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MONOTONE AND 1-1 SETS

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Abstract

An infinite subset of ω is monotone (1-1) if every recursive function is eventually monotone on it (eventually constant on it or eventually 1-1 on it). A recursively enumerable set is co-monotone (co-1-1) just if its complement is monotone (1-1). It is shown that no implications hold among the properties of being cohesive, monotone, or 1-1, though each implies r-cohesiveness and dense immunity. However it is also shown that co-monotone and co-1-1 are equivalent, that they are properly stronger than the conjunction of r-maximality and dense simplicity, and that they do not imply maximality.

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A subset $X \subseteq \omega$ is said to be "monotone" if it is infinite and every total recursive function is eventually nondecreasing on it. More formally we must have for every total recursive function f, a number t such that for $x, y > t$ in X, $x < y$ implies $f(x) \leqslant f(y)$. Furthermore a subset $X \subseteq \omega$ will be said to "1-1" if it is infinite and every total recursive function is either eventually constant or eventually 1-1 on it. This requires that for every total recursive function f, there must be a number t in X such that either $f(x) = f(t)$ for all $x > t$ in X or $f(x) \neq f(y)$ for all $x > y > t$ in X. An r.e. set is said to be co-monotone (co-1-1) just if its complement is monotone (1-1). Owings (1966) was the first to observe that every maximal set was both co-monotone and co-1-1. This led us to study these sets in their own right and to establish their position in the implication lattice of simplicity notions for r.e. sets. The first study of monotone and 1-1 sets was made by Madan (1975). After some preliminary observations about these sets, Section 2 will go on to demonstrate the independence of the notions of cohesiveness, 1-1 and monotone.

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Section 3 will take up the co-r.e. results where it will be shown that co-1-1 and co-monotone are equivalent and that they lie strictly between the join of r-maximal and dense simple, and maximal. For standard notation and facts concerning r.e. sets the reader is referred to Rogers (1967). A summary of definitions and facts concerning simplicity notions is contained in Robinson (1967). Recent developments are described in Lerman and Soare (1980).

THEOREM 1.1. *Monotone and* 1-1 *sets are recursively invariant.*

PROOF. It suffices to show for any $X \subseteq \omega$ and any recursive permutation ϕ of ω that $\phi(X)$ is not 1-1 if X is not, and not monotone if X is not. As $\phi(X)$ is finite when X is, we may assume that X is infinite.

If X is not 1-1, there must be a recursive function f which is not eventually constant on X and not eventually 1-1 on X. In this case $f(\phi^{-1})$ provides a counterexample to $\phi(X)$ being 1-1.

If X is not monotone, there must be a recursive function g which is not eventually monotone on X. There are therefore infinitely many pairs x, y with $x < y$ and $g(y) < g(x)$. For each such pair, if $\phi(x) < \phi(y)$ then $g(\phi^{-1}(\phi(x))) > g(\phi^{-1}(\phi(y)))$ and if $\phi(y) < \phi(x)$ then $\phi^{-1}(\phi(y)) > \phi^{-1}(\phi(x))$. Hence either ϕ^{-1} or $g(\phi^{-1})$ is order reversing on $\phi(X)$ infinitely often, and $\phi(X)$ is not monotone.

THEOREM 1.2. *Monotone or* 1-1 *implies r-cohesive and dense immune.*

PROOF. If X is infinite and not r-cohesive then a counterexample to its being either 1-1 or monotone is provided by the characteristic function of a recursive set R that splits X so that $R \cap X$ and $R' \cap X$ are both infinite. Hence monotone or 1-1 implies r-cohesive.

Conversely, suppose that X is infinite and not dense immune. Then as shown in the proof of Theorem 3 of Robinson (1967), there is a strictly increasing recursive function f such that $|\{x \in X \mid f(n) < x \leqslant f(n+1)\}| \geqslant n$ for infinitely many values of n. Thus if we define $g(x) = n$ for $f(n) < x \leqslant f(n+1)$ and all n, then g shows that X is not 1-1. If, on the other hand, we let $h(x) = f(n+1) - x$ for $f(n) < x \leqslant f(n+1)$ and all n, then h shows that X is not monotone.

This establishes in particular that co-monotone and co-1-1 lie above r-maximal and dense simple. The fact that they are below maximal is a co-r.e. result left to Section 3. We now take up the independence of cohesiveness from monotone and 1-1.

2. Independence

It is shown here first that cohesive does not imply 1-1 or monotone. This is followed by showing that monotone does not imply 1-1 and vice versa. The fact that monotone or 1-1 do not imply cohesive is a consequence of the co-r.e. result of the next section that co-monotone and co-1-1 are strictly below maximal in the implication lattice of simplicity notions.

LEMMA 2.1. *There exists a total recursive finite-*1 *function f such that for any finite partition $\{A_i\}_{i=1}^k$ of ω, there is an infinite member A_i of the partition, on which f is neither eventually constant nor eventually* 1-1.

PROOF. Define the finite-1 function f as follows;

$$f(x) = \begin{cases} 0 & \text{if } x = 0, x = 1; \\ n+1 & \text{if } 2^{n+1} \leqslant x < 2^{n+2}, n \in \omega. \end{cases}$$

Every finite partition must have an infinite member, and f cannot be eventually constant on any infinite set as f is finite-1, so the only possibility for contradicting the proposition of the lemma is that f is eventually 1-1 on every infinite member of the partition. However, this is not possible as f takes the value n on sets of cardinality 2^n, and so for $2^n > k$, there must be in the same set of the partition at least two numbers on which f agrees. There are therefore infinitely many counter-examples to f being 1-1 on the same A_i, distributed among the finitely many infinite members of the partition. f must therefore fail to be eventually 1-1 on at least one of these infinite members.

