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ON THE LAW OF THE ITERATED LOGARITHM FOR INFINITE DIMENSIONAL ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT Let $\{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be independent Ornstein-Uhlenbeck processes and $X(t,n) = \sum_{i=1}^{n} X_i(t)$ In this paper the law of iterated logarithm for $X(t,n)$ is considered The results obtained improve those of Csorgő and Lin(1988) and Schmuland(1987)

A real valued stationary Gaussian process $\{X(t), -\infty < t < \infty\}$ will be called an *Ornstein-Uhlenbeck process* with coefficients γ and λ ($\gamma \geq 0, \lambda > 0$) if $\mathrm{E}X(t) = 0$ and $\mathrm{E}X(s)X(t) = (\gamma/\lambda)\exp(-\lambda|t-s|)$. Let

$$\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$$

be a sequence of independent Ornstein-Uhlenbeck processes with coefficients γ_k and λ_k. The process $Y(\cdot)$ was first studied by Dawson(1972) as the stationary solution of the infinite array of stochastic differential equations:

$$dX_k(t) = -\lambda_k X_k(t)\,dt + (2\gamma_k)^{1/2} dW_k(t), \quad k = 1, 2, \ldots,$$

where $\{W_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ are independent Wiener processes. The properties of $Y(\cdot)$ have been extensively studied in the literature. Since $\mathrm{E}X_k^2(t) = \gamma_k/\lambda_k$, it is clear that for every fixed t, $Y(t)$ is almost surely in ℓ^2 if and only if $\sum_{k=1}^{\infty} \gamma_k/\lambda_k < \infty$. The continuity properties of $Y(\cdot)$ were investigated by Dawson(1972), Schmuland(1987), Iscoe and McDonald(1986), Fernique(1989), Csáki, Csörgő and Shao(1991). Csörgő and Lin(1988) studied $Y(\cdot)$ in terms of the path behaviour of the two-time parameter stochastic process $\{X(t,n), -\infty < t < \infty, n = 1, 2, \ldots\}$, where $X(t,n) = \sum_{k=1}^{n} X_k(t)$, $X(t,0) = 0$ for all $t \in R$ and established P. Lévy type moduli of continuity, large increment rates for the latter process and the following law of the iterated logarithm:

THEOREM A. *Let* $\lambda_N^* = \max_{1\leq i\leq N} \lambda_i$, *and* $\sigma_N = \sigma(N) = \sum_{i=1}^{N} \gamma_i/\lambda_i$. *Assume that*

(1) $$(\log \lambda_N *)/\log\log N \longrightarrow 0, \text{ as } N \longrightarrow \infty,$$

and that the non-decreasing sequence $\{T_N\}$ *satisfies*

(2) $$\log T_N/\log\log N \longrightarrow 0, \text{ as } N \longrightarrow \infty.$$

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Suppose also that for every $\epsilon > 0$ *there exist* $1 < \theta_1 < \theta_2$ *such that*

(3) $$\limsup_{k\to\infty} \sigma(\theta_1^{k+1})/\sigma(\theta_1^k) \le 1+\epsilon$$

and

(4) $$\limsup_{k\to\infty} \sigma(\theta_2^{k})/\sigma(\theta_2^{k+1}) \le \epsilon.$$

Then, with $\beta_N^* = \Big(2(\sum_{\imath=1}^N \gamma_\imath/\lambda_\imath)\log\log N\Big)^{1/2}$, *we have*

$$\limsup_{N\to\infty} |X(T_N,N)|/\beta_N^* = \limsup_{N\to\infty} \max_{1\le n\le N} \sup_{|t|\le T_N} |X(t,n)|/\beta_N^* = 1 \ a.s.$$

Schmuland(1987), using Dirichlet form-techniques, proved that if $\gamma_k/\lambda_k \equiv 1$ and $\sum_{\imath=1}^n \gamma_\imath/(2n\log\log n) \longrightarrow 0$ as $n\to\infty$, then

(5) $$P\Big\{\limsup_{n\to\infty} X(t,n)/(2n\log\log n)^{1/2} = 1 \text{ for all } t\in R\Big\} = 1.$$

It is not difficult to see that (3) and (4), in fact, imply that there exists positive constants α_1, α_2, c_1 and c_2 such that

(6) $$\sigma_n/n^{\alpha_1} \le c_1\sigma_m/m^{\alpha_1}$$

and

(7) $$\sigma_n/n^{\alpha_2} \ge c_2\sigma_m/m^{\alpha_2}$$

for each $1\le n\le m$.

Unfortunately, conditions (1) and (2) in Theorem A are too restrictive to be satisfied even for $\lambda_k = k^\alpha$, or $\lambda_k = \log^\alpha(1+k)$ $(\alpha>0)$, or $T_N = \log N$. The aim of this note is to relax the conditions of Theorem A and that of Schmuland(1987) as well.

Let $\{T_N, n\ge 1\}$ be a non-decreasing sequence of positive numbers. Put

$$\sigma_N = \sigma(N) = \sum_{\imath=1}^N \gamma_\imath/\lambda_\imath,\ \ \Gamma_N = \sum_{\imath=1}^N \gamma_\imath,$$

$$\beta_N = \Big(2\sigma_N\big(\log(\Gamma_N T_N/\sigma_N) + \log\log\sigma_N\big)\Big)^{1/2},$$

where and in the sequel, $\log x = \ln\big(\max(x,e)\big)$, ln is the natural logarithm.
For $0<\epsilon<1$, define $\theta_n(\epsilon)$ as the solution of the equation

(8) $$\sum_{\imath=1}^n \frac{\gamma_\imath}{\lambda_\imath} e^{-2\lambda_\imath\theta_n(\epsilon)} = \epsilon\sigma_n.$$

THEOREM 1. *Assume that*

(9) $$T_N\Gamma_N/\sigma_N + \sigma_N \longrightarrow \infty,\ \ as\ \ N\to\infty.$$

Then, we have

(10) $$\limsup_{N\to\infty} \max_{1\le n\le N} \sup_{|t|\le T_N} |X(t,n)|/\beta_N \le 1 \ a.s.$$

THEOREM 2. *Assume that* (9) *is satisfied and that there exists a positive constant C such that*

$$\sigma_N \leq C\sigma_{N-1} \text{ for every } N \geq 1, \tag{11}$$

$$\log \frac{T_N \Gamma_N}{\sigma_N} \leq (1 + C\epsilon) \log \frac{T_N}{\theta_N(\epsilon)} + C\epsilon \log\log \sigma_N, \tag{12}$$

for every $0 < \epsilon < 1$ *as* $N \longrightarrow \infty$. *Then, we have*

$$\limsup_{N\longrightarrow\infty} \sup_{0\leq t\leq T_N} |X(t,N)|/\beta_N = 1 \text{ a.s.} \tag{13}$$

$$\limsup_{N\longrightarrow\infty} \max_{1\leq n\leq N} \sup_{|t|\leq T_N} |X(t,n)|/\beta_N = 1 \text{ a.s.} \tag{14}$$

If, in addition, we also have

$$\log\log \sigma_N = o\Big(\log \frac{T_N\Gamma_N}{\sigma_N}\Big), \text{ as } N \longrightarrow \infty. \tag{15}$$

