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A CONCEPT OF SYNCHRONICITY ASSOCIATED WITH CONVEX FUNCTIONS IN LINEAR SPACES AND APPLICATIONS

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Abstract

A concept of synchronicity associated with convex functions in linear spaces and a Chebyshev type inequality are given. Applications for norms, semi-inner products and convex functions of several real variables are also given.

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1. Introduction

The Jensen inequality for convex functions plays a crucial role in the theory of inequalities due to the fact that other inequalities such as the arithmetic–geometric mean inequality, Hölder and Minkowski inequalities, and Ky Fan's inequality can be obtained as particular cases of it.

Let C be a convex subset of the linear space X and f be a convex real-valued function on C. If $\mathbf{p} = (p_1, \ldots, p_n)$ is a probability sequence and $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \le \sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

is well known in the literature as *Jensen's inequality*.

For refinements of the Jensen inequality and applications related to Ky Fan's inequality, the arithmetic–geometric mean inequality, the generalized triangle inequality and the f-divergence measures, see [5–11, 13–16, 21].

Assume that $f : X \to \mathbb{R}$ is a *convex function* on the real linear space X. Since for any vectors $x, y \in X$ the function

$$g_{x,y} : \mathbb{R} \to \mathbb{R}, \quad g_{x,y}(t) := f(x + ty)$$

is convex, it follows that the limits

$$\nabla_{+(-)} f(x)(y) := \lim_{t \to 0+(-)} \frac{f(x+ty) - f(x)}{t}$$

exist and are called the *right (left) Gâteaux derivatives* of the function f at the point x in the direction y.

It is obvious that, for any $t > 0 > s$,

$$\begin{aligned} \frac{f(x+ty) - f(x)}{t} &\geq \nabla_+ f(x)(y) = \inf_{t>0} \left[\frac{f(x+ty) - f(x)}{t} \right] \\ &\geq \sup_{s<0} \left[\frac{f(x+sy) - f(x)}{s} \right] = \nabla_- f(x)(y) \\ &\geq \frac{f(x+sy) - f(x)}{s} \end{aligned} \tag{1.2}$$

for any $x, y \in X$ and, in particular,

$$\nabla_- f(u)(u-v) \geq f(u) - f(v) \geq \nabla_+ f(v)(u-v) \tag{1.3}$$

for any $u, v \in X$. We call this the *gradient inequality* for the convex function f. It will be used frequently in the following in order to obtain various results related to Jensen's inequality.

The following properties are also of importance:

$$\nabla_+ f(x)(-y) = -\nabla_- f(x)(y) \tag{1.4}$$

and

$$\nabla_{+(-)} f(x)(\alpha y) = \alpha \nabla_{+(-)} f(x)(y) \tag{1.5}$$

for any $x, y \in X$ and $\alpha \geq 0$.

The right Gâteaux derivative is *subadditive* while the left one is *superadditive*, that is,

$$\nabla_+ f(x)(y+z) \leq \nabla_+ f(x)(y) + \nabla_+ f(x)(z) \tag{1.6}$$

and

$$\nabla_- f(x)(y+z) \geq \nabla_- f(x)(y) + \nabla_- f(x)(z) \tag{1.7}$$

for any $x, y, z \in X$. Some natural examples can be provided by the use of normed spaces.

Assume that $(X, \|\cdot\|)$ is a real normed linear space. The function $f : X \to \mathbb{R}$, $f(x) := \frac{1}{2}\|x\|^2$ is a convex function which generates the *superior* and *inferior semi-inner products*

$$\langle y, x \rangle_{s(i)} := \lim_{t \to 0+(-)} \frac{\|x+ty\|^2 - \|x\|^2}{t}.$$

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For a comprehensive study of the properties of these mappings in the geometry of Banach spaces see the monograph [10].

For the convex function $f_p : X \to \mathbb{R}$, $f_p(x) := \|x\|^p$ with $p > 1$,

$$\nabla_{+(-)} f_p(x)(y) = \begin{cases} p\|x\|^{p-2}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

for any $y \in X$. If $p = 1$, then

$$\nabla_{+(-)} f_1(x)(y) = \begin{cases} \|x\|^{-1}\langle y, x\rangle_{s(i)} & \text{if } x \neq 0 \\ +(-)\|y\| & \text{if } x = 0 \end{cases}$$

for any $y \in X$. This class of functions will be used to illustrate the inequalities obtained in the general case of convex functions defined on an entire linear space.

