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FINITE GROUPS WHICH ARE AUTOMORPHISM GROUPS OF INFINITE GROUPS ONLY

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ABSTRACT. The object of this paper is to exhibit an infinite set of finite semisimple groups H, each of which is the automorphism group of some infinite group, but of no finite group. We begin the construction by choosing a finite simple group S whose outer automorphism group and Schur multiplier possess certain specified properties. The group H is a certain subgroup of Aut S which contains S. For example, most of the PSL's over a non-prime finite field are candidates for S, and in this case, H is generated by all of the inner, diagonal and graph automorphisms of S.

1. **Introduction**. Our main purpose is the construction of an infinite class of groups which are the automorphism groups of infinite groups, but not of any finite group. The simplest examples of such groups are the quaternion group Q_8, the dicyclic group DC_{12} of order 12 and the binary tetrahedral group BT_{24} of order 24. Each of these groups occurs as the automorphism group of a suitable torsion-free abelian group, as is shown by Hallett and Hirsch (see Fuchs [1], p. 272) in their classification of finite groups which are automorphism groups of torsion-free groups. Straightforward arguments show that none of the groups Q_8, DC_{12}, or BT_{24} is the automorphism group of a finite group. In the first case, this has been pointed out by de Vries and de Miranda [4].

The groups which we construct have a somewhat different structure. Our main result, Theorem A below, has as a consequence that *if* $S = \mathrm{PSL}(7, p^3)$ *where* $p \not\equiv 1 \equiv p^3$ (mod 7) *and if* H *is the subgroup of* Aut S *generated by the inner, diagonal and graph automorphisms then* (a) H *is not the automorphism group of any finite group and* (b) H *is the automorphism group of continuously many non-isomorphic countable groups*. Notice that H is isomorphic to the semidirect product $\mathrm{PGL}(7, p^3) \rtimes \langle \tau \rangle$ where τ is the transpose-inverse automorphism.

It is well-known that if $S = \mathrm{PSL}(7, p^3)$ as above, then the outer automorphism group Out S has cyclic Sylow subgroups and the Schur multiplier $M(S)$ is cyclic. Let S be any finite simple group with these properties and let H be a group such that $S \leq H \leq$ Aut S where we identify S and Inn S. We will show first that $H \cong$ Aut G for a finite group G if and only if H/S is self-normalizing in Out S. In addition, certain groups H where H/S is abelian are not automorphism groups of any group. Since Out S is metacyclic, the above facts limit the possibilities for groups H which are automorphism

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groups of infinite groups only. We will prove

THEOREM A. *Let S be a non-abelian finite simple group such that* Out S *has cyclic Sylow subgroups and* $M(S)$ *is cyclic. Then* Out $S = \langle a \rangle \rtimes \langle b \rangle$, *the semidirect product. If the action of b on* $\langle a \rangle$ *is neither the identity nor inversion, then there is a subgroup H with* Inn $S < H <$ Aut S *such that*

(*a*) *H is not the automorphism group of any finite group, and*

(*b*) *H is the automorphism group of continuously many non-isomorphic countable groups.*

Finally, we produce some slightly more complicated examples of automorphism groups of infinite groups only. These are

THEOREM B. *Let p be a prime such that* $p = 2q + 1$ *where q is an odd prime. Let S be a complete non-abelian finite simple group with* $M(S) = 0$. *Then there exists a subgroup H of the permutational wreath product S wr* Sym(p) *containing the base group which is the automorphism group of an infinite group but not of any finite group.*

2. **General results**. We will begin by classifying those groups G for which Aut G is finite and semisimple (i.e., it contains no non-trivial abelian normal subgroups). The theorems below are just minor modifications of the work of D. J. S. Robinson [3] and the proofs consist of splitting the theorems into separate cases and then showing that some of them cannot occur.