THEOREM 2.1. *Cohesive does not imply* 1-1.

PROOF. A cohesive set A that is not 1-1 will be constructed. The nth element of the set will be x_n. Let f be as constructed in Lemma 2.1. Let T be the binary tree of all finite sequences of 0's and 1's, and associate with each $\sigma \in T$ a subset B_σ of ω as follows; $B_\phi = \omega$, and for σ extending σ^* of length n, if $\sigma = \langle \sigma^*, 0 \rangle$ let $B_\sigma = B_{\sigma^*} \cap W_n$ and if $\sigma = \langle \sigma^*, 1 \rangle$ let $B_\sigma = B_{\sigma^*} \cap W_n'$. Let $T' = \{\sigma \mid \sigma \in T, B_\sigma$ is infinite and f is neither eventually constant nor eventually 1-1 on $B_\sigma\}$.

We will now show that T' is a subtree of T containing paths of arbitrarily large length. To show that T' is a subtree, suppose $\sigma \in T'$ and extends σ'. This implies that $B_\sigma \subseteq B_{\sigma'}$ and so if B_σ puts σ in T', $B_{\sigma'}$ puts σ' in T'. T' is therefore a subtree. For the paths of arbitrarily large length, note that the set of B_σ's with σ of length n is a finite partition of ω and by Lemma 2.1 one of these σ's must be in T'. By the König Infinity Lemma, T' must contain an infinite path, let the finite segments of this be given by $D = \{\sigma_n \mid n \in \omega\}$.

The sequence of sets B_{σ_n} is by construction a decreasing sequence of infinite sets eventually contained in W_e or W_e' depending on whether σ_e ends in a 0 or 1 and f is neither eventually constant nor eventually 1-1 on each B_{σ_n}. Choose now an infinite sequence of pairs of numbers x_{2n}, x_{2n+1} with $x_{2n} < x_{2n+1} < x_{2n+2}$ such that $x_{2n}, x_{2n+1} \in B_{\sigma_n}$, $f(x_{2n}) = f(x_{2n+1})$ and $f(x_{2n+2}) > f(x_{2n+1})$. This is possible as f is not eventually 1-1 on each B_{σ_n} and f has infinite range on each B_{σ_n} as it is finite-1. The set $A = \{x_i \mid i \in \omega\}$ is then a cohesive set that is not 1-1.

COROLLARY 2.1. *There exists a cohesive set that is not monotone.*

PROOF. This follows by changing the definition of f in Lemma 2.1 and using the construction of Theorem 2.1. Define g by $g(0) = 1$, $g(1) = 0$ and for $2^{n+1} \leqslant x < 2^{n+2}$ let $g(x) = 2^{n+2} - (x - 2^{n+1})$; g is then decreasing whenever f of Lemma 2.1 was constant. The set A of Theorem 2.1 is therefore also a cohesive set that is not monotone.

THEOREM 2.2. *Monotone does not imply* 1-1.

PROOF. It is sufficient to construct a set A on which some recursive function is neither eventually constant nor eventually 1-1 and such that every total recursive function f is either eventually monotone on A or bounded on A. This is because if a recursive function g is bounded on A but not eventually constant on A then there must exist values $u_1 \neq u_2$ such that both the sets $g^{-1}(u_1)$, $g^{-1}(u_2)$ meet A in infinite subsets. Let $f(x) = x$ when $g(x) = u_1$, and zero otherwise. f must also be eventually monotone or bounded on A. This however is impossible if $g(x) = u_1$ infinitely often in A and $g(x) \neq u_1$ infinitely often in A. Hence g must be eventually constant on A and therefore also eventually monotone on A.

For the construction of A, let f_n, $n \in \omega$, be an enumeration of all total recursive functions, and let $\{A_n\}_{n\in\omega}$, be a recursive partition of ω, with each A_n infinite. The set A will be constructed to meet infinitely many A_n's in a set of cardinality 2 and hence the recursive function h that assigns the value n to all elements of A_n will provide a counterexample to A being 1-1. We shall construct a sequence of infinite sets, M_s, $B_{m,s}$, and numbers k_s for $m, s \in \omega$. M_s will contain indices of A_m's having subsets on which each f_n, $n \leqslant s$, can be kept bounded or monotone. $B_{m,s}$ will pick out a subset of A_m on which this is possible. A will meet A_{k_s} in a set of cardinality 2 and these elements will be x_{2s}, x_{2s+1}.

Let $M_0 = \omega$, $B_{m,0} = A_m$ for $m \in \omega$, $k_0 = 0$ and let $x_0 < x_1$ be two elements of A_0 such that $f_0(x_1) \geqslant f_0(x_0)$. Now proceeding by induction on s, suppose we have defined M_s, $B_{m,s}$ for $m \in \omega$, k_r and x_{2r}, x_{2r+1} for $r \leqslant s$.