In the recent paper [12] the following refinement and reverse of the Jensen inequality in terms of the gradient were obtained.

THEOREM 1.1. *Let $f : X \to \mathbb{R}$ be a convex function defined on a linear space X. Then for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ we have the inequality*

$$\begin{aligned} \sum_{k=1}^n p_k \nabla_- f(x_k)(x_k) &- \sum_{k=1}^n p_k \nabla_- f(x_k)\left(\sum_{i=1}^n p_i x_i\right) \\ &\geq \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right) \\ &\geq \sum_{k=1}^n p_k \nabla_+ f\left(\sum_{i=1}^n p_i x_i\right)(x_k) - \nabla_+ f\left(\sum_{i=1}^n p_i x_i\right)\left(\sum_{i=1}^n p_i x_i\right) \geq 0. \end{aligned} \tag{1.8}$$

A particular case of interest is for $f(x) = \|x\|^p$ where $(X, \|\cdot\|)$ is a normed linear space. Then for any $p \geq 1$, for any n-tuple of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ with $\sum_{i=1}^n p_i x_i \neq 0$ we have the inequality

$$\begin{aligned} \sum_{i=1}^n p_i \|x_i\|^p &- \left\|\sum_{i=1}^n p_i x_i\right\|^p \\ &\geq p\left\|\sum_{i=1}^n p_i x_i\right\|^{p-2}\left[\sum_{k=1}^n p_k \left\langle x_k, \sum_{j=1}^n p_j x_j\right\rangle_s - \left\|\sum_{i=1}^n p_i x_i\right\|^2\right] \\ &\geq 0. \end{aligned} \tag{1.9}$$

If $p \geq 2$ the inequality holds for any n-tuple of vectors and probability distribution.

Also, for any $p \geq 1$, for any n-tuple of vectors

$$\mathbf{x} = (x_1, \ldots, x_n) \in X^n \setminus \{(0, \ldots, 0)\}$$

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and any probability distribution $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{P}^n$ we have the inequality

$$\begin{aligned} p\Big[\sum_{k=1}^{n} p_k \|x_k\|^p - \sum_{k=1}^{n} p_k \|x_k\|^{p-2}\Big\langle \sum_{j=1}^{n} p_j x_j, x_k \Big\rangle_i \Big] \\ \geq \sum_{i=1}^{n} p_i \|x_i\|^p - \Big\| \sum_{i=1}^{n} p_i x_i \Big\|^p. \end{aligned} \tag{1.10}$$

For related inequalities for norms and inner products, see [1–4, 20, 22].

Motivated by the above results we introduce in this paper a class of sequences associated with convex functions in linear spaces and establish a Chebyshev type inequality and some new inequalities for convex functions. Applications for norms, semi-inner products and convex functions of several real variables are also given.

2. ∇f-Synchronicity

Consider $f : X \to \mathbb{R}$ a convex function on the linear space X. We also assume that $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are two n-tuples of vectors with $u_i, v_i \in X$, $i \in \{1, \ldots, n\}$.

DEFINITION 2.1. We say that v is ∇f-synchronous with u if

$$\nabla_- f(u_k)(v_k - v_j) \geq \nabla_+ f(u_j)(v_k - v_j) \tag{2.1}$$

for any $k, j \in \{1, \ldots, n\}$. If the inequality is reversed in (2.1) for each $k, j \in \{1, \ldots, n\}$, then we say that v is ∇f-asynchronous with u.

We notice that in general, if v is ∇f-asynchronous with u, this does not imply that u is ∇f-synchronous with v.

As general examples of such convex functions we can consider $f(x) = \|x\|^p$, $p \geq 1$, where $(X, \|\cdot\|)$ is a normed linear space. Since (see introduction)

$$\begin{aligned} \nabla_- f(x)(y) &= p\|x\|^{p-2}\langle y, x\rangle_i \quad \text{for } x, y \in X \text{ with } x \neq 0; \\ \nabla_- f(0)(y) &= \begin{cases} 0 & \text{if } p > 1 \\ -\|y\| & \text{if } p = 1, \end{cases} \quad \text{for } y \in X; \\ \nabla_+ f(x)(y) &= p\|x\|^{p-2}\langle y, x\rangle_s \quad \text{for } x, y \in X \text{ with } x \neq 0; \\ \nabla_+ f(0)(y) &= \begin{cases} 0 & \text{if } p > 1 \\ \|y\| & \text{if } p = 1, \end{cases} \quad \text{for } y \in X, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_s$ is the *superior semi-inner product* and $\langle \cdot, \cdot \rangle_i$ is the *inferior semi-inner product*, we can define the following concepts of synchronicity for the two n-tuples of vectors $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$.