We will begin by constructing the group $G(Q, F, \epsilon)$ where Q is a finite group with trivial center, F is a torsion-free abelian group and ϵ is an element of Ext (Q_{ab}, F) with the property that $C_{\mathrm{Aut}\,F}(\epsilon) = 1$. Clearly Ext $(Q_{\mathrm{ab}}, F) \cong H^2(Q, F)$ and so we may consider ϵ to be an element of the second cohomology group. Now choose a central extension $F \rightarrowtail G \twoheadrightarrow Q$ with cohomology class ϵ and define $G(Q, F, \epsilon)$ to be G.

Henceforth, we will denote the Aut F-orbit to which ϵ belongs by $\epsilon^{\mathrm{Aut}\,F}$ and the stabilizer in Aut Q of that orbit by $\mathrm{St}_{\mathrm{Aut}\,Q}(\epsilon^{\mathrm{Aut}\,F})$.

THEOREM 2.1. *Let G be a group such that* Aut G *is finite and semisimple. Then there exists* $Q \leq$ Aut G *such that*

(*a*) *if G is infinite, then there exists F and* ϵ *as above such that* $G \cong G(Q, F, \epsilon)$, *or*

(*b*) *if G is finite, then there exists* $K \leq M(Q)_{q'}$, *where* $q = |Q_{\mathrm{ab}}|$ *and an elementary abelian 2-group D whose order is not 4 such that* $G \cong (\hat{Q}/K) \times D$ *where* $\hat{Q}$ *is the unique stem extension of* $M(Q)_{q'}$ *by Q. Furthermore, if* $D \neq 1$, *then* $q = |Q_{\mathrm{ab}}|$ *and* $|M(Q)_{q'} : K|$ *are odd.*

THEOREM 2.2. *Let* Q, F, ϵ, *K and D be as in Theorem 2.1. Then*

(*i*) Aut $(G(Q, F, \epsilon)) \cong \mathrm{St}_{\mathrm{Aut}\,Q}(\epsilon^{\mathrm{Aut}\,F})$ *where* $\epsilon^{\mathrm{Aut}\,F}$ *is the* Aut F-*orbit to which* ϵ *belongs.*

(*ii*) Aut $(\hat{Q}/K \times D) \cong N_{\mathrm{Aut}\,Q}(K) \times$ Aut D.

Moreover, these isomorphisms arise from the obvious induced mappings.

Next, we look at the automorphism group of a semisimple group (i.e., one which has no abelian normal subgroups).

LEMMA 2.3. *Let H be a finite semisimple group with completely reducible radical R. Then* Aut H *is a finite semisimple group with completely reducible radical isomorphic to R. If we identify H and* Aut H *with subgroups of* Aut R *in the natural way, then* Aut $H = N_{\mathrm{Aut}\,R}(H)$.

PROOF. Let $H_0 = \mathrm{Inn}\, H$ and let R_0 be the completely reducible radical of H_0. If A is an abelian normal subgroup of Aut H, then $A \cap H_0 = 1$ and hence $A \leq C_{\mathrm{Aut}\,H}(H_0) = 1$ since $Z(H_0) = 1$. It follows that Aut H is semisimple.

Let S denote the completely reducible radical of Aut H. Clearly $R_0 \leq S$. Since S is completely reducible as an Aut H-operator group and R_0 is Aut H invariant, it follows that $S = R_0 \times R_1$ for some Aut H invariant subgroup R_1. Hence $[H_0, R_1] \leq H_0 \cap R_1 = 1$ and $R_1 \leq C_{\mathrm{Aut}\,H}(H_0) = 1$. Therefore $S = R_0$. We may identify H and Aut H with subgroups of Aut R and the result follows. □

COROLLARY 2.4. *Let H be a finite semisimple group with completely reducible radical R. If H is self-normalizing as a subgroup of* Aut R, *then H is complete and hence it is the automorphism group of a finite group.*

