Isogonals of a Triangle.

By J. S. MACKAY, M.A., LL.D.

DEFINITION.—*If two angles have the same vertex and the same bisector, the sides of either angle are isogonal* to each other with respect to the other angle.*

Thus the isogonal of AP with respect to ∠BAC is the image of AP in the bisector of ∠BAC. It is indifferent whether the bisector of the interior ∠BAC be taken, or the bisector of the angle adjacent to it; the isogonal of AP remains the same.

It follows from the definition that

(1) The internal and the external bisectors of ∠BAC are their own isogonals.

(2) The line joining the orthocentre of a triangle to any vertex is isogonal to the line joining the circumcentre to that vertex.

(3) Any internal median of a triangle is isogonal to the corresponding symmedian.

(4) The tangents to the circumcircle of ABC at A, B, C are isogonal to the external medians.

[The external medians are the parallels to the sides of a triangle drawn through the opposite vertices.

The reason for the giving of this name will be found in the *Proceedings of the Edinburgh Mathematical Society*, Vol. I., p. 16 (1894).]

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* This terminology was proposed by Mr G. de Longchamps in his *Journal de Mathématiques Élémentaires*, 2nd series, Vol. V., p. 245 (1886).

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§ 1.

(*a*) *If P, Q be any two points taken on a pair of lines isogonal with respect to angle BAC, the distances of P from AB, AC are inversely proportional* to those of Q from AB, AC.*

FIGURE 29.

If the quadrilateral AQ_2QQ_1 be revolved through two right angles round the bisector of $\angle B$ as an axis, it will become homothetic to the quadrilateral AP_1PP_2; therefore

$$PP_1 : PP_2 = QQ_2 : QQ_1$$

(*a′*) *If P, Q be any two points and if the distances of P from AB, AC be inversely proportional to those of Q from AB, AC, then AP, AQ are isogonal with respect to $\angle BAC$.*

This may be proved indirectly.

(1) *The points P_1 Q_1 Q_2 P_2 are concyclic*†

Since P_1P_2 Q_1Q_2 are antiparallel with respect to $\angle BAC$; therefore P_1 P_2 Q_1 Q_2 are concyclic.

(2) *The centre of the circle $P_1Q_1Q_2P_2$ is the mid point of PQ.*

For the perpendicular to P_1Q_1 at its mid point goes through the centre of the circle; and this perpendicular bisects PQ.

So does the perpendicular to P_2Q_2 at its mid point.

(3) *P_1P_2 is perpendicular to AQ*
and Q_1Q_2 ,, ,, ,, AP.

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* Sir James Ivory in Leybourn's *Mathematical Repository*, new series, Vol. I., Part II., p. 19 (1806). The mode of proof is due to Professor Neuberg. See his excellent memoir on the Recent Geometry of the Triangle in Rouché and Comberousse's *Traité de Géométrie*, First Part, p. 438 (1891).

† This and the two following theorems are due to Steiner. See Gergonne's *Annales*, XIX., 37-64 (1828), or Steiner's *Gesammelte Werke*, I., 191-210 (1881). The proof given of (1) is Professor Neuberg's. See the reference in the preceding note.

For AP is a diameter of the circumcircle of AP_1P_2; therefore the isogonal of AP with respect to $\angle P_1AP_2$ is the perpendicular * from A to P_1P_2

(4) The circumcentre of either of the triangles AP_1P_2 AQ_1Q_2 and the orthocentre of the other are collinear with the point A.

(5) *Triangle PP_1P_2 is inversely similar † to QQ_2Q_1.*

This follows from the demonstration of § 1; or it may be thus proved:

$$\angle PP_1P_2 = \angle PAP_2 = \angle QAQ_1 = \angle QQ_2Q_1.$$

Similarly $\angle PP_2P_1 = \angle QQ_1Q_2.$

(6) *If PP_1 QQ_2 meet at D*
and PP_2 QQ_1 „ „ E,
then AD, AE are isogonals with respect to $\angle BAC$.

FIGURE 30.

Join P_1Q_2 P_2Q_1.

Since P_1Q_1 P_2Q_2 are concyclic,

therefore $\angle AQ_2P_1 = \angle AQ_1P_2$

therefore their complements are equal

that is $\angle P_1Q_2D = \angle P_2Q_1E.$

Similarly $\angle Q_2P_1D = \angle Q_1P_2E$;

therefore triangles P_1Q_2D, P_2Q_1E are similar;

therefore $P_1Q_2 : P_1D = P_2Q_1 : P_2E.$

Now triangles AP_1Q_2, AP_2Q_1 are similar;

therefore $AP_1 : P_1Q_2 = AP_2 : P_2Q_1.$

Hence $AP_1 : P_1D = AP_2 : P_2E$

and $\angle P_1AD = \angle P_2AE.$

The same result might be arrived at by revolving the quadrilateral AQ_2QQ_1 through two right angles round the bisector of $\angle BAC$.

