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LETTER TO THE EDITOR

Dear Editor,

Correction to Bingham and Doney (1988)

In Bingham and Doney (1988) the following four typographical errors and one calculation error have been found.

(a) In the proof of Proposition 3, the new variable ρ_n should read as ρ_{n-1}.

(b) In the statement of Proposition 3 and the last line of its proof, the exponential order of the ρ_{n-1} term should read as $2n-3$ instead of $2n-1$. (This assertion is proved in Appendix A, but it can be easily checked from Bingham and Doney's expression.)

(c) The $\frac{1}{16}$ factor in the equation of item (ii) on page 129 should read as $1/16\pi$.

(d) The integral over ϕ in item (x) on page 130 should read as

$$\int_{\psi}^{\pi/2} \phi \sin\phi \cos\phi \, \mathrm{d}\phi,$$

i.e. the lower bound is ψ instead of 0.

(e) The integral in item (ix) on page 130 is missing a $\cos\psi$ term, i.e. it should read as

$$\frac{1}{8\pi^2}\int_0^{\pi/2} \psi \sin^3\psi \left(-\frac{1}{4}\pi + \frac{1}{2}\frac{\psi}{\sin^2\psi} + \frac{1}{2}\frac{\cos\psi}{\sin\psi}\right)\cos\psi \, \mathrm{d}\psi.$$

Evaluating this integral yields

$$\frac{5}{2048} - \frac{1}{64\pi^2}$$

(see Appendix A), and the corresponding third moment μ_3 becomes

$$\mu_3 = \frac{63}{512} - \frac{7}{32\pi^2} = 0.100\,88.$$

Appendix A

In this appendix we present the details of corrections (b) and (e).

(b) The Jacobian $J = \partial(v_1, \dots, v_n)/\partial(\rho_1, \dots, \rho_{n-1}, t)$ is illustrated as follows:

$$\begin{pmatrix} 2t\rho_1\rho_2^2\cdots\rho_{n-1}^2 & 2t\rho_2\rho_1^2\cdots\rho_{n-1}^2 & \cdots & 2t\rho_{n-1}\rho_1^2\cdots\rho_{n-2}^2 & \rho_1^2\rho_2^2\cdots\rho_{n-1}^2 \\ -2t\rho_1\rho_2^2\cdots\rho_{n-1}^2 & 2t\rho_2(1-\rho_1^2)\cdots\rho_{n-1}^2 & \cdots & 2t\rho_{n-1}(1-\rho_1^2)\cdots\rho_{n-2}^2 & (1-\rho_1^2)\rho_2^2\cdots\rho_{n-1}^2 \\ 0 & -2t\rho_2\rho_3^2\cdots\rho_{n-1}^2 & \cdots & 2t\rho_{n-1}(1-\rho_2^2)\cdots\rho_{n-2}^2 & (1-\rho_2^2)\rho_3^2\cdots\rho_{n-1}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2t\rho_{n-1}(1-\rho_{n-2}^2) & (1-\rho_{n-2}^2)\rho_{n-1}^2 \\ 0 & 0 & \cdots & -2t\rho_{n-1} & 1-\rho_{n-1}^2 \end{pmatrix}.$$

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Performing some elementary row operations, i.e. replacing row 2 by the sum of row 1 and row 2, replacing row 3 by the sum of the transformed row 2 and row 3, and so on until the nth row, we obtain the following upper triangular matrix J':

$$J' = \begin{pmatrix} 2t\rho_1\rho_2^2\cdots\rho_{n-1}^2 & 2t\rho_2\rho_1^2\cdots\rho_{n-1}^2 & \cdots & 2t\rho_{n-1}\rho_1^2\cdots\rho_{n-2}^2 & \rho_1^2\rho_2^2\cdots\rho_{n-1}^2 \\ 0 & 2t\rho_2\rho_3^2\cdots\rho_{n-1}^2 & \cdots & 2t\rho_{n-1}\rho_2^2\cdots\rho_{n-2}^2 & \rho_2^2\rho_3^2\cdots\rho_{n-1}^2 \\ 0 & 0 & \cdots & 2t\rho_{n-1}\rho_3^2\cdots\rho_{n-2}^2 & \rho_3^2\rho_4^2\cdots\rho_{n-1}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2t\rho_{n-1} & \rho_{n-1}^2 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Hence,