Then

$$\lim_{N\longrightarrow\infty} \sup_{0\leq t\leq T_N} |X(t,N)|/\beta_N = 1 \text{ a.s.} \tag{16}$$

$$\lim_{N\longrightarrow\infty} \max_{1\leq n\leq N} \sup_{|t|\leq T_N} |X(t,n)|/\beta_N = 1 \text{ a.s.} \tag{17}$$

THEOREM 3. *Assume that* (11) *is satisfied. Moreover, suppose that*

$$\log(\Gamma_N/\sigma_N) = o(\log\log \sigma_N) \text{ as } N \longrightarrow \infty, \tag{18}$$

and

$$\sigma_N \longrightarrow \infty \text{ as } N \longrightarrow \infty. \tag{19}$$

Then, we have

$$P\Big\{\limsup_{N\longrightarrow\infty} X(t,N)/(2\sigma_N \log\log \sigma_N)^{1/2} = 1 \text{ for all } t \in R\Big\} = 1. \tag{20}$$

Before stating our corollaries, we introduce the following notations:

$$\lambda_N^* = \max_{\iota\leq N} \lambda_\iota,\ m_n(1,\epsilon) = \max\Big\{\ell : \sum_{\iota=\ell}^{n} \frac{\gamma_\iota}{\lambda_\iota} \geq (1-\epsilon)\sigma_n\Big\},$$

$$m_n(2,\epsilon) = \min\Big\{\ell : \sum_{\iota=1}^{\ell} \frac{\gamma_\iota}{\lambda_\iota} \geq (1-\epsilon)\sigma_n\Big\},\ \lambda'_N(\epsilon) = \max\Big\{\min_{m_n(1,\epsilon)\leq\iota\leq N} \lambda_\iota,\ \min_{1\leq\iota\leq m_n(2,\epsilon)} \lambda_\iota\Big\}.$$

A sequence $\{a_n\}$ is called *quasi-increasing* if there exists a positive constant C such that

$$a_k \leq Ca_n \text{ for each } k \leq n.$$

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COROLLARY 1. *Assume that* (9) *and* (11) *are satisfied and that there exists a positive constant C such that*

$$\lambda_N^* \leq C\big(\lambda_N^{'}(\epsilon)\big)^{1+C\epsilon} \log^{C\epsilon} \sigma_N$$

for every $0 < \epsilon < 1$. *Then,* (13) *and* (14) *hold. If we also have* (15), *then* (16) *and* (17) *are true.*

COROLLARY 2. *Assume that* (11) *is satisfied and that there is positive constants* α *and C such that* $\sigma(n)/n^{\alpha}$ *is quasi-increasing and* $\lambda_{2k}^* \leq C \min_{k \leq \iota \leq 2k} \lambda_\iota$ *for each* $k \geq 1$. *Then,* (13) *and* (14) *hold. If we also have* (15), *then* (16) *and* (17) *are true.*

COROLLARY 3. *Assume that* (11) *is satisfied and that*

$$\log \frac{T_N \Gamma_N}{\sigma_N} = o(\log\log \sigma_N) \text{ as } N \longrightarrow \infty.$$

Then, we have (13), (14) *and*

$$\limsup_{N \longrightarrow \infty} X(T_N, N)/(2\sigma_N \log\log \sigma_N)^{1/2} = 1 \text{ a.s.} \tag{21}$$

COROLLARY 4. *Assume that* (9) *and* (11) *are satisfied and that* $\lambda_n \sigma_n^{1-\alpha}$ *and* $\sigma_n^{1/\alpha}/\lambda_n$ *are quasi-increasing for some* $0 < \alpha < 1$. *Then,* (13) *and* (14) *hold. If we also have* (15), *then* (16) *and* (17) *are true.*

The proof of theorems is based on the following lemmas.

LEMMA 1 (FERNIQUE(1964)). *Let* $G(t)$ *be a Gaussian process on* $[0, 1]$ *with*

$$E\big(G(t) - G(s)\big)^2 \leq \Lambda^2(|t - s|)$$

where Λ *is continuous, non-decreasing and satisfies* $\int_1^\infty \Lambda(e^{-y^2})\,dy < \infty$ *and also* $EG^2(t) \leq \Gamma^2$. *Then, for every* $x > 0$

$$P\Big\{\sup_{0 \leq t \leq 1} |G(t)| > x\Big(\Gamma + 4\int_1^\infty \Lambda(e^{-y^2})\,dy\Big)\Big\} \leq d\int_x^\infty e^{-y^2/2}\,dy,$$

where d is an absolute constant.

LEMMA 2. *For every* $0 < \epsilon < 1$, *there exists a constant* $C = C(\epsilon)$ *such that*

$$P\{\sup_{|t| \leq T} |X(t, n)| \geq x\sigma_n^{1/2}\} \leq C\Big(1 + \frac{T\Gamma_n}{\sigma_n}\Big) \exp\Big(-\frac{(1-\epsilon)}{2}x^2\Big). \tag{22}$$

PROOF. Note that

$$\mathrm{EX}^2(t, n) = \sigma_n \tag{23}$$

and

$$E\big(X(t, n) - X(s, n)\big)^2 = 2\sum_{\iota=1}^{n} \frac{\gamma_\iota}{\lambda_\iota}(1 - e^{-\lambda_\iota|t-s|}) \leq 2\Gamma_n|t - s| \tag{24}$$

for every t and s. Put

$$\delta = \frac{\epsilon}{32(1-\epsilon)^{1/2}}, \quad \theta = \frac{\delta^2 \sigma_n}{\Gamma_n}.$$

Then

$$(25) \qquad P\Big\{\sup_{|t|\le T} |X(t,n)| \ge x\sigma_n^{1/2}\Big\} \le 2\Big(\Big[\frac{T}{\theta}\Big]+1\Big) P\Big\{\sup_{0\le t\le\theta} |X(t,n)| \ge x\sigma_n^{1/2}\Big\}$$
$$= 2\Big(\Big[\frac{T}{\theta}\Big]+1\Big) P\Big\{\sup_{0\le t\le 1} |X(t\theta,n)| \ge x\sigma_n^{1/2}\Big\}.$$

By (23) and (24), we have

$$(26) \qquad \int_1^\infty (2\theta\Gamma_n e^{-y^2})^{1/2}\, dy \le 4\delta\sigma_n^{1/2}.$$

Using the Fernique lemma, we find

$$(27) \qquad P\Big\{\sup_{0\le t\le 1} |X(t\theta,n)| \ge x\sigma_n^{1/2}\Big\}$$
$$\le P\Big\{\sup_{0\le t\le 1} |X(t\theta,n)| \ge \frac{x}{1+16\delta}\Big(\sigma_n^{1/2} + 4\int_1^\infty (2\theta\Gamma_n e^{-y^2})^{1/2}\, dy\Big)\Big\}$$
$$\le d\int_{\frac{x}{1+16\delta}}^\infty e^{-t^2/2}\, dt$$
$$\le d\exp\Big(-\frac{x^2}{2(1+16\delta)^2}\Big)$$
$$\le d\exp\Big(-\frac{(1-\epsilon)x^2}{2}\Big).$$

Now (22) follows from (27) and (25).