Let H be a finite semisimple group such that $H \cong \mathrm{Aut}\, G$ for some group G. Let R be the completely reducible radical of H. By Theorem 2.1, $G \cong G(Q, F, \epsilon)$ or $G \cong \hat{Q}/K \times D$. Assume that H has no direct factor of type $\mathrm{PSL}(r, 2)$ and so by Theorem 2.2, $H \leq \mathrm{Aut}\, Q$. It follows that $C_H(Q) = 1$. Clearly $Q \cong \mathrm{Inn}\, G$ and so with the natural identifications $Q \leq H \leq \mathrm{Aut}\, R$. Since R is a completely reducible H-operator group and $R \cap Q$ is a normal admissible subgroup, $R = (R \cap Q) \times R_1$ for some H-invariant subgroup R_1. Therefore $R_1 \leq C_H(Q) = 1$ and hence $R \leq Q$. We deduce from this that Aut $Q = N_{\mathrm{Aut}\,R}(Q)$. With the natural identifications, this proves

LEMMA 2.5. *Assume that H is a finite semisimple group with no direct factor of type* $\mathrm{PSL}(r, 2)$. *Let R be the completely reducible radical of H. If $H \cong$* Aut G, *then*

$$R \leq Q = \mathrm{Inn}\, G \leq H \leq \mathrm{Aut}\, Q \leq \mathrm{Aut}\, R.$$

Lemma 2.5 implies that when we are checking if a group is an automorphism group we need only consider normal subgroups Q of H which contain the completely reducible radical R in either $G(Q, F, \epsilon)$ or $\hat{Q}/K \times D$. In order for H to be the automorphism group of an infinite group, there must exist non-trivial Q, F and ϵ which satisfy the necessary conditions for $G(Q, F, \epsilon)$ to exist.

LEMMA 2.6. *Let Q be a finite group with trivial center. Then there exists an infinite group $G(Q, F, \epsilon)$ if and only if Q_{ab} is not an elementary abelian 2-group. In addition, if Q_{ab} is not an elementary abelian 2-group, then there are continuously many non-isomorphic countably infinite groups $G(Q, F, \epsilon)$ with* $|\mathrm{Aut}\, F| = 2$.

PROOF. If Q_{ab} is an elementary abelian 2-group, then the inversion automorphism of F operates trivially on Ext (Q_{ab}, F) and since $C_{\mathrm{Aut}\,F}(\epsilon) = 1$ this implies that F is trivial.

Conversely assume that Q_{ab} has exponent $m > 2$. We will construct continuously many torsion-free abelian groups F_π as follows.

Let π be an infinite set of primes none of which divide m. Define F_π to be the additive group of rationals of the form $t/(p_1 \ldots p_k)$ where the p_i are distinct primes contained in π and t is an integer. The only automorphisms of F_π are given by multiplication by 1 or -1.

Writing $Q_{ab} \simeq \operatorname{Dr}_{i=1,\ldots,k} Z_{e_i}$ where e_i divides e_{i+1} and $e_k = m$, we have

$$\operatorname{Ext}(Q_{ab}, F_\pi) \cong \operatorname{Dr}_{i=1,\ldots,k} (F_\pi / e_i F_\pi).$$

Define ϵ_π to be the element of $\operatorname{Ext}(Q_{ab}, F_\pi)$ which corresponds to $(1 + e_1F_\pi, \ldots, 1 + e_kF_\pi)$ in the above Aut F_π-isomorphism. Then $C_{\operatorname{Aut} F_\pi}(\epsilon_\pi) = C_{\operatorname{Aut} F_\pi}(1 + mF_\pi) = 1$ since $m > 2$ and Aut F_π is multiplication by either 1 or -1. Hence $G(Q, F_\pi, \epsilon_\pi)$ is countably infinite for each π. □

3. **Automorphism groups which are subgroups of Aut S for a simple group S.** Let S be a non-abelian finite simple group whose multiplier $M(S)$ is cyclic and whose outer automorphism group Out S has cyclic Sylow subgroups. We are interested in those subgroups H satisfying $S \le H \le \operatorname{Aut} S$. It is clear that there is a correspondence between such groups H and subgroups $\bar{H}$ of Out S by means of the canonical surjection Aut $S \twoheadrightarrow$ Out S. Therefore we will classify all subgroups of Out S.