We define sets $K_{s+1,t}$, $D_{m,s+1,t}$ for $t \leqslant s+1$, and let $M_{s+1} = K_{s+1,s+1}$ and $B_{m,s+1} = D_{m,s+1,s+1}$. Inducting on t let $K_{s+1,0} = M_s$, $D_{m,s+1,0} = B_{m,s}$ and suppose that $K_{s+1,t}$, $D_{m,s+1,t}$ are defined. $K_{s+1,t+1}$, $D_{m,s+1,t+1}$ are then defined as follows: Let

$$E_{t,s+1} = \{m \mid m \in K_{s+1,t} \text{ and } |\{y \mid y \in D_{m,s+1,t} \,\&\, f_t(y) \geqslant \max\{f_t(x_i) \mid i \leqslant 2s+1\}\}| = \infty\}.$$

If $E_{t,s+1}$ is infinite, let $K_{s+1,t+1} = E_{t,s+1}$ and for every m in $E_{t,s+1}$, set $D_{m,s+1,t+1} = \{y \mid y \in D_{m,s+1,t} \,\&\, f_t(y) \geqslant \max\{f_t(x_i) \mid i \leqslant 2s+1\}\}$ and for $m \notin E_{t,s+1}$, $D_{m,s+1,t+1} = D_{m,s+1,t}$.

On the other hand if $E_{t,s+1}$ is finite, let $K_{s+1,t+1} = K_{s+1,t} - E_{t,s+1}$ and set $D_{m,s+1,t+1} = \{y \mid y \in D_{m,s+1,t} \,\&\, f_t(y) \leqslant \max\{f_t(x_i) \mid i \leqslant 2s+1\}\}$ for $m \notin E_{t,s+1}$ and $D_{m,s+1,t+1} = D_{m,s+1,t}$ for $m \in E_{t,s+1}$.

Let k_{s+1} be the least $m \in M_{s+1}$ that is greater than k_s. Now by construction, if $E_{t,s+1}$ is finite for $t \leqslant s$, then f_t is bounded on $B_{k_{s+1},s+1}$ by $\max\{f_t(x_i) \mid i \leqslant 2s+1\}$, on the other hand if $E_{t,s+1}$ is infinite then $f_t(y) \geqslant \max\{f_t(x_i) \mid i \leqslant 2s+1\}$ for $y \in B_{k_{s+1},s+1}$. Choose $y_1 < y_2$ in $B_{k_{s+1},s+1}$ such that $f_t(y_2) \geqslant f_t(y_1)$ for all $t \leqslant s$; this is possible as any infinite set has an infinite subset on which all of a finite number of functions are monotone. Set $x_{2(s+1)} = y_1$ and $x_{2(s+1)+1} = y_2$ and this completes the induction step and the construction.

The set A is clearly not 1-1 as A meets A_{k_s} in a set of 2 elements. Every total recursive function is either monotone or bounded on A, for let f_{t_0} be some recursive function. If there exists $s_0 > t_0$ such that E_{t_0,s_0} is finite then for all $s \geqslant s_0$, $B_{k_s,s} \subseteq B_{k_{s_0},s_0}$ and so $f_{t_0}(y) \leqslant \max\{f_{t_0}(x_i) \mid i \leqslant 2s_0 - 1\}$ for all $y \in A$ and $y > x_{2s_0-1}$. On the other hand if $E_{t_0,s}$ is always infinite for $s \geqslant t_0$ then the choices of x_{2s}, x_{2s+1} in $B_{k_s,s}$ keep f_{t_0} monotone on A.

A is therefore not 1-1, and every recursive function is either monotone or bounded on A. This, as was shown earlier, implies that A is a monotone set that is not 1-1.

This establishes that monotone does not imply 1-1. For the converse, we will need the following lemma. An important distinction between the requirements of a recursive function being monotone and 1-1 will be exploited in this construction. This is that the image of a recursive function monotone on a set is Turing reducible to the set. The same is not true of a recursive function 1-1 on a set. Non-monotonicity can therefore be engineered into a construction by attempting to keep the range not Turing reducible to the domain. This is the strategy of our next construction.

LEMMA 2.2. *Let f be* 1-1, *recursive and suppose that* $f(S) \not\leq_T S$ *for some* $S \subseteq \omega$. *If g is recursive and* $f(S \cap g^{-1}(k)) \leq_T S$ *for all k then there is a subset* $A \subseteq S$ *such that g is* 1-1 *on A and* $f(A) \not\leq_T A$.

PROOF. The notation is simplified by taking $S = \omega$, the relativisation to arbitrary S being easily accomplished. Let T be the binary tree of finite sequences of 0's and 1's. For any finite set F, let T'_F be the subtree of all finite sequences σ such that (i) the function g is 1-1 on $\{n \mid \sigma(n) = 0\}$ and (ii) for all n, $\sigma(n) = 0$ implies that $g(n) \notin F$. Sequences of T'_F are finite segments of subsets of ω on which the function g is 1-1 and F contains a finite set of g values forbidden to these sets. We shall define the segments σ_e, $e \in \omega$ of an infinite path in T' and sets F_e, $e \in \omega$, such that $\sigma_{e'} \in T_{F'_e}$ for all $e' \geq e$. The set A is defined by $m \in A$ if and only if $\sigma_m(m) = 0$. The object of the choice of σ_e is to ensure that $f(A) \neq \{e\}^A$.