LEMMA 3. *Let $0 < \epsilon < \frac{1}{2}$, $\theta_n(\epsilon)$ be the solution of the equation* (8).*Then, there is a positive $C(\epsilon)$ such that*

$$(28) \qquad P\Big\{\sup_{0\le t\le T} |X(t,n)| \le x\sigma_n^{1/2}\Big\} \le \Big(1 - C(\epsilon)\exp\Big(-\frac{x^2}{2(1-2\epsilon)}\Big)\Big)^{T/\theta_n(\epsilon)}$$

for each $x > 0$.

PROOF. Let $\{W_i(t), 0 \le t < \infty\}_{i=1}^\infty$ be independent standard Wiener processes. Noting that

$$\{X(t,n), 0\le t\le T\} \text{ and } \Big\{\sum_{i=1}^n \Big(\frac{\gamma_i}{\lambda_i}\Big)^{1/2} \frac{W_i(e^{2\lambda_i t})}{e^{\lambda_i t}}, 0 \le t \le T\Big\}$$

have the same distribution, we have

$$(29) \qquad P\Big\{\sup_{0\le t\le T} |X(t,n)| \le x\sigma_n^{1/2}\Big\} \le P\Big\{\sup_{0\le j\le[\frac{T}{\theta_n}]} |X(j\theta_n,n)| \le x\sigma_n^{1/2}\Big\}$$
$$= P\Big\{\max_{0\le j\le[\frac{T}{\theta_n}]} \Big|\sum_{i=1}^n \Big(\frac{\gamma_i}{\lambda_i}\Big)^{1/2} \frac{W_i(e^{2j\lambda_i\theta_n})}{e^{j\lambda_i\theta_n}}\Big| \le x\sigma_n^{1/2}\Big\},$$

where $\theta_n = \theta_n(\epsilon)$. Set

$$U_j = \sum_{i=1}^{n}\Big(\frac{\gamma_i}{\lambda_i}\Big)^{1/2}\frac{W_i(e^{2j\lambda_i\theta_n})}{e^{j\lambda_i\theta_n}},\quad V_j = \sum_{i=1}^{n}\Big(\frac{\gamma_i}{\lambda_i}\Big)^{1/2}\frac{W_i(e^{2(j-1)\lambda_i\theta_n})}{e^{j\lambda_i\theta_n}}.$$

It is easy to see that

$$U_j - V_j \sim N\Big(0, \sum_{i=1}^{n}\frac{\gamma_i}{\lambda_i}(1-e^{-2\lambda_i\theta_n})\Big).$$

Whence

$$U_j - V_j \sim N\big(0, (1-\epsilon)\sigma_n\big) \tag{30}$$

by the definition of θ_n.Thus, by (30), we obtain

(31)

$$\begin{aligned}
&P\Big\{\max_{0\le j\le[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}\Big\}\\
&= P\Big\{\max_{0\le j<[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}, |U_{[\frac{T}{\theta_n}]} - V_{[\frac{T}{\theta_n}]} + V_{[\frac{T}{\theta_n}]}| \le x\sigma_n^{1/2}\Big\}\\
&= \int_{-\infty}^{\infty} P\{|U_{[\frac{T}{\theta_n}]} - V_{[\frac{T}{\theta_n}]} + y| \le x\sigma_n^{1/2}\}\, dP\Big\{V_{[\frac{T}{\theta_n}]} < y, \max_{0\le j<[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}\Big\}\\
&= \int_{-\infty}^{\infty}\Big(\Phi\Big(\frac{x\sigma_n^{1/2}-y}{((1-\epsilon)\sigma_n)^{1/2}}\Big) - \Phi\Big(\frac{-x\sigma_n^{1/2}-y}{((1-\epsilon)\sigma_n)^{1/2}}\Big)\Big)\, dP\\
&\qquad \Big\{V_{[\frac{T}{\theta_n}]} < y, \max_{0\le j<[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}\Big\}\\
&\le \int_{-\infty}^{\infty}\Big(\Phi\Big(\frac{x}{(1-\epsilon)^{1/2}}\Big) - \Phi\Big(\frac{-x}{(1-\epsilon)^{1/2}}\Big)\Big)\, dP\Big\{V_{[\frac{T}{\theta_n}]} < y, \max_{0\le j<[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}\Big\}\\
&= \Big(1 - \frac{2}{\sqrt{2\pi}}\int_{\frac{x}{(1-\epsilon)^{1/2}}}^{\infty} e^{-t^2/2}\, dt\Big)P\Big\{\max_{0\le j<[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}\Big\}\\
&\le \big(1 - C(\epsilon)e^{-\frac{x^2}{2(1-2\epsilon)}}\big)P\Big\{\max_{0\le j<[\frac{T}{\theta_n}]}|U_j| \le x\sigma_n^{1/2}\Big\},
\end{aligned}$$

here we have used the following facts on the Wiener Process:

i) $U_{[\frac{T}{\theta_n}]} - V_{[\frac{T}{\theta_n}]}$ and $\{V_{[\frac{T}{\theta_n}]}, U_j, 0 \le j < [\frac{T}{\theta_n}]\}$ are independent,

ii) $\Phi(x-y) - \Phi(-x-y) \le \Phi(x) - \Phi(-x)$ for every $y \in R$ and $x \ge 0$,

iii) for each $\delta > 0$, there is a $C(\delta) > 0$ such that

$$\int_x^{\infty} e^{-t^2/2}dt \ge C(\delta)\exp\Big(-\frac{x^2(1+\delta)}{2}\Big) \text{ for every } x \ge 0.$$

By recurrence, we conclude from (29) and (31) that (28) holds true.

From (28) it is easy to see that

LEMMA 4. *Let $0 < \epsilon < \frac{1}{2}$, $\theta_n(\epsilon)$ be the solution of the equation* (8). *Then, there is a positive $C(\epsilon)$ such that*

$$P\Big\{\sup_{0\le t\le T}|X(t,n)| \ge x\sigma_n^{1/2}\Big\} \ge C(\epsilon)\Big(1 + \frac{T}{\theta_n}\Big)\exp\Big(-\frac{x^2}{2(1-2\epsilon)}\Big) \tag{32}$$

for each $x \geq \left(2(1-2\epsilon)\log \frac{T}{\theta_n}\right)^{1/2}$.

