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BOUNDARY LINKS AND MUTATIONS

GYO TAEK JIN AND KI HYOUNG KO

ABSTRACT. Mutants of boundary links may not be boundary links, not even homology boundary links. Hence mutants of homology boundary links may not be homology boundary links.

A *tangle* in a link L in S^3 is a part of L in one side of a 2-sphere intersecting L transversely at 4 points. The three types of moves given by 180° rotations as shown in Figure 1 on a tangle are called mutations. A knot or link obtained by a mutation is called a *mutant* of the given knot or link.

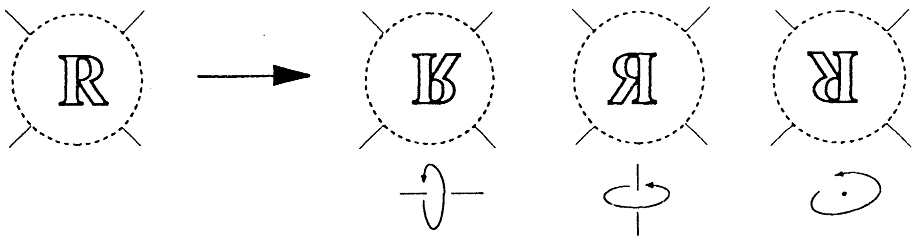

Figure 1.

It is known that mutations preserve the homfly polynomial and Kauffman's two-variable polynomial, and hence the Alexander-Conway polynomial and the Jones polynomial [4,6,8]. But mutations do not preserve knot-cobordism classes [5]. Moreover, mutations on a two-component link may change both of the knot-cobordism classes of its components as well as its link-cobordism class.

In Figure 2, $K_1 \cup K_2$ is a slice link, hence link-cobordant to the unlink. Its mutant $L_1 \cup L_2$ is not null-cobordant since it has non-trivial Cochran sequences $\{0, 1, 1, \ldots\}$ and $\{0, -1, -1, \ldots\}$ [1]. Note also that L_1 and L_2 are trefoils, hence not null-corbordant.

A link $L = L_1 \cup \cdots \cup L_n$ is called a *boundary link* if there are disjoint oriented surfaces $V_1, \ldots, V_n$ such that $\partial V_i = L_i$ for all i. L is called a $\mathbb{Z}_2$*-boundary link* if V_i's are allowed to be non-orientable [3].

PROPOSITION 1. *Mutants of two-component $\mathbb{Z}_2$-boundary links are $\mathbb{Z}_2$-boundary links.*

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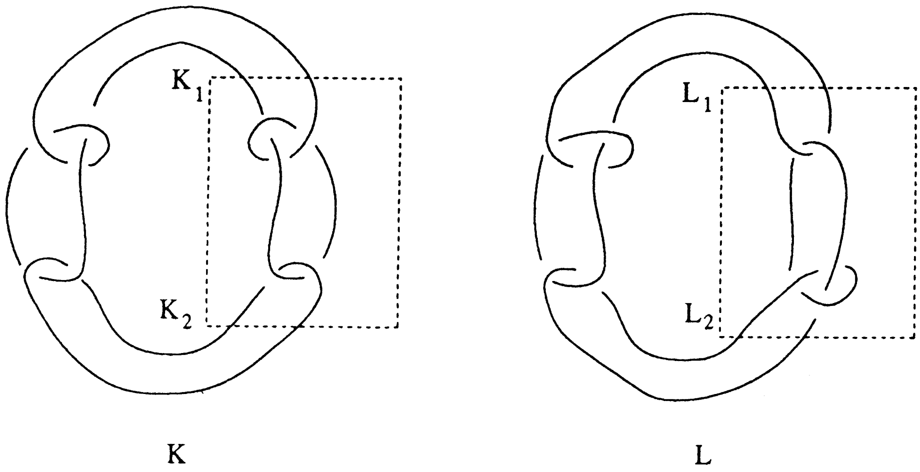

Figure 2.