It is well-known that any finite group G with cyclic Sylow subgroups has presentation

$$(*) \qquad \langle a, b \mid a^m = 1 = b^n,\ b^{-1}ab = a^r \rangle$$

where $r^n \equiv 1 \pmod m$, m is odd, $1 \le r < m$ and m and $n(r-1)$ are coprime (Robinson [2], p. 281). If G is non-cyclic, then all subgroups of G are conjugate to a subgroup of the form $\langle a^k, b^\ell \rangle$ where $k|m$ and $\ell|n$. A subgroup K of G is self-normalizing if and only if it is conjugate to a subgroup of the form $\langle a^k, b\rangle$ where $k|m$ or equivalently if and only if n divides $|K|$.

If $H \cong \operatorname{Aut} G$, then any conjugate of H in Aut S is also isomorphic to Aut G. Hence we need only consider when those subgroups H which correspond to subgroups $\bar{H} = \langle a^k, b^\ell \rangle$ of Out $S = \langle a, b \rangle$ are automorphism groups.

Since $\bar{H}$ has cyclic Sylow subgroups, $M(\bar{H}) = 0$. In conjunction with the following lemma this will imply that $M(H)$ is cyclic for any group H satisfying $S \le H \le \operatorname{Aut} S$.

LEMMA 3.1. *Let N be a normal subgroup of a group Q such that $M(N)$ is cyclic, N is perfect and $M(Q/N) = 0$. Then $M(Q)$ is cyclic. In addition, if $M(N) = 0$, then $M(Q) = 0$.*

PROOF. If $M(Q) \cong C \le \tilde{Q}' \cap Z(\tilde{Q})$ where $\tilde{Q}/C \simeq Q$, and $\tilde{N}/C \simeq N$, then $\tilde{N} = \tilde{N}'C$ since N is perfect. Therefore, $\tilde{N}/\tilde{N}' \le Z(\tilde{Q}/\tilde{N}) \cap (\tilde{Q}/\tilde{N}')'$ and since $(\tilde{Q}/\tilde{N}')/(\tilde{N}/\tilde{N}') \cong Q/N$ and $M(Q/N) = 0$, we have $\tilde{N} = \tilde{N}'$. Then $C \le \tilde{N}' \cap Z(\tilde{N})$ and since $\tilde{N}/C \cong N$ and $M(N)$ is cyclic $(M(N) = 0)$, $M(Q) \cong C$ is cyclic $(M(Q) \cong C = 0)$ as required. □

Let H be a group satisfying $S \leq H \leq \text{Aut } S$ where S is as above and is not of type $\text{PSL}(r, 2)$. Assume that $H \cong \text{Aut } G$ for some finite group G. By Theorems 2.1, 2.2 and Lemma 2.5, $H \cong N_{\text{Aut } Q}(K)$ where $S \leq Q \leq H$ and $K \leq M(Q)$. Since $M(Q)$ is cyclic by Lemma 3.1, Aut Q normalizes every subgroup K of $M(Q)$. Hence $N_{\text{Aut } Q}(K) = \text{Aut } Q$. Since Out S has cyclic Sylow subgroups, normality is transitive in Out S (Robinson [2], p. 392). Therefore the subgroup $H = \text{Aut } Q$ is self-normalizing in Aut S. The above argument along with Corollary 2.4 proves

LEMMA 3.2. *A group H with the above properties is the automorphism group of a finite group if and only if it is self-normalizing in* Aut S.

Hence the group H of Lemma 3.2 is the automorphism group of a finite group if and only if $\bar{H}$ is conjugate to $\langle a^k, b \rangle$ where $k | m$.