LEMMA 5. *For each* $0 < \epsilon < \frac{1}{2}$, *there is a constant* $C = C(\epsilon)$ *such that*

$$P\Big\{\max_{1\leq n\leq N}\sup_{|t|\leq T}|X(t,n)| \geq x\sigma_N^{1/2}\Big\} \leq C\Big(1+\frac{T\Gamma_N}{\sigma_N}\Big)\exp\Big(-\frac{(1-2\epsilon)x^2}{2}\Big). \tag{33}$$

PROOF. (33) will follow from Lemma 2 and

$$\begin{aligned} P\Big\{\max_{1\leq n\leq N}\sup_{|t|\leq T}|X(t,n)| \geq x\sigma_N^{1/2}\Big\} \\ \leq 4\Big(1+\frac{T\Gamma_N}{\sigma_N}\Big)P\Big\{\sup_{|t|\leq\sigma_N/\Gamma_N}|X(t,N)| \geq x(1-\epsilon)\sigma_N^{1/2}\Big\} \end{aligned} \tag{34}$$

for every x sufficiently large. Let

$$B = \sigma_N/\Gamma_N,\ E_1 = \Big\{\sup_{|t|\leq B}|X(t,1)| \geq x\sigma_N^{1/2}\Big\},$$

$$E_i = \Big\{\max_{j<i}\sup_{|t|\leq B}|X(t,j)| < x\sigma_N^{1/2} \leq \sup_{|t|\leq B}|X(t,i)|\Big\}, \qquad i = 2,\ldots,N.$$

Noting that

$$\begin{aligned} \Big\{\max_{1\leq n\leq N}\sup_{|t|\leq B}|X(t,n)| \geq x\sigma_N^{1/2}\Big\} = \bigcup_{n=1}^{N} E_n &\subset \Big\{\sup_{|t|\leq B}|X(t,N)| \geq x(1-\epsilon)\sigma_N^{1/2}\Big\} \\ &\quad \cup \bigcup_{n=1}^{N-1}\Big(E_n \cap \Big\{\sup_{|t|\leq B}|X(t,N)| < x(1-\epsilon)\sigma_N^{1/2}\Big\}\Big) \\ &\subset \Big\{\sup_{|t|\leq B}|X(t,N)| \geq x(1-\epsilon)\sigma_N^{1/2}\Big\} \\ &\quad \cup \bigcup_{n=1}^{N-1}\Big(E_n \cap \Big\{\sup_{|t|\leq B}|X(t,N)-X(t,n)| \geq \epsilon x\sigma_N^{1/2}\Big\}\Big) \end{aligned}$$

and that $\{X(t,N)-X(t,n),\ |t|\leq B\}$ and E_n are independent, we have

$$\begin{aligned} P\Big\{&\max_{1\leq n\leq N}\sup_{|t|\leq B}|X(t,n)| \geq x\sigma_N^{1/2}\Big\} \\ &\leq P\Big\{\sup_{|t|\leq B}|X(t,N)| \geq x(1-\epsilon)\sigma_N^{1/2}\Big\} \\ &\quad + \sum_{n=1}^{N-1} P\Big\{\sup_{|t|\leq B}|X(t,N)-X(t,n)| \geq \epsilon x\sigma_N^{1/2}\Big\}P(E_n) \\ &\leq P\Big\{\sup_{|t|\leq B}|X(t,N)| \geq x(1-\epsilon)\sigma_N^{1/2}\Big\} \\ &\quad + \sum_{n=1}^{N-1} d\Big(1+\frac{B\sum_{\iota=1+n}^{N}\gamma_\iota}{\sum_{\iota=1+n}^{N}\gamma_\iota/\lambda_\iota}\Big)\exp\Big(-\frac{\epsilon^2x^2\sigma_N}{4\sum_{\iota=1+n}^{N}\gamma_\iota/\lambda_\iota}\Big)P(E_n) \end{aligned}$$

$$\le P\Big\{\sup_{|t|\le B}|X(t,N)|\ge x(1-\epsilon)\sigma_N^{1/2}\Big\}+2\,d\exp\Big(-\frac{\epsilon^2x^2}{4}\Big)\sum_{n=1}^{N-1}P(E_n)$$
$$\le P\Big\{\sup_{|t|\le B}|X(t,N)|\ge x(1-\epsilon)\sigma_N^{1/2}\Big\}+\frac{1}{2}P\Big\{\max_{1\le n\le N}\sup_{|t|\le B}|X(t,n)|\ge x\sigma_N^{1/2}\Big\}$$

provided $x\ge 4\big(\log(8d)\big)/\epsilon$. In the last but second inequality we have used the fact that $f(y)=ye^{-ay}$ is decreasing on $[1/a,\infty)$ for each $a>0$ fixed, and d is an absolute constant as in Lemma 2. The above inequality yields

$$P\Big\{\max_{1\le n\le N}\sup_{|t|\le B}|X(t,n)|\ge x\sigma_N^{1/2}\Big\}\le 2P\Big\{\sup_{|t|\le B}|X(t,N)|\ge x(1-\epsilon)\sigma_N^{1/2}\Big\}$$

for $x\ge 4\big(\log(8d)\big)/\epsilon$, as desired.

PROOF OF THEOREM 1. It suffices to show that for each $0<\epsilon<1/8$

(35) $$\limsup_{N\to\infty}\max_{1\le n\le N}\sup_{|t|\le T_N}|X(t,n)|/\beta_N\le 1+8\epsilon \text{ a.s.}$$

For $k\ge 0$, put

$$H_k=\{N:(1+\epsilon)^k<\beta_N\le(1+\epsilon)^{k+1}\},$$
$$M_k=\max\{N:N\in H_k\}.$$

Clearly, (9) implies that $\beta_N\longrightarrow\infty$ as $N\longrightarrow\infty$. So, we have

(36) $$\limsup_{N\to\infty}\max_{1\le n\le N}\sup_{|t|\le T_N}|X(t,n)|/\beta_N$$
$$\le\limsup_{k\to\infty}\max_{N\in H_k}\max_{1\le n\le N}\sup_{|t|\le T_N}|X(t,n)|/\beta_N$$
$$\le(1+\epsilon)\limsup_{k\to\infty}\max_{1\le n\le M_k}\sup_{|t|\le T_{M_k}}|X(t,n)|/(1+\epsilon)^{k+1}.$$

From the definition of M_k, we find that

$$\frac{(1+\epsilon)^{2k}}{2\big(\log(T_{M_k}\Gamma_{M_k}/\sigma_{M_k})+\log\log\sigma_{M_k}\big)}<\sigma_{M_k}\le\frac{(1+\epsilon)^{2(k+1)}}{2\big(\log(T_{M_k}\Gamma_{M_k}/\sigma_{M_k})+\log\log\sigma_{M_k}\big)}.$$

Whence, for each $k\ge 1$

(37) $$\sigma_{M_k}\ge(1+\epsilon)^k \text{ or } \Big(\frac{T_{M_k}\Gamma_{M_k}}{\sigma_{M_k}}+e\Big)\ge\exp\Big(\frac{1}{4}(1+\epsilon)^k\Big).$$

Using Lemma 5, we deduce

$$(38)\quad P\Big\{\max_{1\le n\le M_k}\sup_{|t|\le T_{M_k}}|X(t,n)|\ge(1+\epsilon)^{k+1}(1+\epsilon)^2\Big\}$$
$$\le P\Big\{\max_{1\le n\le M_k}\sup_{|t|\le T_{M_k}}|X(t,n)|\ge\beta_{M_k}(1+\epsilon)^2\Big\}$$
$$\le C(\epsilon)\Big(1+\frac{T_{M_k}\Gamma_{M_k}}{\sigma_{M_k}}\Big)\exp\Big(-(1+\epsilon)\Big(\log\frac{T_{M_k}\Gamma_{M_k}}{\sigma_{M_k}}+\log\log\sigma_{M_k}\Big)\Big)$$
$$\le C(\epsilon)\Big(1+\frac{T_{M_k}\Gamma_{M_k}}{\sigma_{M_k}}\Big)^{-\epsilon}(\log\sigma_{M_k})^{-(1+\epsilon)}$$
$$\le C(\epsilon)k^{-(1+\epsilon)}$$

by (37). Now (35) follows from (36), (38) and the Borel-Cantelli lemma. This completes the proof of Theorem 1.

