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ON FINITE LOOPS WHOSE INNER MAPPING GROUPS ARE ABELIAN II

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If the inner mapping group of a loop is a finite Abelian group, then the loop is centrally nilpotent. We first investigate the structure of those finite Abelian groups which are not isomorphic to inner mapping groups of loops and after this we show that if the inner mapping group of a loop is isomorphic to the direct product of two cyclic groups of the same odd prime power order p^n, then our loop is centrally nilpotent of class at most $n+1$.

1. INTRODUCTION

In this paper, which is a continuation of [**9**], we are interested in the following two questions:

1. Which Abelian groups are/are not isomorphic to inner mapping groups of loops?
2. If Q is a finite loop with an Abelian inner mapping group $I(Q)$, then Q is centrally nilpotent. How does the structure of $I(Q)$ influence the nilpotency class of Q?

The purpose of this paper is to show that the following result holds:

Let $k > l \geqslant 0$ be integers and let p be an odd prime number. If Q is a finite loop, then *the inner mapping group* $I(Q)$ *is never isomorphic to the direct product* $C_{p^k} \times C_{p^l}$.

By using this result we are able to show that if $I(Q)$ is isomorphic to $C_{p^n} \times C_{p^n}$, then Q is a centrally nilpotent loop of class at most $n+1$. The reader interested in the relation between loop theory and group theory is advised to look at articles [**8, 9, 11**]. Some of the main links between the two theories are explained in Sections 2 and 4 of this paper. Section 5 contains some remarks and a brief discussion on similarities between capable groups and inner mapping groups of loops.

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2. Basic definitions and results

If Q is a loop, then the two mappings $L_a(x) = ax$ and $R_a(x) = xa$ are permutations on Q for every $a \in Q$. The permutation group $M(Q) = \langle L_a, R_a : a \in Q \rangle$ is called the multiplication group of Q. The stabiliser of the neutral element of Q is denoted by $I(Q)$ and $I(Q)$ is called the inner mapping group of Q (if Q is a group, then $I(Q)$ is just the group of inner automorphisms of Q). If we write $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$, then the commutator subgroup $[A, B] \leqslant I(Q)$ and A and B are left transversals to $I(Q)$ in $M(Q)$. If $1 < K \leqslant I(Q)$, then K is not a normal subgroup of $M(Q)$.

If we replace $M(Q)$ by G and $I(Q)$ by H, then in group theory we are dealing with the following situation: H is a subgroup of G and A and B are two left transversals to H in G. We assume that $[A, B] \leqslant H$ and we say that A and B are H-connected transversals. By H_G we denote the core of H in G, that is, the largest normal subgroup of G contained in H. If $H_G = 1$, we say that H is core-free in G. The relation between multiplication groups of loops and connected transversals is given by the following result that was proved by Kepka and Niemenmaa [**11**] in 1990.

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H satisfying $H_G = 1$ and H-connected transversals A and B such that $G = \langle A, B \rangle$.*

In Lemmas 2.2 and 2.3 (for the proofs, see [**11**, Lemma 2.5] and [**7**, Lemma 1.4]) we assume that A and B are H-connected transversals in G.

Lemma 2.2. *If $H_G = 1$, then $N_G(H) = H \times Z(G)$.*

Lemma 2.3. *If $H_G = 1$, then $Z(G) \subseteq A \cap B$.*

In the following lemmas we assume that $G = \langle A, B \rangle$. As usual, p denotes a prime number and C_n denotes a cyclic group of order n.

Lemma 2.4. *If $H \cong C_p \times C_p$, then $G' \leqslant N_G(H)$.*

Lemma 2.5. *If H is a cyclic subgroup of G, then $G' \leqslant H$.*

Lemma 2.6. *If G is a finite group and H is Abelian, then H is subnormal in G.*

For the proofs, see [**12**, Lemma 4.2], [**7**, Theorem 2.2], [**8**, Lemma 2.1].

Lemma 2.7. *Let H be Abelian and $H_G = 1$. Then the core of $HZ(G)$ in G contains $Z(G)$ as a proper subgroup.*

Proof: By Lemmas 2.2 and 2.6, $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. Then assume that the core of $HZ(G)$ in G equals $Z(G)$. By Lemma 2.2,

$$N_{G/Z(G)}(HZ(G)/Z(G)) = HZ(G)/Z(G) \times Z(G/Z(G)).$$

If we write $M/Z(G) = Z\big(G/Z(G)\big)$, then $N_G\Big(H\big(Z(G)\big)\Big) = HM$, where M is normal in G, $Z(G)$ is a proper subgroup of M and $H \cap M = 1$. Now $HM = CH = DH$, where $C \subseteq A$ and $D \subseteq B$. By Lemma 2.3, $Z\big(G/Z(G)\big) \subseteq AZ(G)/Z(G) \cap BZ(G)/Z(G)$. Thus it is clear that $M \subseteq CZ(G) \cap DZ(G)$. If $m \in M$, then $m = cz_1 = dz_2$, where $c \in A$, $d \in B$ and z_1 and z_2 are from $Z(G)$. If $x \in A \cup B$, then $[x, m] \in M \cap H = 1$. Thus $C_G(m) \geqslant \langle A, B\rangle = G$. But then $M = Z(G)$, a contradiction. The proof is complete. ∎

Finally, we need the following well known result on commutator calculus ([**6**, pp. 253–254]).