PROOF. Let $V_1 \cup V_2$ be a disjoint union of surfaces bounded by the link $K_1 \cup K_2$ and let S be a 2-sphere enclosing a tangle T. We may assume that $S \cap (V_1 \cup V_2)$ is a union of finite number of circles and two arcs which are mutually disjoint. By cutting and pasting on $V_1 \cup V_2$ we can remove circles which do not separate the two arcs, one by one from the innermost ones. Suppose such circles are all removed. If any two components in $S \cap (V_1 \cup V_2)$ contained in V_i are adjacent, we can add a one-handle to V_i along a path in S joining them. Therefore, no matter whether the arcs are from the same surface or not, the sequence of components of $S \cap (V_1 \cup V_2)$ from one arc to the other is from V_1 and V_2 alternatingly. By an isotopy, make S round and $S \cap (V_1 \cup V_2)$ symmetric. Then any mutation on the tangle T will create a new pair of disjoint surfaces bounded by the mutant. ∎

This construction does not work for oriented surfaces. In Figure 3, $L = L_1 \cup L_2$ is a mutant of the boundary link $K = K_1 \cup K_2$. But it is not a boundary link. It is not even a *homology boundary link* [9]. According to the following theorem, it is enough to show that there is no epimorphism of $\pi_1(S^3 \setminus L)$ onto a free group F of rank 2.

THEOREM [9]. *$L = L_1 \cup \cdots \cup L_n$ is a homology boundary link with n components if and only if there exists a homomorphism $f\colon \pi_1(S^3 \setminus L) \longrightarrow F(n)$ onto a free group of rank n. Furthermore, L is a boundary link if and only if there exist meridians $\alpha_1, \ldots, \alpha_n$ of $L_1, \ldots, L_n$ such that $f(\alpha_1), \ldots, f(\alpha_n)$ freely generate $F(n)$.*

A computation from Figure 4 shows that $\pi_1(S^3 \setminus L)$ has the following Wertinger presentation:

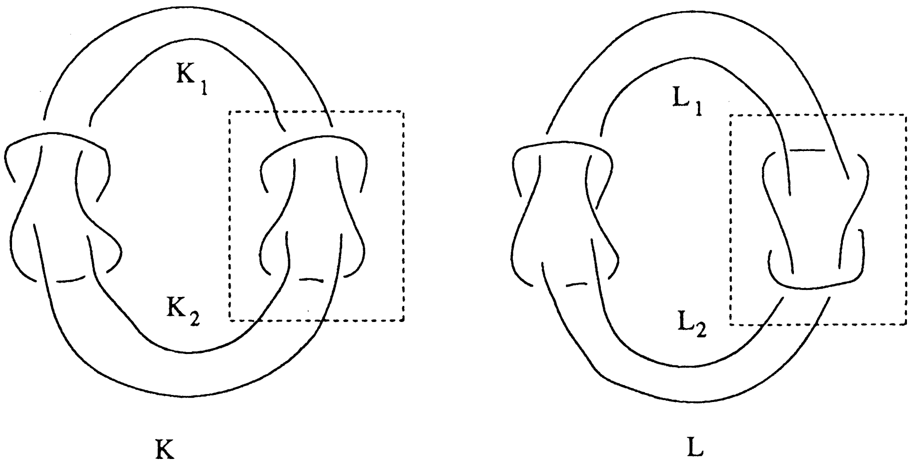

Figure 3.

$$\begin{aligned}\pi_1(S^3 \setminus L) = \langle a,b,c,d,e,f \mid & bd = d^{-1}bde, \\ & cd = d^{-1}cdf, \\ & f^{-1}d^{-1}cdf = e^{-1}d^{-1}bde, \\ & ea = a^{-1}eab, \\ & fa = a^{-1}fac, \\ & c^{-1}a^{-1}fac = b^{-1}a^{-1}eab\rangle \\ = \langle a,b,c,d \mid & c^{-1}d^{-1}cdc = b^{-1}d^{-1}bdb, \\ & d^{-1}b^{-1}dbda = a^{-1}d^{-1}b^{-1}dbdab, \\ & d^{-1}c^{-1}dcda = a^{-1}d^{-1}c^{-1}dcdac, \\ & c^{-1}a^{-1}d^{-1}c^{-1}dcdac = b^{-1}a^{-1}d^{-1}b^{-1}dbdab\rangle\end{aligned}$$

Suppose there is an epimorphism $f: \pi_1(S^3 \setminus L) \longrightarrow F = F(2)$. If $f(b) \neq f(c)$, then $f(c^{-1}d^{-1}cdc) = f(b^{-1}d^{-1}bdb)$ would be a nontrivial relation in F which is impossible since F is free. Therefore f must factor through the quotient group

$$G = \pi_1(S^3 \setminus L)/\langle bc^{-1}\rangle = \langle a,c,d \mid d^{-1}c^{-1}dcda = a^{-1}d^{-1}c^{-1}dcdac\rangle.$$