We will now consider groups H where $\bar{H}$ is of the form $\langle a^\ell \rangle$ or $\langle b^k \rangle$ for $\ell \neq 1$ and $k \neq 1$. If H is such a group, then it is not the automorphism group of any group. This follows from

LEMMA 3.3. *Let S be a non-abelian finite simple group which is not of type PSL$(r, 2)$ and which has cyclic multiplier. Let H be a subgroup of* Aut S *containing S such that H/S is cyclic. If there exists a subgroup X of* Aut S *properly containing H such that $[H, X] \leq S$, then H is not an automorphism group.*

PROOF. Suppose that $H \cong \text{Aut } G$ for some group G. If G is finite, then $H \cong N_{\text{Aut } Q}(K)$ where $S \leq Q \leq H$ and $K \leq M(Q)$ by Theorems 2.1 and 2.2. By Lemma 3.1, $M(Q)$ is cyclic and hence $H \cong N_{\text{Aut } Q}(K) = \text{Aut } Q$. Since $X \leq \text{Aut } Q = H$, we get a contradiction.

If G is infinite, then $H \cong \text{St}_{\text{Aut } Q}(\epsilon^{\text{Aut } F})$ by Theorems 2.1 and 2.2. Since $Q/S = Q_{\text{ab}}$ and $[H, X] \leq S$, we deduce that X operates trivially on Q_{ab} and hence on Ext (Q_{ab}, F). This gives the contradiction that $X \leq H$. Hence H is not an automorphism group. □

One consequence of this lemma is that any non-abelian finite simple group S not of type $\text{PSL}(r, 2)$ with $M(S)$ and Out S cyclic has no subgroups H satisfying $S \leq H < \text{Aut } S$ which are automorphism groups.

In order to find automorphism groups of infinite groups only we must concentrate on groups S with Out S metacyclic, but not cyclic. Theorem A will provide us with a group H satisfying $S \leq H < \text{Aut } S$ with $\bar{H} = \langle a, b^k \rangle$ for $k \neq 1$ which is the automorphism group of infinite groups only.

PROOF OF THEOREM A. It is clear that Out S has presentation (*), and so it is a semidirect product.

Let Q be the subgroup of Aut S such that $\bar{Q}$ is $\langle a \rangle$. Since $Q_{\text{ab}} = Q/S$, we have that $m = |Q_{\text{ab}}|$ is odd. By Lemma 2.6, we have continuously many groups $G_\pi = G(Q, F_\pi, \epsilon_\pi)$ with $|\text{Aut } F_\pi| = 2$. Therefore the Aut F_π-orbit of ϵ_π is $\{\epsilon_\pi, -\epsilon_\pi\}$ and by Theorem 2.2, Aut $(G_\pi) \cong \text{St}_{\text{Aut } Q}(\{\epsilon_\pi, -\epsilon_\pi\})$. Since Q is normal in Aut S, it follows from Lemma 2.3 that Aut $(G_\pi) \cong \text{St}_{\text{Aut } S}(\{\epsilon_\pi, -\epsilon_\pi\})$.

Define $H = \mathrm{St}_{\mathrm{Aut}\,S}(\{\epsilon_\pi, -\epsilon_\pi\})$. It is clear that $S \leq H \leq \mathrm{Aut}\,S$ and that $H \cong \mathrm{Aut}\,(G_\pi)$ for any such set π. Hence H is a semisimple group which is isomorphic to the automorphism group of continuously many non-isomorphic countable groups. Finally, since b acts on $Q_{\mathrm{ab}} = \langle a \rangle$ as multiplication by an integer r not equal to 1 or -1, it cannot stabilize $\{\epsilon_\pi, -\epsilon_\pi\}$. Hence $b \notin \bar{H}$ and $\bar{H} = \langle a, b^\ell \rangle$ where $\ell \neq 1$. It follows by lemma 3.2 that H is not the automorphism group of any finite group. □

We are now in a position to give some concrete examples.