Let $p \geq 1$ and $u, v \in X^n$ be as above. We say that v *is* $p - \nabla$-*synchronous with* u if

$$\|u_k\|^{p-2}\langle v_k - v_j, u_k\rangle_i \geq \|u_j\|^{p-2}\langle v_k - v_j, u_j\rangle_s \tag{2.2}$$

for any $k, j \in \{1, \ldots, n\}$.

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We observe that for $p \in [1, 2)$ we should assume that $u_k \neq 0$ for $k \in \{1, \dots, n\}$. For $p = 2$, the relation (2.2) reduces to

$$\langle v_k - v_j, u_k \rangle_i \geq \langle v_k - v_j, u_j \rangle_s \quad \text{for any } k, j \in \{1, \dots, n\}. \tag{2.3}$$

If $(X, \|\cdot\|)$ is a smooth normed space, meaning that the norm is Gâteaux differentiable on any $x \in X$, $x \neq 0$, and if we denote by $[\cdot, \cdot]$ the semi-inner product generating the norm $\|\cdot\|$ (see [10, pp. 19–20]), then the fact that v is $p - \nabla$-synchronous with u means that

$$\|u_k\|^{p-2}[v_k - v_j, u_k] \geq \|u_j\|^{p-2}[v_k - v_j, u_j] \tag{2.4}$$

for any $k, j \in \{1, \dots, n\}$. For $p = 2$,

$$[v_k - v_j, u_k] \geq [v_k - v_j, u_j] \quad \text{for any } k, j \in \{1, \dots, n\}. \tag{2.5}$$

Moreover, if the norm $\|\cdot\|$ is generated by an inner product $\langle \cdot, \cdot \rangle$, then v is $p - \nabla$-synchronous with u means that

$$\langle v_k - v_j, \|u_k\|^{p-2}u_k - \|u_j\|^{p-2}u_j \rangle \geq 0 \quad \text{for any } k, j \in \{1, \dots, n\}, \tag{2.6}$$

while for $p = 2$, it reduces to

$$\langle v_k - v_j, u_k - u_j \rangle \geq 0 \quad \text{for any } k, j \in \{1, \dots, n\}, \tag{2.7}$$

which is the concept of *synchronous sequences* in inner product spaces that was introduced in [18]. For some inequalities for synchronous sequences in inner product spaces, see [17, 18].

As natural examples of synchronous sequences in inner product spaces, we can consider the sequences $\{x_i\}_{i \in \mathbb{N}}$ and $\{Ax_i\}_{i \in \mathbb{N}}$ where $A : X \to X$ is a positive linear operator on X, that is, $\langle Ax, x \rangle \geq 0$ for any $x \in X$.

For a convex function $f : X \to \mathbb{R}$ we define $\tilde{\nabla} f(\cdot)(\cdot)$ as

$$\tilde{\nabla} f(x)(y) := \tfrac{1}{2}[\nabla_- f(x)(y) + \nabla_+ f(x)(y)], \tag{2.8}$$

where $x, y \in X$. We observe that for f as above, we have the *homogeneity property*:

$$\tilde{\nabla} f(x)(\alpha y) = \alpha \tilde{\nabla} f(x)(y) \quad \text{for any } x, y \in X, \tag{2.9}$$

and any $\alpha \in \mathbb{R}$.

The following inequality for $\nabla - f$-synchronous sequences holds.

THEOREM 2.2. *Assume that v is $\nabla - f$-synchronous with u and $\mathbf{p} = (p_1, \dots, p_n)$ is a probability distribution. Then*

$$\sum_{i=1}^{n} p_i \tilde{\nabla} f(u_i)(v_i) \geq \sum_{i,j=1}^{n} p_i p_j \tilde{\nabla} f(u_i)(v_j). \tag{2.10}$$

PROOF. Since $\nabla_+(\cdot)(\cdot)$ is subadditive in the second variable, then

$$\nabla_+ f(u_j)(v_i - v_j) \geq \nabla_+ f(u_j)(v_i) - \nabla_+ f(u_j)(v_j) \tag{2.11}$$