We first establish the following proposition crucial to the induction step of the construction of σ_e.

$$\forall T'_F \forall \sigma \in T'_F$$

either

$$\exists m \exists \alpha \exists G \big(\sigma \subseteq \alpha \in T'_{F\cup G} \,\&\, \forall \beta \in T'_{F\cup G}\big(\alpha \subseteq \beta \to \{e\}^{\beta}(m)\ddagger\big)\big)$$

or

$$\exists m \exists \alpha \big(\sigma \subseteq \alpha \,\&\, \alpha \in T'_F \,\&\, \{e\}^{\alpha}(m)\downarrow$$
$$\text{and disagrees with the representing function of } \{f(k) \mid \alpha(k) = 0\}\big).$$

We shall show that if both parts of the disjunct are false then $f(\omega)$ is recursive. Given m, the negation of the first part of the disjunct implies that for some α with $\alpha \in T'_F$ extending σ, $\{e\}^{\alpha}(m)\downarrow$ and this is an r.e. search. When such an α is discovered, if $\{e\}^{\alpha}(m) = 0$, then the negation of the second part of the disjunct implies that $m \in f(\omega)$. If $\{e\}^{\alpha}(m) = 1$ then for all α' extending α, with $\alpha' \in T'_F$, $\{e\}^{\alpha'}(m) = 1$ and so again using the negation of the second disjunct, one has $m \neq f(k)$ when $\alpha'(k) = 0$. This implies that if m is in the range of f, then $m = f(k)$ for a k to which an extension of α by α' cannot be made in T'_f. This happens only if k has a forbidden g value, that is, $g(k) \in \{g(x) \mid \alpha(x) = 0\} \cup F$. Now let $D = \{g(x) \mid \alpha(x) = 0 \text{ and } \sigma(x)\downarrow\}$ and by the negation of the first part of the disjunct replacing G by D seek $\alpha' \in T'_{F\cup D}$ such that $\{e\}^{\alpha'}(m)\downarrow$ and $\sigma \subseteq \alpha'$. If $\{e\}^{\alpha'}(m) = 0$, $m \in f(\omega)$, but if $\{e\}^{\alpha'}(m) = 1$ then by the same argument as for α, applying the negation of the second disjunct to F and not $F \cup D$, one deduces that if $m = f(k)$, $g(k) \in \{g(x) \mid \alpha'(x) = 0\} \cup F$. Hence $g(k) \in \{g(x) \mid \alpha(x) = 0$ and $\alpha'(x) = 0\} \cup F$, but we forbade α' from taking on any g value that occurred in α after its extension from σ. Hence $g(k) \in \{g(x) \mid \alpha(x) = 0\} \cup F$. This is a

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prespecified finite set of g values and since $f(\omega \cap g^{-1}(k))$ is recursive for all k, this is a recursive set which may be searched for m.

The definition of the sequence σ_e can now be accomplished. Let σ_{-1} be the empty sequence, and F_{-1} the empty set. Suppose σ_{e-1} and F_{e-1} are defined. Applying the proposition just demonstrated to σ_{e-1} and F_{e-1}, if the first part of the disjunct holds, let $\sigma_e = \alpha$ and $F_e = F_{e-1} \cup G$. If the first part fails, let α be as in the second part and set $\sigma_e = \alpha$, $F_e = F_{e-1}$. Defining A by $m \in A$ if and only if $\sigma_m(m) = 0$, gives by construction a set on which g is 1-1 and $f(A) \nleq_T A$.

THEOREM 2.3. 1-1 *does not imply monotone.*

PROOF. Let f_n, $n \in \omega$, be an enumeration of the recursive functions and let f be any 1-1 recursive function for which $f(\omega)$ is not recursive. We define a decreasing sequence of subsets A_n of ω such that $f(A_n) \nleq_T A_n$ and f_n is constant or 1-1 on A_n. Let $A_{-1} = \omega$, and suppose A_{n-1} is defined and $f(A_{n-1}) \nleq_T A_{n-1}$. If there exists k such that $f(A_{n-1} \cap f_n^{-1}(k)) \nleq_T A_{n-1}$ then let $A_n = A_{n-1} \cap f_n^{-1}(k)$ and since $A_n \leqslant_T A_{n-1}$, $f(A_n) \nleq_T A_n$. f_n is constant on A_n in this case. On the other hand if for all k, $f(A_{n-1} \cap F_n^{-1}(k)) \leqslant_T A_{n-1}$ then by Lemma 2 there exists $A_n \subseteq A_{n-1}$ such that f_n is 1-1 on A_n and $f(A_n) \nleq_T A_n$.

Now choose an infinite sequence of pairs of numbers $x_{2n}, x_{2n+1} \in A_n$ with $x_{2n} < x_{2n+1} < x_{2n+2}$ such that $f(x_{2n+1}) < f(x_{2n})$. This is possible for as $f(A_n) \nleq_T A_n$, f is not eventually monotone on A_n. The set $A = \{x_n \mid n \in \omega\}$ has f not eventually monotone on it and as A is eventually contained in each A_n, every total recursive function is eventually constant on it or eventually 1-1 on it. A is therefore a 1-1 set that is not monotone.

3. Co-r.e. results

The first result of this section is that co-monotone and co-1-1 are implied by maximal. This is the original observation due to Owings from which this work began. It is clear from section one that co-monotone and co-1-1 imply both r-maximal and dense simple. We show next that co-monotone implies co-1-1 and vice versa. It is then shown that co-1-1 is preserved under major subsets and so lies strictly below maximal. Finally we show that r-maximal and dense simple do not imply co-monotone.