PROOF OF THEOREM 2. Noting that σ_N is non-decreasing, we have

$$\sigma_N\longrightarrow\sigma \text{ as } N\longrightarrow\infty,$$

where $0<\sigma\le\infty$. If $0<\sigma<\infty$, then (9) implies $T_N\Gamma_N/\sigma_N\longrightarrow\infty$ and hence (15) is satisfied. So we only need to consider two cases: one is $\sigma=\infty$, the other is (15) being satisfied. We formulate the proof below in two steps, which together with (10) will imply our statements.

STEP 1. Suppose $\sigma=\infty$, then, for each $0<\epsilon<1/(4C^2)$

$$(39)\quad \limsup_{N\to\infty}\sup_{0\le t\le T_N}|X(t,N)|/\beta_N\ge 1-\epsilon^{1/2}\text{ a.s.}$$

Let

$$N_1=1,\ N_{k+1}=\min\Big\{n:\sigma_n\ge\Big(\frac{8C^2}{\epsilon^2}\Big)^k\Big\},\quad k=1,2,\ldots.$$

From condition (11), we get

$$(40)\quad \Big(\frac{8C^2}{\epsilon^2}\Big)^k<\sigma_{N_{k+1}}\le C\Big(\frac{8C^2}{\epsilon^2}\Big)^k.$$

Clearly, $\sigma=\infty$ implies $N_k\uparrow\infty$ as $k\longrightarrow\infty$. Then

$$(41)\quad \limsup_{N\to\infty}\sup_{0\le t\le T_N}|X(t,N)|/\beta_N\ge\limsup_{k\to\infty}\sup_{0\le t\le T_{N_k}}|X(t,N_k)|/\beta_{N_k}$$
$$\ge\limsup_{k\to\infty}\sup_{0\le t\le T_{N_k}}|X(t,N_k)-X(t,N_{k-1})|/\beta_{N_k}$$
$$-\limsup_{k\to\infty}\sup_{0\le t\le T_{N_k}}|X(t,N_{k-1})|/\beta_{N_k}.$$

Using Lemma 5 again, we have

$$P\Big\{\sup_{0\le t\le T_{N_k}}|X(t,N_{k-1})|/\beta_{N_k}\ge\frac{\epsilon}{2}\Big\}$$
$$\le C(\epsilon)\Big(1+\frac{T_{N_k}\Gamma_{N_{k-1}}}{\sigma_{N_{k-1}}}\Big)\exp\Big(-\frac{\epsilon^2\sigma_{N_k}}{9\sigma_{N_{k-1}}}\Big(\log\frac{T_{N_k}\Gamma_{N_k}}{\sigma_{N_k}}+\log\log\sigma_{N_k}\Big)\Big)$$
$$\le C(\epsilon)\Big(1+\frac{T_{N_k}\Gamma_{N_{k-1}}}{\sigma_{N_{k-1}}}\Big)\Big(1+\frac{T_{N_k}\Gamma_{N_k}}{\sigma_{N_k}}\Big)^{-2}\log^{-2}\sigma_{N_k}$$
$$\le C(\epsilon)k^{-2}$$

by (40). This implies that

$$\limsup_{k\to\infty}\sup_{0\le t\le T_{N_k}}|X(t,N_{k-1})|/\beta_{N_k}\le\frac{\epsilon}{2}\text{ a.s.}\tag{42}$$

To estimate $|X(t,N_k)-X(t,N_{k-1})|/\beta_{N_k}$, we let $\theta_k^*(\epsilon)$ be the solution of the equation

$$\sum_{l=1+N_{k-1}}^{N_k}\frac{\gamma_l}{\lambda_l}e^{-2\lambda_l\theta_k^*(\epsilon)}=\epsilon\sum_{l=1+N_{k-1}}^{N_k}\frac{\gamma_l}{\lambda_l}$$

and let

$$\beta_k'=\Big(2\Big(\sum_{l=1+N_{k-1}}^{N_k}\frac{\gamma_l}{\lambda_l}\Big)\Big(\log(T_{N_k}/\theta_k^*(\epsilon)\Big)+\log\log\sum_{l=1+N_{k-1}}^{N_k}\frac{\gamma_l}{\lambda_l})\Big).$$

Then, in terms of (32), we obtain

$$P\Big\{\sup_{0\le t\le T_{N_k}}|X(t,N_k)-X(t,N_{k-1})|/\beta_k'\ge(1-2\epsilon)^{1/2}\Big\}$$
$$\ge C(\epsilon)\Big(1+\frac{T_{N_k}}{\theta_k^*(\epsilon)}\Big)\exp\big(-(\beta_k')^2/2\big)$$
$$\ge C(\epsilon)\log^{-1}\Big(\sum_{l=1+N_{k-1}}^{N_k}\frac{\gamma_l}{\lambda_l}\Big)$$
$$\ge C(\epsilon)k^{-1}$$

by (40) again. Therefore, we have

$$\limsup_{k\to\infty}\sup_{0\le t\le T_{N_k}}|X(t,N_k)-X(t,N_{k-1})|/\beta_k'\ge(1-2\epsilon)^{1/2}\text{ a.s.,}\tag{43}$$

since $\{\sup_{0\le t\le T_{N_k}}|X(t,N_k)-X(t,N_{k-1})|,k\ge 1\}$ are independent random variables. On the other hand, it follows from the definitions of $\theta_{N_k}(\epsilon/2)$ and θ_k^* that

$$\frac{1}{4}\epsilon\sigma_{N_k}=\sum_{l=1}^{N_k}\frac{\gamma_l}{\lambda_l}e^{-2\lambda_l\theta_{N_k}(\frac{\epsilon}{4})}$$
$$\ge\sum_{l=1+N_{k-1}}^{N_k}\frac{\gamma_l}{\lambda_l}e^{-2\lambda_l\theta_{N_k}(\frac{\epsilon}{4})}$$

and

$$\frac{1}{2}\epsilon\sigma_{N_k} = \frac{\epsilon}{2}(\sigma_{N_k} - \sigma_{N_{k-1}} + \sigma_{N_{k-1}}) \le \frac{\epsilon}{2}(\sigma_{N_k} - \sigma_{N_{k-1}}) + \frac{\epsilon}{4}\sigma_{N_k}.$$

From the latter, we find that $\sigma_{N_k} \le 4(\sigma_{N_k} - \sigma_{N_{k-1}})$. Hence

$$\sum_{\imath=1+N_{k-1}}^{N_k} \frac{\gamma_\imath}{\lambda_\imath} e^{-2\lambda_\imath \theta_{N_k}(\frac{\epsilon}{4})} \le \sum_{\imath=1+N_{k-1}}^{N_k} \frac{\gamma_\imath}{\lambda_\imath} e^{-2\lambda_\imath \theta_k^*(\epsilon)},$$

which is equivalent to say that $\theta_{N_k}(\frac{\epsilon}{4}) \ge \theta_k^*(\epsilon)$. Combining the above results with the assumption (12), we finally conclude that

$$\text{(44)} \qquad \limsup_{k\to\infty} \sup_{0\le t\le T_{N_k}} |X(t,N_k) - X(t,N_{k-1})| / \beta_{N_k} \ge \frac{(1-2\epsilon)}{(1+C\epsilon)^2} \text{ a.s.}$$

This proves (39) by (41), (42) and (44).