LEMMA 2.8. *If $[a, b]$ commutes with a and b, then $(ab)^n = a^n b^n [b, a]^{\binom{n}{2}}$.*

3. Main Theorems

Throughout this section we assume that G is a finite group, H is an Abelian subgroup of G and there exist H-connected transversals A and B such that $G = \langle A, B\rangle$.

THEOREM 3.1. *Let $H \cong C_{p^k} \times C_{p^l}$ and assume that p is an odd prime number and $k > l \geqslant 0$. Then H_G is not trivial.*

PROOF: Our proof is by induction on k. If $k = 1$, then the claim follows from Lemma 2.5. Then assume that $k \geqslant 2$ and G is a minimal counter example. As we now assume that $H_G = 1$, it follows from Lemmas 2.2 and 2.6 that $N_G(H) = H \times Z(G)$ and $Z(G)$ is not trivial. Then let $z \in Z(G)$ and assume that $|z| = r$, where r is a prime number and consider the groups $G/\langle z\rangle$ and $H\langle z\rangle/\langle z\rangle$. If K denotes the core of $H\langle z\rangle$ in G, then $\langle z\rangle < K \leqslant H\langle z\rangle$. If $r \neq p$, then the Sylow p-subgroup P of K is normal in G. But then $H_G > 1$, which is not possible.

Thus we may assume that $r = p$. Since the Frattini subgroup $\Phi(K) \leqslant H$ is normal in G, it follows that K is elementary Abelian. If $|K| = p^3$, then the core of HK/K in G/K is not trivial (as G is a minimal counter example) and the core of HK in G is larger than K. Thus it follows that $K = \langle x\rangle \times \langle z\rangle$, where $x \in H$ and $|x| = p$. We may also conclude that $k = l + 1$.

Since $C_G(K)$ is normal in $N_G(K) = G$, it follows that $C_G(x) = C_G(K)$ is normal in G. By Lemma 2.2,

$$N_{G/K}(HK/K) = N_{G/K}\big(H\langle z\rangle/K\big) = H\langle z\rangle/K \times Z(G/K).$$

If we write $Z(G/K) = M/K$, then $N_G\big(H\langle z\rangle\big) = HM$ and $H \cap M = \langle x\rangle$. Clearly, M is normal in G. By Lemma 2.7, W is equal to the core of HM in G contains M as a proper subgroup. As $K < M$, we conclude that $W = H_1M$, where $\langle x\rangle < H_1 \leqslant H$. Now $H/H_1 = H/H \cap W \cong HW/W$. As $H \cong C_{p^k} \times C_{p^{k-1}}$ and the core

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of
$HW/W = HM/W$ in G/W is trivial, it follows that $H_1 \cong C_{p^t} \times C_{p^{t-1}}$ and $k \geqslant t \geqslant 2$.

Now $T = W \cap C_G(x) = H_1M \cap C_G(x) = H_1(M \cap C_G(x))$ is normal in G. It is easy to see that $T' \leqslant W' \leqslant (HM)' \leqslant K$ and thus $T' \leqslant Z(T)$. Now p is odd and thus by Lemma 2.8, it follows that $E = \langle g \in T \mid g^p = 1 \rangle = \{g \in T \mid g^p = 1\}$. Obviously, E is normal in G. If Y denotes the core of HE in G, then $Y = H_2E$, where $H_2 \leqslant H_1$. It is clear that H_2 contains p-elements whose order is at least p^2. If $x \in Y$, then $x = yw$, where $y \in H_2$ and $w \in E$. Now $x^p = (yw)^p = y^pw^p[w,y]^{\binom{p}{2}} = y^p$ (since p is odd). Thus Y^p is a nontrivial normal subgroup of G and as $Y^p \leqslant H$, we have a contradiction with $H_G = 1$. ∎

Now we write $N_1(H) = N_G(H)$ and $N_{i+1}(H) = N_G(N_i(H))$ for each $i \geqslant 1$. By Lemma 2.6, the chain of subgroups $N_i(H)$ reaches G in a finite number of steps in the case that H is Abelian. The following theorem shows how many steps (at most) are needed in a special case.

Theorem 3.2. *Let p be an odd prime number. If $H \cong C_{p^t} \times C_{p^t}$, then $G' \leqslant N_t(H)$.*

Proof: If $t = 1$, then the claim is true by Lemma 2.4. Assume then that the claim is true for each $k < t$ and let $H \cong C_{p^t} \times C_{p^t}$. If $H_G > 1$, then from Theorem 3.1 it follows that $H/H_G \cong C_{p^l} \times C_{p^l}$, where $l < t$. Thus $(G/H_G)' \leqslant N_l(H/H_G)$ and we conclude that $G' \leqslant N_l(H) \leqslant N_t(H)$.