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* This mode of proof is given by Professor Fuhrmann in his *Synthetische Beweise planimetrischer Sätze*, p. 93 (1890).

† See Ivory's paper already cited, p. 20.

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§ 2.

(a) *If ABC be a triangle, and if AP, AQ be isogonal with respect to A, then* *

$$BP\cdot BQ : CP\cdot CQ = AB^2 : AC^2$$

FIGURE 31.

About APQ circumscribe a circle, cutting AB, AC in F, E; join FE.

Because	$\angle BAP = \angle CAQ$
therefore	arc FP = arc EQ
therefore	FE is parallel to BC
therefore	$AB : BF = AC : CE$
therefore	$AB^2 : AB\cdot BF = AC^2 : AC\cdot CE$
therefore	$AB^2 : BP\cdot BQ = AC^2 : CP\cdot CQ$.

A second demonstration will be found in C. Adams's *Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks*, p. 1 (1846), and a third in Professor Fuhrmann's *Synthetische Beweise*, p. 94 (1890).

(a′) *If ABC be a triangle and BC be divided at P and Q so that*

$$BP\cdot BQ : CP\cdot CQ = AB^2 : AC^2$$

then † *AP, AQ are isogonals with respect to A.*

This may be proved indirectly.

(1) If AQ be the internal or the external median from A, then $BQ = CQ$, and the theorem becomes ‡

$$BP : CP = AB^2 : AC^2.$$

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* Pappus's *Mathematical Collection*, VI. 12. The same theorem differently stated is more than once proved in Book VII. among the lemmas which Pappus gives for Apollonius's treatise on *Determinate Section*. The proof in the text is taken from Pappus.

† In Pappus's *Mathematical Collection*, VI. 13, there is proved the theorem:

If $BP\cdot BQ : CP\cdot CQ > AB^2 : AC^2$

then $\angle BAP > \angle CAQ$.

‡ Adams (see the reference to him on this page) gives (1)-(4), (6), (8). His proof of (4) is different from that in the text.

(2) *If AQ be the internal or the external median from A and* $\angle$ *BAC be right, then AP is perpendicular to BC.*

FIGURES 32, 33.

Since $\angle ACB = \angle CAQ = \angle BAP$

therefore $\angle ACB + \angle CAP = \angle BAP + \angle CAP$

$= \text{a right angle.}$

(3) If AP and AQ coincide, then AP is either the internal or the external bisector of $\angle$ A, and the theorem becomes

$$BP^2 : CP^2 = AB^2 : AC^2$$

or

$$BP : CP = AB : AC$$

a known result, namely, Euclid VI. 3, or the cognate theorem.

(4) $$BP \cdot CP : BQ \cdot CQ = AP^2 : AQ^2.$$

This follows from the theorem of § 2 by considering APQ as the triangle and AB, AC as the isogonals.

(5) *If AP, AQ which are isogonal with respect to* $\angle$ *BAC meet the circumcircle of ABC in R, S, then* $AP \cdot AS = AQ \cdot AR$.

FIGURE 34.

For triangles ACR, AQB are similar

therefore $AQ \cdot AR = AB \cdot AC.$

Similarly $AP \cdot AS = AB \cdot AC.$

(6) RS is parallel to BC.

(7) The distances from the mid point of any side of a triangle to the points where two isogonals from the opposite vertex meet the circumcircle are equal.*

For the perpendicular which bisects BC bisects RS.

* Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 59 (1885).

(8) If APR becomes the diameter of the circumcircle ABC then AQ becomes perpendicular to BC, and

$$AQ \cdot AR = AB \cdot AC,$$

a theorem of Brahmegupta's.

See Chasles's *Aperçu*, 2nd ed., pp. 420-447.

(9) If AP, AQ coincide, then AP becomes either the internal or the external bisector of $\angle$ A.
Hence in the first case

$$\begin{aligned} AB \cdot AC &= AP \cdot AS \\ &= AP \cdot PS + AP^2 \\ &= BP \cdot PC + AP^2; \end{aligned}$$

and in the second case

$$\begin{aligned} AB \cdot AC &= AP \cdot AS \\ &= AP \cdot PS - AP^2 \\ &= BP \cdot PC - AP^2. \end{aligned}$$

(10) *In triangle ABC, AP, AQ are isogonals with respect to A ; through B draw BE parallel to AP meeting CA in E ;*
„ C „ CF „ „ AQ „ BA „ F;
*then EF is antiparallel * to BC with respect to A.*

FIGURE 35.