STEP 2. If, in addition, (15) is satisfied, then for each $0 < \epsilon < \frac{1}{8}$

$$\text{(45)} \qquad \liminf_{N\to\infty} \sup_{0\le t\le T_N} |X(t,N)| / \alpha_N \ge 1 - 4\epsilon \text{ a.s.},$$

where $\alpha_N = (2\sigma_N \log \frac{T_N\Gamma_N}{\sigma_N})^{1/2}$.

Let $1 < \theta < 1 + \frac{\epsilon^2}{6}$. Define

$$A_k = \{N : \theta^k\sigma_1 \le \sigma_N < \theta^{k+1}\sigma_1\}, \quad k = 0, 1, \ldots,$$

$$B_\jmath = \Big\{N : \theta^\jmath \le \frac{T_N\Gamma_N}{\sigma_N} + 1 < \theta^{\jmath+1}\Big\}, \quad \jmath = 0, 1, \ldots,$$

$$L_{k,\jmath} = \min\{N : N \in A_kB_\jmath\},\ L_{k,\jmath}^* = \max\{N : N \in A_kB_\jmath\},$$

$$\Gamma_{k,\jmath} = \sum_{\imath=1+L_{k,\jmath}}^{L_{k,\jmath}^*} \gamma_\imath,\ \sigma_{k,\jmath} = \sum_{\imath=1+L_{k,\jmath}}^{L_{k,\jmath}^*} \frac{\gamma_\imath}{\lambda_\imath}.$$

Clearly, (15) implies that $T_N\Gamma_N/\sigma_N \to \infty$ and that $A_kB_\jmath = \emptyset$ if $k \ge \theta^{\epsilon\jmath}$, when $\jmath$ is sufficiently large. Thus, we have

$$\begin{aligned}
\text{(46)} \quad & \liminf_{N\to\infty} \sup_{0\le t\le T_N} |X(t,N)|/\alpha_N \\
&\ge \liminf_{\jmath\to\infty} \inf_{N\in B_\jmath} \sup_{0\le t\le T_N} |X(t,N)|/\alpha_N \\
&\ge \liminf_{\jmath\to\infty} \inf_{0\le k\le\theta^{\epsilon\jmath}} \inf_{N\in B_\jmath A_k} \sup_{0\le t\le T_N} |X(t,N)|/\alpha_N \\
&\ge \liminf_{\jmath\to\infty} \inf_{0\le k\le\theta^{\epsilon\jmath}} \inf_{N\in B_\jmath A_k} \sup_{0\le t\le T_{L_{k,\jmath}}} \frac{|X(t,N)|}{(2\theta^{k+1}\log\theta^{\jmath+1})^{1/2}} \\
&\ge \liminf_{\jmath\to\infty} \inf_{0\le k\le\theta^{\epsilon\jmath}} \sup_{0\le t\le T_{L_{k,\jmath}}} \frac{|X(t,L_{k,\jmath})|}{(2\theta^{k+1}\log\theta^{\jmath+1})^{1/2}} \\
&\quad - \limsup_{\jmath\to\infty} \sup_{0\le k\le\theta^{\epsilon\jmath}} \sup_{L_{k,\jmath}\le N\le L_{k,\jmath}^*} \sup_{0\le t\le T_{L_{k,\jmath}}} \frac{|X(t,N) - X(t,L_{k,\jmath})|}{(2\theta^{k+1}\log\theta^{\jmath+1})^{1/2}}.
\end{aligned}$$

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Similarly to (33), we can obtain that

$$P\Big\{\sup_{L_{k,j}\le N\le L^*_{k,j}}\ \sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,N)-X(t,L_{k,j})|}{(2\theta^{k+1}\log\theta^{j+1})^{1/2}}\ge\epsilon\Big\}$$
$$\le C(\epsilon)\Big(1+\frac{T_{L_{k,j}}\Gamma_{k,j}}{\sigma_{k,j}}\Big)\exp\Big(-\frac{\epsilon^2\theta^{k+1}\log\theta^{j+1}}{2\sigma_{k,j}}\Big).$$

Since xe^{-x} is decreasing on $[1,\infty)$ and $\sigma_{k,j}=\sigma_{L^*_{k,j}}-\sigma_{L_{k,j}}\le(\theta-1)\theta^k$, the above inequality is bounded by

$$C(\epsilon)\Big(1+\frac{T_{L^*_{k,j}}\Gamma_{L^*_{k,j}}}{\sigma_{L^*_{k,j}}}\Big)\exp\Big(-\frac{\epsilon^2\log\theta^{j+1}}{2(\theta-1)}\Big)\le C(\epsilon)\theta^{j+1}\exp(-3\log\theta^{j+1})\le C(\epsilon)\theta^{-2j}$$

for every j sufficiently large. Therefore

$$P\Big\{\sup_{0\le k\le\theta^{\epsilon j}}\ \sup_{L_{k,j}\le N\le L^*_{k,j}}\ \sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,N)-X(t,L_{k,j})|}{(2\theta^{k+1}\log\theta^{j+1})^{1/2}}\ge\epsilon\Big\}\le C(\epsilon)\theta^{-j},$$

which follows that

$$\limsup_{j\to\infty}\ \sup_{0\le k\le\theta^{\epsilon j}}\ \sup_{L_{k,j}\le N\le L^*_{k,j}}\ \sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,N)-X(t,L_{k,j})|}{(2\theta^{k+1}\log\theta^{j+1})^{1/2}}\le\epsilon \text{ a.s.} \tag{47}$$

On the other hand, using (28), we have

$$P\Big\{\sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,L_{k,j})|}{(2\theta^{k+1}\log\theta^{j+1})^{1/2}}\le\frac{1-2\epsilon}{\theta}\Big\}$$
$$\le P\Big\{\sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,L_{k,j})|}{(2\sigma_{L_{k,j}}\log\theta^{j+1})^{1/2}}\le 1-2\epsilon\Big\}$$
$$\le\Big(1-C(\epsilon)\exp\big(-(1-2\epsilon)\log\theta^{j+1}\big)\Big)^{T_{L_{k,j}}/\theta_{L_{k,j}}(\epsilon)}$$
$$\le\exp\Big(-\frac{C(\epsilon)T_{L_{k,j}}}{\theta^{(1-2\epsilon)j}\theta_{L_{k,j}}(\epsilon)}\Big)$$
$$\le\exp\big(-C(\epsilon)\theta^{\epsilon j}\big)$$

by (12) and (15), for every sufficiently large j, and hence

$$P\Big\{\inf_{0\le k\le\theta^{\epsilon j}}\ \sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,L_{k,j})|}{(2\theta^{k+1}\log\theta^{j+1})^{1/2}}\le\frac{1-2\epsilon}{\theta}\Big\}\le\theta^{\epsilon j}\exp\big(-C(\epsilon)\theta^{\epsilon j}\big)\le\theta^{-j}$$

provided that j is sufficiently large, which implies immediately

$$\liminf_{j\to\infty}\ \inf_{0\le k\le\theta^{\epsilon j}}\ \sup_{0\le t\le T_{L_{k,j}}}\frac{|X(t,L_{k,j})|}{(2\theta^{k+1}\log\theta^{j+1})^{1/2}}\ge\frac{1-2\epsilon}{\theta} \text{ a.s.} \tag{48}$$

by the Borel-Cantelli lemma.