THEOREM 3.1. *Maximal implies co-monotone and co-*1-1.

PROOF. Let $M(s)$, $s \in \omega$, be an enumeration of a maximal set M and let f be any total recursive function. Define the r.e. set

$$W = \{x \mid \exists s(x \notin M(s) \,\&\, \forall y < x(y \notin M(s) \to f(y) < f(x)))\}.$$

If f has an infinite range on M' then $M' \subseteq {}^*W$ and f is eventually both 1-1 and monotone on M'. On the other hand if f has a finite range on M' then f must be eventually constant on M'. M is therefore both co-monotone and co 1-1.

The following lemma turns out to be very important and is frequently used in the results of this section.

LEMMA 3.1. *If W is r.e. and $\{A(i)\}$ is a disjoint recursive array, then either* (i) *or* (ii) *holds*:

(i) *there is a disjoint recursive array $\{B(i)\}$ with $\bigcup_i B(i) = \bigcup_i A(i)$ and $B(i) \cap W' = \varnothing$ for all i*;

(ii) *there is a recursive function $b(i)$ such that $\forall x(x \in A(i) \cap W' \to x \leqslant b(i))$ for almost all i.*

PROOF. Let $W(s)$ and $A(i, s)$ be enumerations of W and $A(i)$. Elements enumerated in $A(i)$ at stage s, $A(i, s+1) - A(i, s)$, are put into $B(a(s, i), s+1)$. Furthermore let $c(s, i) = \mu x_{x \leqslant s}(x \in B(i, s) - W(s))$. Let $a(0, i) = i$ and at stage $s+1$ let

$$a(s+1, i) = \begin{cases} a(s, i) - 1 & \text{if } \exists j < a(s, i)(c(s+1, j) \neq c(s, j)) \\ a(s, i) & \text{otherwise} \end{cases}$$

The $B(i)$'s form disjoint recursive array and note that $a(s, i) = j + i \leqslant j + s$. It is clear that $\exists i \forall s(a(s, i) > k)$ just if $\lim_s c(s, j)$ exists for all $j \leqslant k$, hence just if $B(j) \cap W' \neq \varnothing$ for all $j \leqslant k$. Thus if $\forall k \exists i \forall s(a(s, i) > k)$ then $\{B(i)\}$ satisfies condition (i). Otherwise, let $n = \mu k \forall i \exists s(a(s, i) \leqslant k)$ and let $m = \mu i \forall s(a(s, i) \geqslant n)$. Let $b(i) = 0$ for $i < m$ and $b(i) = \mu s(a(s, i) = n)$ for $i \geqslant m$. $b(i)$ is recursive and if $i \geqslant m$ and $x \in A(i) \cap W'$ then $x \leqslant b(i)$, for if not then x will be enumerated into $A(i)$ at a stage greater than x and hence exceeding $b(i)$ and so will be put into $B(n)$ which would then meet W' and this contradicts the nonexistence of $\lim_s c(s, n)$. Hence $b(i)$ satisfies condition (ii).

THEOREM 3.2. *Co-monotone implies co*-1-1.

PROOF. Suppose W is r.e., co-monotone and not co-1-1. Let f be total recursive such that f is neither eventually 1-1 nor eventually constant on W'. Let $A(k)$ be the disjoint recursive array given by $A(k) = f^{-1}(k)$. Apply Lemma 3.1 to W and the array $A(k)$. Since the union of the $A(k)$'s is ω, condition (i) would contradict the r-cohesiveness of W' and so condition (ii) must hold. But then any recursive function g defined to be strictly decreasing on $A(k) \cap \{x \mid x \leqslant b(k)\}$ for each k will contradict M' monotone. Hence co-monotone implies co-1-1.

The proof in the other direction uses the following lemma.

LEMMA 3.2. *Let R be recursive, X a* 1-1 *subset of R, and f a recursive function which is* 1-1 *on R and for which $f(R)$ is recursive. f is then eventually monotone on X.*

PROOF. We present the details for the case $R = \omega$. The proof is easily relativised to any recursive set R.

We define disjoint recursive arrays $C(i)$ and $D(i)$ for all i by induction as follows:

$$C(i) = \left\{\mu x\left(x \notin \bigcup_{j<i} D(j)\right)\right\} \cup \left\{x \notin \bigcup_{j<i} D(j) \mid x < \max \bigcup_{j<i} D(j)\right\},$$

$$D(i) = C(i) \cup \left\{y \notin \bigcup_{j<i} D(j) \mid \exists x(x \in C(i) \,\&\, x < y \,\&\, f(y) < f(x))\right\}.$$

Note that for each $x \in C(i)$ we can determine which numbers below $f(x)$ are in the range of f, since the range is recursive, and hence $D(i)$ is recursive. The construction gives $0 < |D(i)| < \infty$ with $\bigcup_i D(i) = \omega$ and $D(i) \cap D(j) = \varnothing$ for $i \neq j$. In addition for each pair x, y with $x < y$ and $f(y) < f(x)$ we have $x \in D(i)$ for some i and $y \in D(i)$ or $D(i-1)$. This follows because if $x \in D(i)$ and $y \notin \bigcup_{j<i} D(j)$ then the construction places y in $D(i)$. This is immediate if $x \in C(i)$; on the other hand if $x \in D(i)$ because $z < x$, $z \in C(i)$ and $f(x) < f(z)$ then $z < y$ and $f(y) < f(z)$ and $y \in D(i)$. If however $y \in \bigcup_{j<i} D(j)$, say $y \in D(k)$ then as $x < y$ the construction puts x into $C(k+1)$ and hence $D(k+1)$.