For $\angle ABE = \angle BAP = \angle CAQ = \angle ACF$;
therefore the points E, B, C, F are concyclic.

The same thing would happen if BE, CF were drawn parallel to AQ, AP.

(11) *In triangle ABC, AP, AQ are isogonal; from P and Q perpendiculars are drawn to BC ; these perpendiculars are intersected at D, E by a perpendicular to AB at B, and at D', E' by a perpendicular to AC at C. To prove* †

$$BD \cdot BE : CD' \cdot CE' = AB^4 : AC^4.$$

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* Mr Emile Vigarié.

† Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 224 (1885) says that this theorem was communicated to him by his friend Mr Th. Valiech.

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FIGURE 36.

Draw AX perpendicular to BC.

The similar triangles BDP, BEQ, ABX give

$$BD : BP = AB : AX$$
$$BE : BQ = AB : AX$$

therefore $$\frac{BD \cdot BE}{BP \cdot BQ} = \frac{AB^2}{AX^2}$$

Similarly $$\frac{CD' \cdot CE'}{CP \cdot CQ} = \frac{AC^2}{AX^2}$$

therefore $$\frac{BD \cdot BE}{CD' \cdot CE'} \cdot \frac{CP \cdot CQ}{BP \cdot BQ} = \frac{AB^2}{AC^2}$$

therefore $$\frac{BD \cdot BE}{CD' \cdot CE'} \cdot \frac{AC^2}{AB^2} = \frac{AB^2}{AC^2}$$

(12) *If in* (11) *AQ be the median* from A,*

then $$BD : CD' = AB^3 : AC^3.$$

FIGURE 36.

For $$BE : BQ = AB : AX$$
and $$CE' : CQ = AC : AX;$$
therefore $$BE : CE' = AB : AC,$$
whence the result follows.

§ 3.

If three straight lines drawn through the vertices of a triangle are concurrent, their isogonals with respect to the angles of the triangle are also concurrent.‡

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* Mr Emile Vigarié in the *Journal de Mathématiques Élémentaires*, 2nd series, IV. 225 (1885).

† Steiner in Gergonne's *Annales*, xix. 37-64 (1828), or Steiner's *Gesammelte Werke*, I. 193 (1831). Ivory in his paper previously cited proves the theorem :

If the isogonals BO, BO′ meet the bisector of $\angle$ A at O, O′,

then $$BO : CO = BO' : CO';$$

and he adds as a corollary that CO, CO′ are isogonals with respect to C.

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FIGURE 37.

Let BO, BO′ be isogonals with respect to B

and CO, CO′ ,, ,, ,, ,, C;

then AO, AO′ are ,, ,, ,, A.

Denote the distances of O from the sides by p_1 p_2 p_3 and those of O′ by q_1 q_2 q_3

Then $p_1 q_1 = p_2 q_2$ and $p_1 q_1 = p_3 q_3$

therefore $p_2 q_2 = p_3 q_3$

therefore AO, AO′ are isogonals with respect to A.

Another demonstration will be found in C. Adams's *Eigenschaften des...Dreiecks*, pp. 7-8 (1846).

Points such as O, O′ determined by the intersection of pairs of isogonal lines will be called *isogonal points*, or simply *isogonals*, with respect to the triangle ABC.

They are sometimes* called *isogonally conjugate points*, or *isogonal conjugates*, but more frequently on the continent of Europe *inverse points* with respect to the triangle ABC.

The designation, *inverse points*, was suggested about the same time in Scotland and in France. See a paper read before the Royal Society of Edinburgh on 20th March 1865, by the Rev. Hugh Martin, and printed in their *Transactions*, xxiv. 37-52: and an article by Mr J. J. A. Mathieu in the *Nouvelles Annales*, 2nd series, IV. 393–407, 481-493, 529-537 (1865).

Perhaps the adoption of the nomenclature proposed by Mr G. de Longchamps in the *Journal de Mathématiques Élémentaires*, 2nd series, V. 109 (1886) would be advantageous.

(1) $\angle BOC + \angle BO'C = 180° + A.$

FIGURE 37.