Now (45) follows from (46)–(48). This completes the proof of Theorem 2.

PROOF OF THEOREM 3. It suffices to show that

$$\forall A > 0, \quad \limsup_{n\to\infty} \sup_{|t|\le A} \frac{|X(t,n)|}{(2\sigma_n \log\log \sigma_n)^{1/2}} \le 1 \text{ a.s.} \tag{49}$$

and

$$\forall \epsilon > 0,\ \forall A > 0, \quad \lim_{n\to\infty} P\Big\{ \bigcup_{|t|\le A} \bigcap_{\iota=n}^{\infty} \{X(t,i) < (1-\epsilon)(2\sigma_\iota \log\log\sigma_\iota)^{1/2}\}\Big\} = 0 \tag{50}$$

hold true.

(49) follows from Theorem 1 and (18) immediately. We now prove (50). Let

$$0 < \epsilon < \frac{1}{4}, \quad n_k = \max\{n : \sigma_n \le a^k\}$$

where $a > 1$ is a constant which will be specified later. Then

$$\frac{a^k}{C} \le \sigma_{N_k} \le a^k.$$

Clearly, (50) is implied by

$$\lim_{k\to\infty} P\Big\{ \bigcup_{|t|\le A} \bigcap_{\iota=k}^{\infty} \{X(t,n_\iota) < (1-\epsilon)(2\sigma_{n_\iota} \log\log\sigma_{n_\iota})^{1/2}\}\Big\} = 0. \tag{51}$$

Noting that

$$\begin{aligned}\{X(t,n_\iota) &< (1-\epsilon)(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\} \\ &\subset \{X(t,n_{\iota-1}) < -\frac{\epsilon}{2}(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\} \\ &\quad \bigcup \Big\{X(t,n_\iota) - X(t,n_{\iota-1}) < \Big(1-\frac{\epsilon}{2}\Big)(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\Big\},\end{aligned}$$

we have

$$\begin{aligned}&\bigcup_{|t|\le A}\bigcap_{\iota=k}^{\infty}\{X(t,n_\iota) < (1-\epsilon)(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\} \\ &\subset \bigcup_{|t|\le A}\bigcap_{\iota=k}^{\infty}\Big\{X(t,n_\iota) - X(t,n_{\iota-1}) < \Big(1-\frac{\epsilon}{2}\Big)(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\Big\} \\ &\quad \bigcup \bigcup_{|t|\le A}\bigcup_{\iota=k}^{\infty}\Big\{X(t,n_{\iota-1}) < -\frac{\epsilon}{2}(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\Big\}.\end{aligned}$$

From Theorem 1 and (18) it follows that

$$P\Big\{\bigcup_{|t|\le A}\bigcup_{\iota=k}^{\infty}\{X(t,n_{\iota-1}) < -\frac{\epsilon}{2}(2\sigma_{n_\iota}\log\log\sigma_{n_\iota})^{1/2}\Big\} \longrightarrow 0 \text{ as } k\longrightarrow\infty$$

provided $a > 8C/\epsilon^2$.

The rest we should do is to prove

$$(52)\qquad P\Big\{\bigcup_{|t|\le A}\bigcap_{i=k}^{\infty}\Big\{X(t,n_i)-X(t,n_{i-1})<\Big(1-\frac{\epsilon}{2}\Big)(2\sigma_{n_i}\log\log\sigma_{n_i})^{1/2}\Big\}\Big\}\longrightarrow 0$$

as $k\longrightarrow\infty$. Let $b:=b_k=1/(Ak^2)$. Then

$$\begin{aligned}(53)\qquad & P\Big\{\bigcup_{|t|\le A}\bigcap_{i=k}^{\infty}\Big\{X(t,n_i)-X(t,n_{i-1})<\Big(1-\frac{\epsilon}{2}\Big)(2\sigma_{n_i}\log\log\sigma_{n_i})^{1/2}\Big\}\Big\}\\
&\le P\Big\{\bigcup_{|t|\le A}\bigcap_{i=k}^{2k}\Big\{X(t,n_i)-X(t,n_{i-1})<\Big(1-\frac{\epsilon}{2}\Big)(2\sigma_{n_i}\log\log\sigma_{n_i})^{1/2}\Big\}\Big\}\\
&\le 4k^2P\Big\{\bigcup_{0\le t\le b}\bigcap_{i=k}^{2k}\Big\{X(t,n_i)-X(t,n_{i-1})<\Big(1-\frac{\epsilon}{2}\Big)(2\sigma_{n_i}\log\log\sigma_{n_i})^{1/2}\Big\}\Big\}\\
&\le 4k^2P\Big\{\bigcap_{i=k}^{2k}\Big\{X(b,n_i)-X(b,n_{i-1})<\Big(1-\frac{\epsilon}{3}\Big)(2\sigma_{n_i}\log\log\sigma_{n_i})^{1/2}\Big\}\Big\}\\
&\quad +4k^2P\Big\{\bigcup_{0\le t\le b}\bigcup_{i=k}^{2k}\Big\{\frac{X(t,n_i)-X(t,n_{i-1})-X(b,n_i)+X(b,n_{i-1})}{(2\sigma_{n_i}\log\log\sigma_{n_i})^{1/2}}<-\frac{\epsilon}{6}\Big\}\Big\}\\
&:=I_1(k)+I_2(k).\end{aligned}$$

Since $\{X(b,n_i)-X(b,n_{i-1}),k\le i\le 2k\}$ are independent , we have

$$\begin{aligned}(54)\qquad I_1(k)&\le 4k^2\prod_{i=k}^{2k}\Big(1-C(\epsilon)\exp\big(-(1-\frac{\epsilon}{6})\log\log\sigma_{N_i}\big)\Big)\\
&\le 4k^2\prod_{i=k}^{2k}\big(1-C(\epsilon)i^{-1+\frac{\epsilon}{6}}\big)\\
&\le 4k^2\exp\Big(-\sum_{i=k}^{2k}C(\epsilon)i^{-1+\frac{\epsilon}{6}}\Big)\\
&\le 4k^2\exp\big(-C(\epsilon)k^{\epsilon/6}\big)\longrightarrow 0,\ \text{as } k\longrightarrow\infty.\end{aligned}$$