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for any $i, j \in \{1, \dots, n\}$. Also, by the fact that $\nabla_-(\cdot)(\cdot)$ is superadditive in the second variable, we have that

$$\nabla_- f(u_i)(v_i) - \nabla_- f(u_i)(v_j) \geq \nabla_- f(u_i)(v_i - v_j) \tag{2.12}$$

for all $i, j \in \{1, \dots, n\}$. Now, by (2.11), (2.12) and by the definition of $\nabla - f$-synchronicity, we deduce that

$$\nabla_- f(u_i)(v_i) - \nabla_- f(u_i)(v_j) \geq \nabla_+ f(u_j)(v_i) - \nabla_+ f(u_j)(v_j),$$

which is equivalent to

$$\nabla_- f(u_i)(v_i) + \nabla_+ f(u_j)(v_j) \geq \nabla_+ f(u_j)(v_i) + \nabla_- f(u_i)(v_j) \tag{2.13}$$

for all $i, j \in \{1, \dots, n\}$.

Therefore, by multiplying (2.13) with $p_i p_j \geq 0$ and summing over i and j from 1 to n, we get

$$\begin{aligned} &\sum_{i=1}^n p_i \nabla_- f(u_i)(v_i) + \sum_{j=1}^n p_j \nabla_+ f(u_j)(v_j) \\ &\quad \geq \sum_{i,j=1}^n p_i p_j \nabla_+ f(u_j)(v_i) + \sum_{i,j=1}^n p_i p_j \nabla_- f(u_i)(v_j). \end{aligned} \tag{2.14}$$

Now, observe that

$$\sum_{j=1}^n p_j \nabla_+ f(u_j)(v_j) = \sum_{i=1}^n p_i \nabla_+ f(u_i)(v_i)$$

and

$$\sum_{i,j=1}^n p_i p_j \nabla_+ f(u_j)(v_i) = \sum_{i,j=1}^n p_i p_j \nabla_+ f(u_i)(v_j),$$

which, by (2.14) divided by 2, provides the desired result (2.10). □

COROLLARY 2.3. *With the assumptions of Theorem 2.2, and if in addition $\tilde{\nabla} f(u_i)(\cdot)$ is additive for any $i \in \{1, \dots, n\}$, then*

$$\sum_{i=1}^n p_i \tilde{\nabla} f(u_i)(v_i) \geq \sum_{i=1}^n p_i \tilde{\nabla} f(u_i)\Big(\sum_{j=1}^n p_j u_j\Big). \tag{2.15}$$

REMARK 2.4. If f is Gâteaux differentiable at the points u_i, $i \in \{1, \dots, n\}$, then $\tilde{\nabla} f(u_i)(\cdot) = \nabla f(u_i)(\cdot)$ and is therefore linear on X. With this assumption, inequality (2.15) becomes

$$\sum_{i=1}^n p_i \nabla f(u_i)(v_i) \geq \sum_{i=1}^n p_i \nabla f(u_i)\Big(\sum_{j=1}^n p_j u_j\Big). \tag{2.16}$$

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Moreover, we note that there are examples of convex functions defined on linear spaces for which $\tilde{\nabla} f(x)(\cdot)$ is linear for any $x \neq 0$ without the function f being Gâteaux differentiable at that point (see, for instance, [10, pp. 44–45]).

Following [19], we consider the *g-semi-inner product* $\langle \cdot, \cdot \rangle_g : X \times X \to \mathbb{R}$ defined by

$$\langle y, x \rangle_g := \tfrac{1}{2}[\langle y, x \rangle_i + \langle y, x \rangle_s], \quad x, y \in X.$$

Using this notation, we have the following conditional inequality for semi-inner products and norms in normed linear spaces.

PROPOSITION 2.5. *Let $(X, \|\cdot\|)$ be a normed linear space, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in X^n$ and $p \geq 1$. If*

$$\|u_k\|^{p-2} \langle v_k - v_j, u_k \rangle_i \geq \|u_j\|^{p-2} \langle v_k - v_j, u_j \rangle_s \tag{2.17}$$

for any $k, j \in \{1, \ldots, n\}$, then

$$\sum_{k=1}^{n} p_k \|u_k\|^{p-2} \langle v_k, u_k \rangle_g \geq \sum_{k,j=1}^{n} p_k p_j \|u_k\|^{p-2} \langle v_j, u_k \rangle_g \tag{2.18}$$

for any $\mathbf{p}$ a probability distribution. If $p < 2$, then we should have in (2.17) all $u_k \neq 0$. If $p = 2$ and

$$\langle v_k - v_j, u_k \rangle_i \geq \langle v_k - v_j, u_j \rangle_s \tag{2.19}$$

for any $k, j \in \{1, \ldots, n\}$, then

$$\sum_{k=1}^{n} p_k \langle v_k, u_k \rangle_g \geq \sum_{k,j=1}^{n} p_k p_j \langle v_j, u_k \rangle_g, \tag{2.20}$$

for any $\mathbf{p}$ a probability distribution.