For $\angle \mathrm{BOC} = \mathrm{A} + \mathrm{ABO} + \mathrm{ACO},$

$= \mathrm{A} + \mathrm{CBO'} + \mathrm{BCO'},$

and $\angle \mathrm{BO'C} = \mathrm{A} + \mathrm{ABO'} + \mathrm{ACO'};$

therefore $\angle \mathrm{BOC} + \angle \mathrm{BO'C} = 2\mathrm{A} + \mathrm{B} + \mathrm{C},$

$= 180° + \mathrm{A}.$

* Professor J. Neuberg's *Mémoire sur le Tétraèdre*, p. 10 (1884).

(2) In triangle ABC, AP_1 BP_2 CP_3 are concurrent at O, and their isogonals AQ_1 BQ_2 CQ_3 are concurrent at O′.

FIGURE 36.

Suppose BP_2 BQ_2 to form one straight line
and CP_3 CQ_3 ,, ,, ,, ,, ,, ;
then the points O O′ coincide.*

There are four cases.

(*a*) If BP_2 CP_3 bisect the interior angles B, C, then AP_1 bisects the interior angle A.

(*b*) If BP_2 CP_3 bisect the exterior angles B, C, then AP_1 bisects the interior angle A.

(*c*) If BP_2 bisects the interior angle B
and CP_3 ,, ,, exterior ,, C,
then AP_1 ,, ,, exterior ,, A.

(*d*) If BP_2 bisects the exterior angle B
and CP_3 ,, ,, interior ,, C,
then AP_1 ,, ,, exterior ,, A.

Hence the six bisectors of the angles of a triangle meet three by three in four points.

FIGURE 36.

(3) By considering AP_1Q_1 as the triangle, and AB, AC as the isogonals

$$BP_1 \cdot CP_1 : BQ_1 \cdot CQ_1 = AP_1^2 : AQ_1^2.$$

Similarly $$CP_2 \cdot AP_2 : CQ_2 \cdot AQ_2 = BP_2^2 : BQ_2^2,$$

and $$AP_3 \cdot BP_3 : AQ_3 \cdot BQ_3 = CP_3^2 : CQ_3^2;$$

therefore $$\frac{BP_1 \cdot CP_2 \cdot AP_3 \cdot CP_1 \cdot AP_2 . BP_3}{BQ_1 \cdot CQ_2 \cdot AQ_3 \cdot CQ_1 \cdot AQ_2\, BQ_3} = \frac{AP_1^2 \cdot BP_2^2 \cdot CP_3^2}{AQ_1^2 \cdot BQ_2^2 \cdot CQ_3^2}.$$

Now $$BP_1 \cdot CP_2 \cdot AP_3 = CP_1 \cdot AP_2 \cdot BP_3$$

and $$BQ_1 \cdot CQ_2 \cdot AQ_3 = CQ_1 \cdot AQ_2 \cdot BQ_3;$$

therefore $$\frac{AP_1 \cdot BP_2 \cdot CP_3}{AQ_1 \cdot BQ_2 \cdot CQ_3} = \frac{BP_1 \cdot CP_2 \cdot AP_3}{BQ_1 \cdot CQ_2 \cdot AQ_3} = \frac{CP_1 \cdot AP_2 \cdot BP_3}{CQ_1 \cdot AQ_2 \cdot BQ_3}.$$

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* C. Adams's *Eigenschaften des...Dreiecks*, p. 8 (1846). Adams gives also (3).

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§ 4.

Positions of two isogonal points with reference to a triangle.

(1) Any point on a side has for isogonal point the opposite vertex.

(2) A vertex has for isogonal point any point on the opposite side.

(3) A point inside the triangle has its isogonal point also inside the triangle.

(4) If a point be outside the triangle and situated in the angle vertically opposite to ∠ BAC, for example, its isogonal point will be outside the triangle and situated in that segment of the circumcircle (remote from A) cut off by BC.

(5) If a point be outside the circumcircle and situated within the angle BAC, for example, its isogonal point will be outside the circumcircle and situated within the same angle.

(6) If a point be on the circumference of the circumcircle, its isogonal point will be at infinity.

The truth of these statements,* which are not quite obvious, may be ascertained by the construction of a few figures. Of the last statement the following proof may be given :—

FIGURE 39.

If AD, BE, CF, be three parallel lines drawn through the vertices of a triangle ABC, their three isogonals will be concurrent at a point on the circumference of the circumcircle.†

Because AD, BE, CF are parallel,

therefore arc AE = arc BD, arc BC = arc EF.

Make arc CP equal to arc BD; join AP, BP, CP.

* They are all given by Mr J. J. A. Mathieu in *Nouvelles Annales*, 2nd series, IV. 403 (1865).

† Professor Eugenio Beltrami in *Memorie del l'Accademia delle Scienze del Istituto di Bologna*, 2nd series, II., 383 (1863).