Now since X is 1-1, it is r-cohesive and so it is contained modulo finite sets in the recursive set $\bigcup_i D(2i)$ or its complement $\bigcup_i D(2i+1)$. In any case almost all pairs x, y in X on which f reverses order must have both x and y in the same $D(i)$. Therefore if f reverses order on infinitely many pairs in X the recursive function that assigns to $x \in D(i)$ the value i, would not be eventually 1-1 on X, contradicting X being a 1-1 set. Hence f must be monotone on X.

THEOREM 3.3. *co*-1-1 *implies co-monotone.*

PROOF. Let W be a co-1-1 set and let f be a total recursive function that is not eventually constant on W'. Let $A(i) = \{x \mid f(x) = i\}$; W co-1-1 implies that for

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almost all i $|A(i) \cap W'| \leqslant 1$ with the equality holding infinitely often. W' r-cohesive implies that condition (ii) of Lemma 3.1 holds for the set W and the array $A(i)$. Let $b(i)$ be the recursive function of condition (ii) of Lemma 3.1, and suppose without loss of generality that $|A(i) \cap W'| \leqslant 1$ for all i. Given an enumeration $W(s)$ of W, define recursive functions h and g as follows:

$$h(i) = \mu s(|\{x \mid x \leqslant b(i) \,\&\, x \in A(i) - W(s)\}| \leqslant 1),$$

$$g(i) = \begin{cases} 2i & \text{if } \{x \mid x \leqslant b(i) \,\&\, x \in A(i) - W(h(i))\} = \varnothing, \\ 2y+1 & \text{where } y = \mu x(x \mid x \leqslant b(i) \,\&\, x \in A(i) - W(h(i)) \text{ otherwise.} \end{cases}$$

Note that $2y + 1 \in$ range g if and only if $g(f(y)) = 2y + 1$. Let $R = \{x \mid 2x + 1 \in \text{range } g\}$; R is recursive, W' is a 1-1 subset of R, f is 1-1 on R, and $f(R)$ is recursive for $i \in f(R)$ just if $\{x \mid x \leqslant b(i) \,\&\, x \in A(i) - W(h(i))\} \neq \varnothing$. Hence by Lemma 3.2, f is eventually monotone on W'. As f was an arbitrary recursive function not eventually constant on W', this implies that W is co-monotone.

The next theorem implies that co-1-1 and hence also co-monotone are strictly below maximal. As is now standard, $A \subseteq {}^*B$ denotes inclusion modulo finite sets, that is that $A - B$ is finite. Recall that C is a major subset of D if $C \subseteq D$, $D - C$ is infinite, and for every r.e. set W we have $D' \subseteq {}^*W$ implies $C' \subseteq {}^*W$.

THEOREM 3.4. *If W is co-1-1 and S is a major subset of W then S is co-1-1.*

PROOF. Let f be any recursive function. If f is eventually constant on W' then for some k, $W' \subseteq {}^*f^{-1}(k)$ and so $S' \subseteq {}^*f^{-1}(k)$, and f is eventually constant on S'. Suppose therefore that f is eventually 1-1 on W. Again, W' r-cohesive gives by condition (ii) of Lemma 3.1, a recursive function $b(i)$ such that $(f(x) = i \,\&\, x \in W') \to x \leqslant b(i)$, say for $i \geqslant m$. As f is eventually 1-1 on W', $|f^{-1}(j) \cap W'| < \infty$ for all j and hence $W' \subseteq {}^* \bigcup_{i \geqslant m} f^{-1}(i)$. Let $W(s)$ be some enumeration of W and for $i \geqslant m$ let

$$s(i) = \mu s(|\{x \mid x \leqslant b(i) \,\&\, f(x) = i \,\&\, x \notin W(s)\}| \leqslant 1).$$

$s(i)$ is partial recursive and for $i \geqslant m$, $s(i)\downarrow$. We now define a recursive set R on which f is 1-1;

$$R = \{x \mid f(x) \geqslant m \,\&\, x \leqslant b(f(x)) \,\&\, x \notin W(s(f(x)))\}.$$

Now $W' \subseteq {}^*R$, so $S' \subseteq {}^*R$ and f is eventually 1-1 on S'. Hence S is co-1-1.

COROLLARY 3.1. *There are co-1-1 sets which are not maximal.*

PROOF. If M is maximal, it is co-1-1. Any major subset of M is a co-1-1 set that is not maximal. For the existence of a major subset see Lachlan (1968, p. 29).

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Finally we take up the question of showing that co-monotone and co-1-1 lie strictly above r-maximal and dense simple.

THEOREM 3.5. *r-maximal and dense simple does not imply co-monotone.*

PROOF. We shall construct an r-maximal dense simple set W that is not co-monotone. At stage s, we define $x(s, n)$ an increasing enumeration of a recursive subset of ω that will be referred to as numbers marked for the complement of W at stage s. All numbers less than or equal to s that are not marked for the complement at stage s are put into $W(s)$. $x(s, n)$ will be shown to converge to $x(n)$, the nth element of W' and the dense simplicity of W will be obtained by ensuring that $x(n)$ eventually dominates every total recursive function.