As a particular case of interest, we state the following result that holds in inner product spaces.

COROLLARY 2.6. *Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n) \in X^n$ and $p \geq 1$. If*

$$\langle v_k - v_j, \|u_k\|^{p-2} u_k - \|u_j\|^{p-2} u_j \rangle \geq 0 \tag{2.21}$$

for any $k, j \in \{1, \ldots, n\}$, then

$$\sum_{k=1}^{n} p_k \|u_k\|^{p-2} \langle v_k, u_k \rangle \geq \left\langle \sum_{j=1}^{n} p_j u_j, \sum_{k=1}^{n} p_k \|u_k\|^{p-2} u_k \right\rangle \tag{2.22}$$

for any $\mathbf{p}$ a probability distribution.

REMARK 2.7. We observe that if the n-tuples u and v above are synchronous, that is,

$$\langle v_k - v_j, u_k - u_j \rangle \geq 0 \quad \text{for any } j, k \in \{1, \ldots, n\}, \tag{2.23}$$

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then we have the Chebyshev type inequality

$$\sum_{k=1}^{n} p_k \langle v_k, u_k \rangle \geq \left\langle \sum_{k=1}^{n} p_k v_k, \sum_{k=1}^{n} p_k u_k \right\rangle. \tag{2.24}$$

This result was first obtained in [18].

3. Inequalities for convex functions

The following result for convex functions may be stated.

THEOREM 3.1. *Let $f : X \to \mathbb{R}$ be a convex function on the linear space X and $x, y \in X^n$. Let $\mathbf{p}$ be a probability distribution so that*

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i.$$

If $x - y$ is $\nabla - f$-synchronous with y and $\tilde{\nabla} f(y_i)(\cdot)$ is additive for each $i \in \{1, \ldots, n\}$, then

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{i=1}^{n} p_i f(y_i). \tag{3.1}$$

PROOF. Since f is convex, then for any $x, y \in X$,

$$f(x) - f(y) \geq \nabla_+ f(y)(x - y) \geq \tilde{\nabla} f(y)(x - y). \tag{3.2}$$

Then from (3.2),

$$f(x_i) - f(y_i) \geq \tilde{\nabla} f(y_i)(x_i - y_i) \tag{3.3}$$

for each $i \in \{1, \ldots, n\}$.

If we multiply (3.3) by $p_i \geq 0$ and then sum over i from 1 to n, we get

$$\sum_{i=1}^{n} p_i f(x_i) - \sum_{i=1}^{n} p_i f(y_i) \geq \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)(x_i - y_i). \tag{3.4}$$

If we now use Corollary 2.3 for $u_i = y_i$ and $v_i = x_i - y_i$, $i \in \{1, \ldots, n\}$, we deduce the inequality

$$\begin{aligned} \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)(x_i - y_i) &\geq \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)\left(\sum_{i=1}^{n} p_i (x_i - y_i)\right) \\ &= \sum_{i=1}^{n} p_i \tilde{\nabla} f(y_i)(0) = 0. \end{aligned} \tag{3.5}$$

Combining (3.4) with (3.5), we deduce the desired inequality (3.1). □

REMARK 3.2. It is clear that if f is Gâteaux differentiable at all the points y_i, $i \in \{1, \ldots, n\}$, then $\tilde{\nabla} f(y_i)(\cdot) = \nabla f(y_i)(\cdot)$, $i \in \{1, \ldots, n\}$, which are linear on X and then inequality (3.1) holds true.

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In the case of Gâteaux differentiable functions, we can state the following result as well.