$$|J| = |J'| = 2^{n-1}t^{n-1}\rho_1\rho_2^3\cdots\rho_{n-1}^{2n-3}.$$

(e) We start as in Bingham and Doney (1988), i.e. the limits are $0 < w < u < 1$ and the Jacobian is $1/u$. To solve

$$\int_0^1\int_0^1 uv^3\sin u \arcsin uv \,\mathrm{d}u\,\mathrm{d}v = \int_0^1 w^3\arcsin w\,\mathrm{d}w\int_w^1 \frac{\arcsin u}{u^3}\,\mathrm{d}u,$$

we first calculate

$$I = \int_w^1 \frac{\arcsin u}{u^3}\,\mathrm{d}u.$$

Letting $\varphi = \arcsin u$, we have

$$\begin{aligned} I &= \int_{\arcsin w}^{\pi/2} \frac{\varphi}{\sin^3\varphi}\cos\varphi\,\mathrm{d}\varphi \\ &= \int_{\arcsin w}^{\pi/2} \varphi\csc^2\varphi c\tan\varphi\,\mathrm{d}\varphi \\ &= -\int_{\arcsin w}^{\pi/2} \varphi c\tan\varphi\,\mathrm{d}(c\tan\varphi) \\ &= -\left(\varphi c\tan^2\varphi\,\Big|_{\arcsin w}^{\pi/2} - \int_{\arcsin w}^{\pi/2} c\tan\varphi(c\tan\varphi - \varphi\csc^2\varphi)\,\mathrm{d}\varphi\right) \\ &= \arcsin w\frac{1-w^2}{w^2} + \int_{\arcsin w}^{\pi/2} c\tan^2\varphi\,\mathrm{d}\varphi - \int_{\arcsin w}^{\pi/2}\frac{\varphi}{\sin^3\varphi}\cos\varphi\,\mathrm{d}\varphi \\ &= \arcsin w\frac{1-w^2}{w^2} + \frac{\sqrt{1-w^2}}{w} - \left(\frac{\pi}{2} - \arcsin w\right) - I. \end{aligned}$$

This implies that

$$I = -\frac{\pi}{4} + \frac{1}{2}\frac{\arcsin w}{w^2} + \frac{1}{2}\frac{\sqrt{1-w^2}}{w}.$$

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Thus, we have

$$\begin{aligned}
&\int_0^1\int_0^1 uv^3 \sin u \arcsin uv \,\mathrm{d}u\,\mathrm{d}v \\
&\quad = \int_0^1 w^3 \arcsin w \,\mathrm{d}w \biggl(-\frac{\pi}{4} + \frac{1}{2}\frac{\arcsin w}{w^2} + \frac{1}{2}\frac{\sqrt{1-w^2}}{w}\biggr) \\
&\quad = \int_0^{\pi/2} \psi \sin^3\psi \biggl(-\frac{\pi}{4} + \frac{1}{2}\frac{\psi}{\sin^2\psi} + \frac{1}{2}\frac{\cos\psi}{\sin\psi}\biggr)\cos\psi \,\mathrm{d}\psi \\
&\quad = -\frac{\pi}{4}I_4 + \frac{1}{2}I_2 + \frac{1}{32}I_{10} \\
&\quad = \frac{5}{256}\pi^2 - \frac{1}{8},
\end{aligned}$$

where I_2, I_4, and I_{10} are given on page 129 of Bingham and Doney. Hence, the value of the integral in item (ix) on page 130 is

$$\frac{1}{8\pi^2}\biggl(\frac{5}{256}\pi^2 - \frac{1}{8}\biggr) = \frac{5}{2048} - \frac{1}{64\pi^2}.$$

References

Bingham, N. H. and Doney, R. A. (1988). On higher-dimensional analogues of the arc-sine law. *J. Appl. Prob.* **25,** 120–131.

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