Since arc CP = arc BD,
therefore $\angle$ CAP = $\angle$ BAD,
and AP is isogonal to AD.

Since arc CP = arc AE,
therefore $\angle$ CBP = $\angle$ ABE,
and BP is isogonal to BE.

Since arc BC = arc EF, arc CP = arc AE,
therefore arc BP = arc AF;
therefore $\angle$ BCP = $\angle$ ACF,
and CP is isogonal to CF.

Hence, if P be a point on the circumcircle of ABC, the point isogonal to it is the point of concurrency of AD, BE, CF.

(1) AD is perpendicular* to the Wallace line P (ABC).

This follows from § 1, (3).

§ 5.

*If three angular transversals cut the opposite sides in three collinear points, their isogonals will also cut the opposite sides in three collinear points.**

FIGURE 40.

Let AD, AD′; BE, BE′; CF, CF′ be pairs of isogonals; then if D, E, F, be collinear, so will D′, E′, F′.

For
$$\frac{BD \cdot BD'}{CD \cdot CD'} = \frac{c^2}{b^2},$$
$$\frac{CE \cdot CE'}{AE \cdot AE'} = \frac{a^2}{c^2},$$
$$\frac{AF \cdot AF'}{BF \cdot BF'} = \frac{b^2}{a^2};$$

therefore
$$\frac{BD \cdot CE \cdot AF}{CD \cdot AE \cdot BF} \cdot \frac{BD' \cdot CE' \cdot AF'}{CD' \cdot AE' \cdot BF'} = 1.$$

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* Professor J. Neuberg in Rouché and Comberousse's *Traité de Géometrié*, First Part, p. 439 (1891).

† Townsend's *Modern Geometry*, I. 181 (1863).

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Now $$\frac{BD \cdot CE \cdot AF}{CD \cdot AE \cdot BF} = 1;$$

therefore $$\frac{BD' \cdot CE' \cdot AF'}{CD' \cdot AE' \cdot BF'} = 1;$$

therefore D′, E′, F′ are collinear.

§ 6.

If O be any point in the plane of triangle ABC, and AO BO CO meet the circumcircle in A_1 B_1 C_1 and D E F be the projections of O on BC CA AB the triangles $A_1B_1C_1$ DEF are directly similar, and the point O of triangle DEF corresponds to that point of $A_1B_1C_1$ which is isogonal to O.*

FIGURE 41.

For the points O F B D are concyclic;

therefore $\angle FDO = \angle FBO$
$= \angle B_1A_1O.$

Similarly $\angle EDO = \angle C_1A_1O.$

The demonstration may be easily seen to apply to the more general case where A_1 B_1 C_1 are taken inverse to O with any other constant of inversion.*

(1) If O be the orthocentre of ABC, it must be the incentre or an excentre of DEF, and therefore the incentre or an excentre of $A_1B_1C_1$.

(2) If O be the circumcentre of ABC, it must be the orthocentre of DEF, and therefore the circumcentre of $A_1B_1C_1$.

(3) If O be the incentre of ABC, it must be the circumcentre of DEF, and therefore the orthocentre of $A_1B_1C_1$.

(4) If O be an excentre of ABC, it must be the circumcentre of DEF, and therefore the orthocentre of $A_1B_1C_1$.

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* Mr E. M. Langley and Professor Neuberg. (1)—(4) are Mr Langley's. See the *Seventeenth General Report of the Association for the Improvement of Geometrical Teaching*, p. 45 (1891.)

§ 7

*If two points be isogonal with respect to a triangle their six projections on the sides of the triangle are concyclic.**

FIGURE 42.

Let O, O′ be isogonal with respect to ABC, and let D, E, F, D′, E′, F′ be their projections on the sides BC, CA, AB.

Then EF is antiparallel to E′F′ with respect to A ;

therefore E,E′ F,F′, are concyclic.

Similarly F,F′ D,D′ ,, ,, ,

and D,D′ E,E′ ,, ,, ;

therefore the six points are concyclic.

§ 8

If O O′ be isogonal points with respect to ABC, and D E F D′E′F′ be their respective projections on BC CA AB, then

AO BO CO are perpendicular to the sides of D′E′F′
AO BO′ CO′ ,, ,, ,, D E F.

FIGURE 42.

This has been established in § 1, (3).

The application of the preceding properties of isogonals to the particular case of medians and symmedians will be taken up in a succeeding paper.

* Steiner in Gergonne's *Annales*, xix. 37-64 (1828).
In the same article will be found also the property of § 8.

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