THEOREM 3.3. *Let $f : X \to \mathbb{R}$ be a convex and Gâteaux differentiable function on the linear space X. Assume that $x, y \in X^n$ and $\mathbf{p}$ is a probability distribution. If $x - y$ is $\nabla - f$-synchronous with y and*

$$\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i \in \bigcap_{i=1}^{n} \ker(\nabla f(y_i)(\cdot)),$$

then

$$\sum_{i=1}^{n} p_i f(x_i) \ge \sum_{i=1}^{n} p_i f(y_i). \tag{3.6}$$

The proof is as for Theorem 3.1 when in (3.5) we take into account that

$$\nabla f(y_i)\left(\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i\right) = 0$$

for all $i \in \{1, \ldots, n\}$ since

$$\sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i y_i \in \bigcap_{i=1}^{n} \ker(\nabla f(y_i)(\cdot)).$$

The following result in smooth normed linear spaces may be stated.

PROPOSITION 3.4. *Let $(X, \|\cdot\|)$ be a smooth normed linear space and let $[\cdot, \cdot]$ be the semi-inner product that generates its norm $\|\cdot\|$. If $x, y \in X^n$ and $p \ge 1$ are such that*

$$\|y_k\|^{p-2}[x_k - y_k - x_j + y_j, y_k] \ge \|y_j\|^{p-2}[x_k - y_k - x_j + y_j, y_j] \tag{3.7}$$

for any $k, j \in \{1, \ldots, n\}$, then, for any probability distribution $\mathbf{p}$ with the property that

$$\sum_{j=1}^{n} p_j x_j = \sum_{j=1}^{n} p_j y_j, \tag{3.8}$$

we have the inequality

$$\sum_{k=1}^{n} p_k \|x_k\|^p \ge \sum_{k=1}^{n} p_k \|y_k\|^p. \tag{3.9}$$

If $p \in [1, 2)$ we shall assume that $y_k \neq 0$ for $k \in \{1, \ldots, n\}$.

If $p = 2$ and

$$[x_k - y_k - x_j + y_j, y_k] \ge [x_k - y_k - x_j + y_j, y_j] \tag{3.10}$$

for any $k, j \in \{1, \ldots, n\}$, then for any probability distribution $\mathbf{p}$ satisfying (3.8),

$$\sum_{k=1}^{n} p_k \|x_k\|^2 \ge \sum_{k=1}^{n} p_k \|y_k\|^2. \tag{3.11}$$

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The case of inner product spaces is incorporated in the following corollary.

COROLLARY 3.5. *Let $(X; \langle\cdot\,,\cdot\rangle)$ be an inner product space. If $x, y \in X^n$ and $p \geq 1$ are such that*

$$\langle x_k - x_j, \|y_k\|^{p-2} y_k - \|y_j\|^{p-2} y_j\rangle \geq \langle y_k - y_j, \|y_k\|^{p-2} y_k - \|y_j\|^{p-2} y_j\rangle \tag{3.12}$$

for any $k, j \in \{1, \ldots, n\}$, then for any $\mathbf{p}$ satisfying (3.8) we have inequality (3.9).

If $p \in [1, 2)$, then we shall assume that $y_k \neq 0$, $k \in \{1, \ldots, n\}$.

If $p = 2$ and

$$\langle x_k - x_j, y_k - y_j\rangle \geq \|y_k - y_j\|^2, \quad \text{for any } k, j \in \{1, \ldots, n\}, \tag{3.13}$$

then for any $\mathbf{p}$ satisfying (3.8), we have inequality (3.11).

REMARK 3.6. We notice that examples of sequences x_k and y_k, $k \in \{1, \ldots, n\}$, satisfying (3.13) can easily be provided by taking $x_k = Ay_k$, $k \in \{1, \ldots, n\}$, where A is a selfadjoint operator on the Hilbert space H satisfying the condition $\langle Az, z\rangle \geq \|z\|^2$, for any $z \in H$, that is, $A \geq I$ in the operator order of the Banach algebra $B(H)$.

4. Applications for convex functions on $\mathbb{R}^m$

Now consider an open and convex set C in the real linear space $\mathbb{R}^m$, $m \geq 1$. For a convex and differentiable function $f : C \to \mathbb{R}$,

$$\nabla f(x)(y) = \left\langle \frac{\partial f(x)}{\partial x}, y\right\rangle, \quad x \in C,\ y \in \mathbb{R}^m, \tag{4.1}$$

where

$$\frac{\partial f(x)}{\partial x} = \left(\frac{\partial f(x)}{\partial x^1}, \ldots, \frac{\partial f(x)}{\partial x^m}\right), \quad x = (x^1, \ldots, x^m) \in C$$

and $\langle\cdot\,,\cdot\rangle$ is the usual inner product in $\mathbb{R}^m$, that is, $\langle u, v\rangle = \sum_{k=1}^m u^i \cdot v^i$, where $u = (u^1, \ldots, u^m)$ and $v = (v^1, \ldots, v^m) \in \mathbb{R}^m$.