Thus we may assume that $H_G = 1$. Then $N_G(H) = H \times Z(G)$ and $Z(G) > 1$. Then consider the groups $G/Z(G)$ and $HZ(G)/Z(G)$. By Lemma 2.7, the core of $HZ(G)$ contains $Z(G)$ as a proper subgroup. Thus we have a normal subgroup L of G such that $Z(G) < L \leqslant HZ(G)$. Now consider the groups G/L and HL/L. By Theorem 3.1, we conclude that $L \cong (C_{p^r} \times C_{p^r}) \times Z(G)$, where $1 \leqslant r \leqslant t$. Thus it follows that $(G/L)' \leqslant N_{t-r}(HL/L)$, hence $G' \leqslant N_{t-r}(HL) = N_{t-r}(HZ(G))$ $= N_{t-r+1}(H) \leqslant N_t(H)$. This completes the proof. ∎

4. Loop theory

We combine the result of Theorem 3.1 with Theorem 2.1 and we have the following result which generalises the result in[9, Corollary 4.1].

Corollary 4.1. *Let Q be a finite loop. Then the inner mapping group $I(Q)$ can not be isomorphic to $C_{p^k} \times C_{p^l}$, where p, k and l are as in Theorem 3.1.*

Theorem 3.2 also has an interpretation in loop theory but in order to present this interpretation we first must introduce the reader the notion of *central nilpotency* in loop theory. If Q is a loop, then the *centre* $Z(Q)$ consists of all elements $a \in Q$ which satisfy the equations $ax.y = a.xy$, $xa.y = x.ay$, $xy.a = x.ya$ and $xa = ax$ for all $x, y \in Q$. It is not difficult to see that $Z(Q)$ is an Abelian subgroup of Q and $Z(Q) \cong Z(M(Q))$. If we put $Z_0 = 1$, $Z_1 = Z(Q)$ and $Z_i/Z_{i-1} = Z(Q/Z_{i-1})$, then we obtain a series of normal subloops of Q. If Z_{n-1} is a proper subloop of Q but $Z_n = Q$, then Q is centrally nilpotent of class n.

Now we write $I_0 = I(Q)$ and $I_i = N_{M(Q)}(I_{i-1})$ for each $i \geqslant 1$. Bruck ([**4**, pp. 278–281]) proved the following criterion for central nilpotency of Q.

Theorem 4.2. *A necessary and sufficient condition that Q be centrally nilpotent of class c, is that $I_c = M(Q)$ but $I_{c-1} \neq M(Q)$.*

If Q is centrally nilpotent of class $\leqslant 2$, then $N_{M(Q)}(I(Q)) = I(Q) \times Z(M(Q))$ is normal in $M(Q)$. As the core of $I(Q)$ in $M(Q)$ is trivial, we conclude that $I(Q)$ is an Abelian group. Kepka and Niemenmaa managed to show in [**12**] that if Q is a finite loop such that $I(Q)$ is Abelian, then Q is centrally nilpotent. Very little is known about how the structure of the Abelian inner mapping group influences the nilpotency class of the loop. By combining Theorem 3.2 with Theorem 4.2 we have the following.

Corollary 4.3. *If Q is a finite loop and $I(Q) \cong C_{p^n} \times C_{p^n}$ (here p is an odd prime), then Q is centrally nilpotent of class $\leqslant n+1$.*

5. Remarks

Recall that if Q is a group then $I(Q)$ is the group of inner automorphisms of Q. Groups, which are isomorphic to inner automorphism groups of groups, are called centre factor groups or capable groups. The question which Abelian groups can occur as capable groups was completely solved by Baer [**3**]. In the finite case his result is as follows:

Let $G = C_1 \times \cdots \times C_n$ be a finite Abelian group written as a product of cyclic groups such that $|C_{i+1}|$ divides $|C_i|$. Then G is a capable group if and only if $n \geqslant 2$ and $|C_1| = |C_2|$.

Our conjecture is that the situation in loop theory is similar and we thus claim the following: If Q is a finite loop and $I(Q) = C_1 \times \cdots \times C_n$ is a finite Abelian group (written as in the result by Baer), then $n \geqslant 2$ and $|C_1| = |C_2|$.

In a recent paper of Ali and Cossey [**1**], the authors show how to construct 4 - tuples (G, H, A, B) which satisfy the conditions of our Theorem 2.1. In the construction H is Abelian, $A = B$ and $F(G) = H \times Z(G)$ (here $F(G)$ denotes the Fitting subgroup of G). Thus the corresponding loop has an Abelian inner mapping group and is centrally nilpotent of class two.

Finally, what do we know about the nonabelian case? There are some recent results (see [**2, 5**]) on nonabelian capable groups (mainly groups which are nilpotent of class two). In loop theory, Niemenmaa [**10**] has shown that if C is a nontrivial finite cyclic group of odd order and D is a dihedral 2 - group, then $I(Q) \cong D \times C$ is not possible.

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