, k)$ must have a blue $P(m-k+1, k)$ and that each edge in the complement of such a path must also be blue. This leads to the existence of a blue edge having one vertex in each of the terminal edges of the path and its remaining $k-2$ vertices in the complement of the path. This complement must then simply be large enough so that this edge meets one of the (blue) edges of the complement in exactly one vertex.

5. Related questions

Given the relationship established here between $r(T(2k-1, k), C(k+1, k))$ and the existence of Steiner systems, there seems to be little hope at present of determining the exact values of $r(T(m, k), C(n, k))$ even for all k-paths or all k-stars. It may be possible, however, to obtain the values for trees containing sufficiently long "suspended" k-paths (see Burr (1979) and Burr and Erdös (1979)). Further progress could no doubt be made on various specific cases, but we have found even the precise determination of $r(S(9, 3), C(4, 3))$ to be surprisingly difficult.

There is no end to list of questions which could be asked about the Ramsey numbers of pairs of k-graphs. As one example, if $D(n, k)$ denotes the k-graph obtained by deleting a single edge from $C(n, k)$, then we can show that $r(P(m, 3), D(4, 3)) = r(S(m, 3), C(4, 3)) = m$. (While this result for paths is fairly simple, the argument for stars is rather involved, and we hope to include it elsewhere.) The values of $r(T(m, 3), D(4, 3))$ are unknown for other k-trees, although again it may be possible to obtain them for trees containing sufficiently long k-paths. As far as we know, $r(D(m, k), C(n, k))$ has not been studied.

Stahl (1975) generalized Chvátal's theorem from trees to forests. For the k-forest $F(kp, k)$ consisting entirely of p disjoint k-edges, we have $r(F(kp, k), C(n, k)) = n + k(p - 1)$. But, unlike the case for graphs (Chartrand, Gould and Polimeni (1979), Faudree, Schelp and Scheehan (1979)), $r(F(3p, 3), C(n, 3)) = r(F(3p, 3), D(n, 3))$, for n sufficiently large. (Here too, block designs enter the picture; these results will be explored elsewhere.) Some results concerning the "diagonal" Ramsey numbers for multiple copies of k-graphs were obtained by Burr, Erdös and Spencer (1975).

Frankl (1977) has shown that for each $k \geqslant 4$ and sufficiently large n, each k-graph with n vertices and more than $\binom{n-2}{k-2}$ edges must contain a $T(2k - 1, k)$. Bounds were found by Duke and Erdös (1977) for the number of k-edges which would assure the existence of an $S(m, k)$, and these bounds have since been improved by Frankl (private communication).

Finally, it follows from Theorem 1 that

$$\lim_{n\to\infty} n^{-1}f(m, n; k) = (m - 1)/(k - 1).$$

One might expect that $\lim_{m\to\infty} m^{-1}f(m, n; k) = (n - t)/(k - 1)$, where $n = p(k - 1) + t$, $1 \leqslant t \leqslant k - 1$, but our results establish this only for $k \leqslant 3$ and for $k \geqslant 4$ when $t = 1$ or $n = k + 1$.

References

L. W. Beineke and R. E. Pippert (1971), 'Properties and characterizations of k trees', *Mathematika* **18**, 141–151.

S. A. Burr (1979), 'Ramsey numbers involving graphs with long suspended', to appear.

S. A. Burr and P. Erdös (1970), 'Generalizations of a Ramsey-theoretic result of Chvatál', to appear.

S. A. Burr, P. Erdös and J. Spencer (1975), 'Ramsey theorems for multiple copies of graphs', *Trans. Amer. Math. Soc.* **209**, 87–89.

G. Chartrand, R. J. Gould and A. D. Polimeni (1979), 'On Ramsey numbers of forests versus nearly complete graphs', *Notices Amer. Math. Soc.* **26**, A-568.

V. Chvátal (1977), 'Tree-complete graph Ramsey numbers', *J. Graph Theory*, **1**, 93.

H. L. DeVries (1977), 'On property *B* and on Steiner systems', *Math. Z.* **153**, 155–159.

R. A. Duke (1975), 'Ramsey numbers of families of 2-complexes', *Proc. 6th S-E Conference on Combinatorics, Graph Theory and Computing* Florida Atlantic University, Boca Raton, Florida, 1975, pp. 265–277 (Congressus Numerantium, No. XIV, Utilitas Math., Winnipeg, Man.)

R. A. Duke and P. Erdös (1977), 'Systems of finite sets having a common intersection', *Proc. 8th S-E Conference on Combinatorics, Graph Theory and Computing* Florida Atlantic University, Boca Raton, Florida, 1977, pp. 247–252. (Congressus Numerantium, No. XIX, Utilities Math., Winnipeg, Man.)

R. A. Duke and F. Harary (1976), 'Generalized Ramsey theory VI: Ramsey numbers for small plexes', *J. Austral. Math. Soc. Ser. A* **22**, 400–410.

R. J. Faudree, R. H. Schelp and J. Sheehan (1979), 'Ramsey numbers for matchings', to appear.

P. Frankl (1977), 'On families of finite sets no two of which intersect in a singleton', *Bull. Austral. Math. Soc.* **17**, 125–135.

E. S. Kramer and D. M. Mesner (1975), 'Admissible parameters for Steiner systems $S(t, k, v)$ with a table for all $(v - t) < 498$, *Utilitas Math.* **7**, 211–222.

N. S. Mendelsohn and S. H. Y. Hung (1972), 'On the Steiner systems $S(3, 4, 14)$ and $S(4, 5, 15)$', *Utilitas Math.* 1, 5–96.

S. Stahl (1975), 'On the Ramsey number of $r(F, K_m)$, where F is a forest', *Canad. J. Math.* **27**, 585–589.

Computer Science Department
City College (CUNY)
New York, New York 10031
U.S.A.

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30322
U.S.A.