Now, if $\mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{R}^m$ and $\mathbf{u} := (u_1, \ldots, u_n) \in C^m$, then we say that $\mathbf{v}$ is $\nabla - f$-synchronous with $\mathbf{u}$ if

$$\left\langle \frac{\partial f(u_k)}{\partial x} - \frac{\partial f(u_j)}{\partial x}, v_k - v_j\right\rangle \geq 0 \quad \text{for any } k, j \in \{1, \ldots, n\}. \tag{4.2}$$

The following result may be stated.

PROPOSITION 4.1. *Let $f : C \to \mathbb{R}$ be a differentiable convex function on the open and convex set $C \subseteq \mathbb{R}^m$. If $\mathbf{v} := (v_1, \ldots, v_n) \in \mathbb{R}^m$ and $\mathbf{u} := (u_1, \ldots, u_n) \in C^m$ are such that $\mathbf{v}$ is $\nabla - f$-synchronous with $\mathbf{u}$, then for any probability distribution $\mathbf{p} = (p_1, \ldots, p_n)$,*

$$\sum_{i=1}^n p_i \left\langle \frac{\partial f(u_i)}{\partial x}, v_i\right\rangle \geq \left\langle \sum_{i=1}^n p_i \frac{\partial f(u_i)}{\partial x}, \sum_{i=1}^n p_i v_i\right\rangle. \tag{4.3}$$

The proof is obvious by Corollary 2.3.

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Now, if $u_k = (u_k^1, \dots, u_k^m)$, $k \in \{1, \dots, n\}$, and $v_k = (v_k^1, \dots, v_k^m)$, then

$$\left\langle \frac{\partial f(u_k)}{\partial x} - \frac{\partial f(u_j)}{\partial x}, v_k - v_j \right\rangle = \sum_{\ell=1}^{m} \left(\frac{\partial f(u_k)}{\partial x^\ell} - \frac{\partial f(u_j)}{\partial x^\ell} \right)(v_k^\ell - v_j^\ell). \tag{4.4}$$

REMARK 4.2. Relation (4.4) shows that a sufficient condition for $\mathbf{v}$ to be $\nabla - f$-synchronous with $\mathbf{u}$ is that all the sequences $\{\partial f(u_k)/\partial x^\ell\}_{k=1,\dots,n}$ and $\{v_k^\ell\}_{k=1,\dots,n}$ are synchronous, where $\ell \in \{1, \dots, m\}$, that is,

$$\left(\frac{\partial f(u_k)}{\partial x^\ell} - \frac{\partial f(u_j)}{\partial x^\ell} \right)(v_k^\ell - v_j^\ell) \geq 0 \quad \text{for any } k, j \in \{1, \dots, n\} \tag{4.5}$$

and for all $\ell \in \{1, \dots, m\}$.

The following result is an obvious consequence of Theorem 3.3.

PROPOSITION 4.3. *Let $f : C \to \mathbb{R}$ be a differentiable convex function on the open and convex set $C \subseteq \mathbb{R}^m$. If $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^m$ and $\mathbf{y} = (y_1, \dots, y_n) \in C^m$ are such that*

$$\left\langle \frac{\partial f(y_k)}{\partial x} - \frac{\partial f(y_j)}{\partial x}, x_k - x_j \right\rangle \geq \left\langle \frac{\partial f(y_k)}{\partial x} - \frac{\partial f(y_j)}{\partial x}, y_k - y_j \right\rangle, \tag{4.6}$$

for each $k, j \in \{1, \dots, n\}$, then for any probability distribution $\mathbf{p} = (p_1, \dots, p_n)$ with

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i y_i, \tag{4.7}$$

we have the inequality

$$\sum_{i=1}^{n} p_i f(x_i) \geq \sum_{i=1}^{n} p_i f(y_i). \tag{4.8}$$

REMARK 4.4. As above, a sufficient condition for (4.6) to hold is that the sequences $\{\partial f(y_k)/\partial x^\ell\}_{k=1,\dots,n}$ and $\{x_k^\ell - y_k^\ell\}_{k=1,\dots,n}$ are synchronous for each $\ell \in \{1, \dots, m\}$.

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