On the other hand, for $0\le t\le b$ and $k\le i\le 2k$ we have

$$\begin{aligned}E\big(X(t,n_i)-X(t,n_{i-1})-X(b,n_i)+X(b,n_{i-1})\big)^2&=2\sum_{j=1+n_{i-1}}^{n_i}\frac{\gamma_i}{\lambda_i}(1-e^{-2\lambda_i(b-t)})\\
&\le 4(b-t)\sum_{j=1+n_{i-1}}^{n_i}\gamma_i\\
&\le\frac{4\Gamma_{n_i}}{A\sigma_{n_i}}k^{-2}\sigma_{n_i}\\
&\le 4k^{-2}\sigma_{n_i}(\log\sigma_{N_i})/A\\
&\le 8\sigma_{n_i}(\log a)/(Ak)\\
&\le\epsilon\sigma_{n_i}/48\end{aligned}$$

provided that k is large enough.
Consequently, using the Fernique lemma again, we get

$$\begin{aligned} I_2(k) &\le C(\epsilon)k^3 \max_{k\le\iota\le 2k}\Big(1+\frac{b\Gamma_{n_\iota}}{\sigma_{n_\iota}}\Big)\exp(-4\log\log\sigma_{n_k}) \\ &\le C(\epsilon)k^3\exp(-4\log\log\sigma_{n_k}) \\ &\le C(\epsilon)k^{-1} \longrightarrow 0 \text{ as } k\longrightarrow\infty. \end{aligned}$$

This proves (52) by (53) and (54), as desired. The proof of Theorem 3 is completed.

PROOF OF COROLLARY 1. It is easy to see that

$$\frac{T_N\Gamma_N}{\sigma_N}\le\lambda_N^*,$$

$$\begin{aligned} 2\epsilon\sigma_n &= \sum_{\iota=1}^{n}\frac{\gamma_\iota}{\lambda_\iota}e^{-2\lambda_\iota\theta_n(2\epsilon)} \\ &\le \sum_{\iota=1}^{m_n(1,\epsilon)-1}\frac{\gamma_\iota}{\lambda_\iota}+\sum_{\iota=m_n(1,\epsilon)}^{n}\frac{\gamma_\iota}{\lambda_\iota}e^{-2\lambda_\iota\theta_n(2\epsilon)} \\ &\le \epsilon\sigma_n+\sum_{\iota=m_n(1,\epsilon)}^{n}\frac{\gamma_\iota}{\lambda_\iota}e^{-2\lambda_\iota\theta_n(2\epsilon)} \\ &\le \epsilon\sigma_n+\Big(\sum_{\iota=m_n(1,\epsilon)}^{n}\frac{\gamma_\iota}{\lambda_\iota}\Big)\exp\Big(-2\min_{m_n(1,\epsilon)\le\iota\le n}\{\lambda_\iota\}\theta_n(2\epsilon)\Big) \\ &\le \epsilon\sigma_n+\sigma_n\exp\Big(-2\min_{m_n(1,\epsilon)\le\iota\le n}\{\lambda_\iota\}\theta_n(2\epsilon)\Big). \end{aligned}$$

The latter implies that

$$\frac{1}{\theta_n(2\epsilon)}\ge 2\min_{m_n(1,\epsilon)\le\iota\le n}\{\lambda_\iota\}/\log(1/\epsilon).$$

Similarly, we have

$$\frac{1}{\theta_n(2\epsilon)}\ge 2\min_{1\le\iota\le m_n(2,\epsilon)}\{\lambda_\iota\}/\log(1/\epsilon).$$

Consequently, we obtain

$$\frac{1}{\theta_n(2\epsilon)}\ge 2\lambda_n'(\epsilon)/\log(1/\epsilon)$$

This indicates that the condition (12) is satisfied. The corollary now follows from Theorems 2 and 3.

PROOF OF COROLLARY 2. Since σ_n/n^α is quasi-increasing, there exists a positive constant C such that

(55) $$\sigma_\ell/\ell^\alpha\le C\sigma_n/n^\alpha$$

for each $\ell\le n$. From (55) we can find that for every $0<\epsilon<\frac{1}{4}$

$$\sigma_\ell\le\epsilon\sigma_n \text{ for each } \ell\le\Big(\frac{\epsilon}{C}\Big)^{1/\alpha}n$$

and hence

$$m_n(1,\epsilon) \geq \left(\frac{\epsilon}{C}\right)^{1/\alpha} n. \tag{56}$$

On the other hand, it is easy to find that from the assumption $\lambda_{2k}^* \leq C \min_{k\leq \iota\leq 2k} \lambda_\iota$, for each $0 < \epsilon < \frac{1}{4}$, there exists a constant $C(\epsilon)$ such that

$$\lambda_n^* \leq C(\epsilon) \min_{\epsilon n\leq \iota\leq n} \lambda_\iota. \tag{57}$$

Thus, the assumption of Corollary 1 is satisfied by (56) and (57) and hence the corollary holds.

The proof of Corollary 3 is trivial and so is omitted here.

PROOF OF COROLLARY 4. By the assumption of quasi-increasing, there is a positive constant C such that for each $k \leq n$

$$\lambda_k \sigma_k^{1-\alpha} \leq C\lambda_n \sigma_n^{1-\alpha}$$

and

$$\sigma_k^{1/\alpha}/\lambda_k \leq C\sigma_n^{1/\alpha}/\lambda_n.$$

Then

$$\begin{aligned} \frac{\Gamma_n}{\sigma_n} &= \Big(\sum_{\iota=1}^n \frac{\gamma_\iota}{\lambda_\iota}\lambda_\iota\Big)/\sigma_n \\ &\leq \Big(\sum_{\iota=1}^n \frac{(\sigma_\iota-\sigma_{\iota-1})}{\sigma_\iota^{1-\alpha}}\lambda_n\sigma_n^{1-\alpha}\Big)/\sigma_n \\ &\leq C\lambda_n/\alpha, \end{aligned}$$

and

$$\begin{aligned} 2\epsilon\sigma_n &= \sum_{\iota=1}^n \frac{\gamma_\iota}{\lambda_\iota} e^{-2\lambda_\iota\theta_n(2\epsilon)} \\ &\leq \epsilon\sigma_n + \sum_{\iota=m_n(1,\epsilon)}^n \frac{\gamma_\iota}{\lambda_\iota} e^{-2\lambda_\iota\theta_n(2\epsilon)} \\ &\leq \epsilon\sigma_n + \sum_{\iota=m_n(1,\epsilon)}^n \frac{\gamma_\iota}{\lambda_\iota}\exp\Big(-\frac{2\lambda_n\sigma_\iota^{1/\alpha}\theta_n(2\epsilon)}{C\sigma_n^{1/\alpha}}\Big) \\ &\leq \epsilon\sigma_n + \sum_{\iota=m_n(1,\epsilon)}^n \frac{\gamma_\iota}{\lambda_\iota}\exp\Big(-\frac{2\lambda_n\epsilon^{1/\alpha}\theta_n(2\epsilon)}{C}\Big) \\ &\leq \epsilon\sigma_n + \sigma_n\exp\Big(-\frac{2\lambda_n\epsilon^{1/\alpha}\theta_n(2\epsilon)}{C}\Big). \end{aligned}$$

Therefore, we have

$$\frac{1}{\theta_n(2\epsilon)} \geq \frac{2\lambda_n\epsilon^{1/\alpha}}{C\log(1/\epsilon)}.$$

This proves that condition (12) is also satisfied and hence the corollary follows from Theorems 2 and 3.

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