The change of variables

$$\begin{cases} x = a \\ y = cd \\ z = d^{-1}c^{-1}dcdacd, \end{cases}$$

or equivalently,

$$\begin{cases} a = x \\ c = y^2xyz^{-1}y^{-1} \\ d = yzy^{-1}x^{-1}y^{-1} \end{cases}$$

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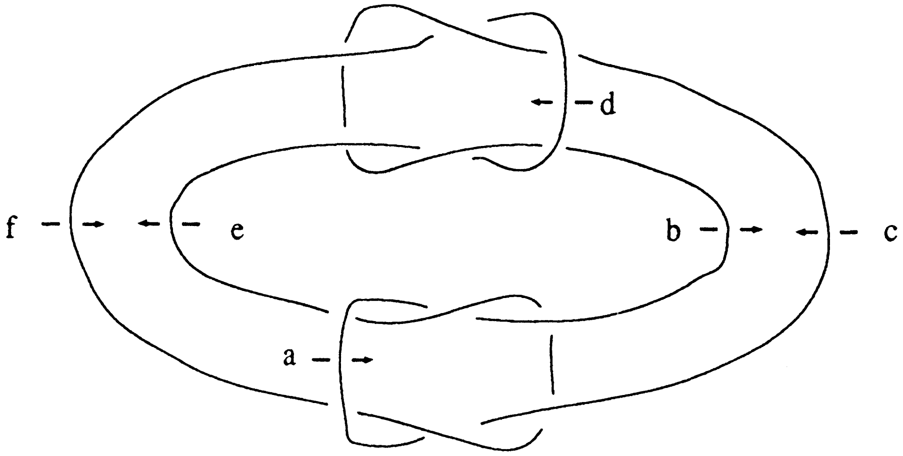

Figure 4.

gives

$$G = \langle x, y, z \mid xz^2 = zyxy \rangle.$$

Let D be the free differential operator [2]. Applying D on the relation $xz^2 = zyxy$, we obtain

$$Dx + xDz + xy^2Dz = Dz + y^2Dy + y^3Dx + xy^3Dy,$$

i.e.,

$$r = (1 - y^3)Dx - (y^2 + xy^3)Dy + (-1 + x + xy^2)Dz = 0,$$

since abelianization of G gives $z = y^2$. Hence f induces an epimorphism $\tilde{f}\colon M \longrightarrow \Lambda \oplus \Lambda$ of Λ-modules where Λ is the ring $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$ of Laurent polynomials in x and y and M is the Λ-module presented by $\langle Dx, Dy, Dz \mid r \rangle$. Then

$$0 \longrightarrow \operatorname{Ker}\phi \hookrightarrow \Lambda \oplus \Lambda \oplus \Lambda \xrightarrow{\phi} \Lambda \oplus \Lambda \longrightarrow 0$$

is a splitting short exact sequence where $\phi\colon \Lambda \oplus \Lambda \oplus \Lambda \longrightarrow \Lambda \oplus \Lambda$ is the composite

$$\begin{array}{ccccc} \Lambda \oplus \Lambda \oplus \Lambda & \xrightarrow{\text{proj}} & M & \xrightarrow{\tilde{f}} & \Lambda \oplus \Lambda. \\ & & \| & & \\ & & \langle Dx, Dy, Dz \rangle / \Lambda r & & \end{array}$$

Therefore $\operatorname{Ker}\phi$ is a projective Λ-module. It is in fact a rank one free Λ-module [7, Corollary 4.12]. Then the 2×3 matrix over Λ representing ϕ is a minor of a 3×3 invertible matrix B satisfying

$$B \begin{pmatrix} 1 - y^3 \\ -y - xy^3 \\ -1 + x + xy^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

or equivalently,

$$\bullet \qquad \begin{pmatrix} 1 - y^3 \\ -y - xy^3 \\ -1 + x + xy^2 \end{pmatrix} = B^{-1} \begin{pmatrix} 0 \\ 0 \\ \lambda \end{pmatrix}$$

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for some $\lambda \in \Lambda$. Therefore

$$I = (1 - y^3, -y - xy^3, -1 + x + xy^2)$$

is a principal ideal of Λ generated by λ. Consider the ring homomorphism

$$h: \mathbb{Z}[x, x^{-1}, y, y^{-1}] \longrightarrow \mathbb{Z}[x, x^{-1}]$$

given by $h(y) = 1$. Then

$$h(I) = (1 + x, 1 - 2x)$$

which is a not a principal ideal, a contradiction. Consequently L is not a boundary link, not even a homology boundary link.

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