Let $\{A_n\}_{n\in\omega}$, be a recursive partition of ω, with A_n infinite for all n. At each stage s, we shall have before us an infinite matrix of numbers, with row n containing an increasing enumeration of $A_{y(s,n)}$ and $y(s, n)$ is an increasing function of f defined as part of the construction. $B(n, s)$ will be a finite subset of $A_{y(s,n)}$ and will contain the elements marked for W' at stage s in row n. We shall ensure that $2 \leqslant |B(n, s)| \leqslant 2^{n+1}$. The number n will be said to require attention at stage s through e if $e < n$, $B(n, s) \not\subseteq W_e(s)$ and

$$2\,|B(n, s) \cap W_e(s)| \geqslant |B(n, s)|\,.$$

We also define an e-state function W by

$$W(x, e, s) = \sum \{2^{e-z} \mid z \leqslant e \text{ and } B(x, s) \subseteq W_e(s)\}$$

and recursive functions $r(s, n)$ by

$$r(s, n) = \text{Max}\{\phi_e(n) \mid e < n \quad \text{and} \quad \phi_e(x) \text{ is defined at stage } s \text{ for all } x \leqslant n\}.$$

A recursive function g that is not monotone on W' will be constructed by defining g at each stage on the numbers marked for W'. Furthermore if a number is put into W and its g value is not yet defined then its g value is set to 0 on its entry into W. g will be defined to be monotonically decreasing on $B(n, s) \subset A_{y(s,n)}$ and as eventually $B(n, s)$ will converge to a set of cardinality exceeding 1, g will not be monotone on W'.

We are now ready to describe the construction:

Stage 0: Let $y(0, n) = n$ for all $n \in \omega$. Mark the first two elements of $A_{y(0,0)}$ for the complement, and inductively mark the first 2^{n+2} elements of $A_{y(0,n+1)}$ that exceed the marked elements of $A_{y(0,n)}$. Set $B(n, 0)$ equal to the marked elements of $A_{y(0,n)}$ and let $x(0, n)$ be an increasing enumeration of all the marked elements. Define $g(x)$ for x a marked element in $A_{y(0,n)}$ by $g(x) = 2^{n+1} - i$ if x is the ith marked element of $A_{y(0,n)}$.

Stage $s + 1$: If there is no n for which $x(s, n) < r(s + 1, n)$, go to stage $s + 2$. Otherwise let $n_0 = \mu n(x(s, n) < r(s + 1, n))$. The construction now proceeds in four parts.

(i) Let $e_0 = \mu e(e < n_0, \phi_e^{s+1}(x)\downarrow$ for $x \leq n_0$, $x(s, n_0) < \phi_e^{s+1}(n_0))$, and let m_0 be such that $x(s, n_0) \in A_{y(s,m_0)}$.

Find 2^{m_0+1} elements in $A_{y(s,m_0)}$ greater than $r(s + 1, n_0)$ and not in the domain of g, or in any W at stage $s + 1$. Mark these elements for the complement and erase all previous marks in $A_{y(s,m_0)}$. Now inductively for $m > m_0$, find 2^{m+1} elements of $A_{y(s,m)}$ that are not in the domain of g, or in W_e or W so far and exceed the newly marked elements of $A_{y(s,m-1)}$ and mark them for W', erasing all marks on the old marked elements of $A_{y(s,m)}$. Let $\hat{B}(m, s) = B(m, s)$ for all $m < m_0$ and for $m \geq m_0$, $\hat{B}(m, s)$ is equal to the newly marked elements of $A_{y(s,m)}$. Define g on the ith element of $\hat{B}(m, s)$ to be $2^{m+1} - i$ for $m \geq m_0$.

(ii) For any $n < m_0$, such that n requires attention at stage through e, let e_1 be the least such e and erase the marks on elements of $\hat{B}(n, s)$ that do not belong to $W_{e_1}(s)$. Correspondingly set $\hat{B}(n, s)$ to equal its intersection with $W_{e_1}(s)$.

(iii) If $\exists e \exists n(n > e,\ W(e, e, s) < W(n, e, s))$, let e_2 be the least such e and n_2 the least associated n. Set $y(s + 1, e_2 + k) = y(s + 1, n_2 + k)$ for all $k \in \omega$. Furthermore for all $k \in \omega$, let $B(e_2 + k, s + 1)$ be the first 2^{e_2+k+1} elements of $\hat{B}(e_2 + k, s)$ if $\hat{B}(e_2 + k, s)$ has more than 2^{e_2+k+1} elements and otherwise let $\hat{B}(e_2 + k, s + 1) = \hat{B}(e_2 + k, s)$. Erase marks accordingly. Also for $e < e_2$ let $B(e, s + 1) = B(e, s)$ and $y(s + 1, e) = y(s, e)$.

(iv) Put into W all $x \leq s$ such that $x \notin \bigcup_n B(n, s + 1)$ and let $x(s + 1, n)$ be an increasing enumeration of $\bigcup_n B(n, s + 1)$.

This completes the construction; we now show that W has the desired properties.

LEMMA 3.5.1. $\forall n \exists s \forall s'(s' > s \rightarrow y(s', n) = y(s, n)$ *and* $B(n, s') = B(n, s))$.