THEOREM 3.4. *Suppose that p and m are primes and r is a positive integer such that* $gcd(r,p-1)=1$, *m is odd, and* $gcd(r,p^m-1)$ *is a square-free integer greater than one. Then the subgroup H of* Aut $(\mathrm{PSL}(r,p^m))$ *generated by all of the inner, diagonal and graph automorphisms is the automorphism group of continuously many countable groups but no finite groups. Notice that* $H \cong \mathrm{PGL}(r,p^m) \rtimes \langle \tau \rangle$ *where* τ *is the transpose-inverse automorphism.*

PROOF. Let $S = \mathrm{PSL}(r,p^m)$. The multiplier $M(S)$ is cyclic and $|\mathrm{Out}\ S| = 2m(gcd(r,p^m-1))$ is square-free. Define the group $\langle a \rangle$ to be the subgroup of all outer diagonal automorphisms and define $\langle b \rangle$ to be the subgroup of outer graph and field automorphisms. It is easy to show that any field automorphism acts in the proper way on $\langle a \rangle$. Hence Theorem A shows the existence of the group H and we may deduce its form from the formula $H = \mathrm{St}_{\mathrm{Aut}\,S}(\{\epsilon, -\epsilon\})$. □

In particular, the groups $\mathrm{PSL}(7,p^3)$ for $p \not\equiv 1 \pmod 7$ and $p^3 \equiv 1 \pmod 7$ provide an infinite class of examples of automorphism groups of infinite groups only.

4. **More groups which are automorphism groups of infinite groups only**. The techniques used to decide which subgroups of Aut S are automorphism groups can be used to construct more complicated groups which are automorphism groups of infinite groups only.

PROOF OF THEOREM B. Let p be a prime such that $p = 2q+1$ where q is an odd prime. Let $P = \langle x \rangle$ be a Sylow p-subgroup of the symmetric group Sym (p). Then Aut $P = \langle \sigma \rangle$ where $x^\sigma = x^r$ for some $r > 1$. There exists an element y of Sym (p) of order $(p-1) = 2q$ such that $y^{-1}xy = x^r$. Let $T = \langle x, y \rangle$. Clearly $T = \langle x \rangle \rtimes \langle y \rangle$. Define $M = \langle x, y^q \rangle$ and note that $|y^q| = 2$. Since Sym (p) has $(p-2)!$ Sylow p-subgroups, $|N_{\mathrm{Sym}(p)}(P)| = p(p-1)$ and so $N_{\mathrm{Sym}(p)}(P) = T$. Clearly $N_{\mathrm{Sym}(p)}(M) = T$.

Let S be a complete non-abelian finite simple group with $M(S) = 0$ and define H to be the permutational wreath product (S wr M). Then H is a semisimple group with completely reducible radical R equal to the base group. Let Q be any normal subgroup of H containing R. It follows that Q equals either R, $R \rtimes \langle x \rangle$, or H. In each of these cases, $M(Q) = 0$ by Lemma 3.1 and hence if H is the automorphism group of a finite group, then $H = \mathrm{Aut}\ Q$ by Theorems 2.1 and 2.2. Clearly H is properly contained in Aut Q for all possible Q, which shows that H is not the automorphism group of any finite group. Also $H_{\mathrm{ab}} \cong M_{\mathrm{ab}}$ has order 2 and so it follows from Lemma 2.6 that if H is to be an automorphism group then Q cannot equal R or H.

We shall choose Q to be $R \rtimes \langle x \rangle$. Lemma 2.6 gives us an infinite group $G(Q, F, \epsilon)$ with $|\text{Aut } F| = 2$ and so

$$\text{Aut } (G(Q, F, \epsilon)) \cong \text{St}_{\text{Aut } Q}(\{\epsilon, -\epsilon\}).$$

Since S is complete, Aut $Q \cong S$ wr T. A quick calculation shows that $H \cong$ Aut $(G(Q, F, \epsilon))$ and hence H is the automorphism group of an infinite group. □

The primes less than 100 to which Theorem B applies are $p = 7, 11, 23, 47, 59$, and 83. It is not known if there are infinitely many such primes.

This paper is an excerpt from the author's doctoral dissertation at the University of Illinois (1983).

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