PROOF. Proceeding by induction on n, suppose that for all $k < n$ and $s \geq s_0$ we have $y(s, k) = y(s_0, k)$ and $B(k, s) = B(k, s_0)$. Let $s_1 \geq s_0$ be such that for all $s \geq s_1$, $r(s, k) = r(s_1, k)$ for all $k \leq 2^{n+2} - 2$. After stage s_1, n may be m_0 of part (i) of the construction on behalf of some $k \leq 2^{n+2} - 2$. But for each such k, n could be m_0 at most once. Hence there exists a stage $s_2 \geq s_1$ such that for $s \geq s_2$, $\hat{B}(n, s) = B(n, s)$ at the end of part (i) of stage s of the construction. Once part (i) is ineffective on row n, the movement of higher rows down to row n in part (iii) does not disturb the domination of part (i). Such movement only occurs on behalf of higher n states of which there are only finitely many. The cutting down of $B(n, s)$ in part (ii) only raises the n state and can occur only finitely often, and so there must exist a stage s_3 such that for all $s \geqslant s_3$, $B(n, s) = B(n, s_3)$ and $y(s, n) = y(s_3, n)$.

LEMMA 3.5.2. $\forall s(|B(n, s)| \geqslant 2)$.

PROOF. Let $B(n_0, s_0) \subset A_{y(s_0, n_0)}$ be the limit of $B(n_0, s)$ with $y(s_0, n_0) = k_0$. For $s \leqslant s_0$, let $h(s)$ be the row number n to which A_{k_0} is assigned at stage s, hence $y(s, h(s)) = k_0$. $h(s)$ is a decreasing function of s with $h(0) = k_0$ and the limiting value of h being n_0. Let $s_1 + 1$ be the largest stage $\leqslant s_0$ such that part (i) of the construction is effective for a row $m_0 \leqslant h(s_1)$, and if there is no such stage let s_1 be 0. At the end of part (i) of stage s_1, $\hat{B}(h(s_1), s_1)$ has a cardinality of $2^{h(s_1)+1}$ and and for all s, if $s_1 < s \leqslant s_0$, then $B(n_0, s_0) \subset B(h(s), s) \subseteq \hat{B}(h(s_1), s_1)$. Let a_s be the cardinality of $B(h(s), s)$. We have $a_{s_1+1} = 2^{h(s_1+1)+1}$; now let $s_1 + 1$ be the greatest stage $s \leqslant s_0$ such that $a_{s+1} = 2^{h(s+1)+1}$. The only way for $B(h(s+1), s+1) \neq B(h(s), s)$ for $s \geqslant s_2 + 1$ is for part (ii) of the construction to give attention to $h(s)$ of stage $s + 1$. This attention is given at most once on account of $e < h(s)$ and cuts the size of $B(h(s), s)$ by a factor of at most $\frac{1}{2}$ its previous size. Reducing a set of cardinality $2^{h(s_2+1)+1}$ by a factor of $\frac{1}{2}$ its previous size, at most $h(s_2 + 1)$ times cannot reduce its cardinality below 2. Hence $|B(n_0, s_0)| \geqslant 2$.

This establishes that W' is infinite, that $x(s, n)$ converges to a function $x(n)$ which by part (i) of the construction must dominate every total recursive function. By construction g is strictly decreasing on $B(n, s)$ and so as $|B(n, s)| \geqslant 2$ in the limit, we have the W is not comonotone. It remains to show that W is r-maximal.

LEMMA 3.5.3. *W is r-maximal.*

PROOF. Suppose R is recursive and W_{e_0}, W_{e_1} are respectively enumerations on R and R', with $e_0 < e_1$. Let $W(n, e_1) = \lim_s W(n, e_1, s)$, and because of the maximisation of the e state function there must exist m_0 such that for all $m \geqslant m_0$, $W(m, e_1) = W(m_0, e_1)$. This implies that for all $m \geqslant m_0$, $B(m) \subseteq W_{e_0}$ or $B(m) \subseteq W_{e_1}$ according as $B(m_0) \subseteq W_{e_0}$ or $B(m_0) \subseteq W_{e_1}$. Since W_{e_0} and W_{e_1} are R and R' one must have either

$$|W_{e_0} \cap B(m_0)| \geqslant \tfrac{1}{2}|B(m_0)| \quad \text{or} \quad |W_{e_1} \cap B(m_0)| \geqslant \tfrac{1}{2}|B(m_0)|$$

and part (ii) of the construction will then force $B(m_0) \subseteq W_{e_0}$ or $B(m_0) \subseteq W_{e_1}$. This implies that $\bigcup_m B(m) \subseteq {}^*W_{e_0}$ or $\bigcup_m B(m) \subseteq {}^*W_{e_1}$ and hence that W' is r-cohesive.

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4. Related questions

If the following conjecture were true then it could be used in the proof of Theorem 2.3 to provide a recursive sequence of sets $\{A_n\}$ in place of the consequence provided by Lemma 2.2.

CONJECTURE. *Suppose f and g are recursive, $f(g^{-1}(n))$ is recursive for all n, and $f(\omega)$ is not recursive. Then there is a recursive set R such that g is* 1-1 *on R and $f(R)$ is not recursive.*

Our construction of co-1-1 sets which are not maximal in Corollary 3.1 guarantees nevertheless a maximal superset. It would be interesting to know whether every co-1-1 set is contained in a maximal set.

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